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ON RECURSIVE FUNCTIONS AND REGRESSIVE ISOLS

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In memory of Dr. Erik Ellentuck

We are interested in regressive isols, recursive functions, and the extension to the isols of recursive functions. Our results in the paper were motivated by an interest to clarify the nature of the domain in the regressive isols of the Myhill-Nerode extension to the isols of a recursive function of one-variable.

In particular, if f is a recursive function, f_Λ the Myhill-Nerode extension of f to the isols, Λ the set of isols, and Λ_R the set of regressive isols, we give a characterization of the set

$$\operatorname{Dom} f_\Lambda = (A \in \Lambda_R \colon f_\Lambda(A) \in \Lambda),$$

and then relate $f_\Lambda(\operatorname{Dom} f_\Lambda)$ to α_R where α is the range of f and $\alpha_R = \Lambda_R \cap \alpha_\Lambda$ where α_Λ is the Nerode extension of α to the isols.

1. Notations. Let ω be the set of non-negative integers (*numbers*). The domain and range of a (partial or total) function g will be denoted δg, ρg respectively. All functions will be assumed total, i.e., their domain $= \omega$, unless otherwise specified. We will say that a function $g\colon \omega \to \omega$ is *increasing* if $x < y$ implies $g(x) \leq g(y)$, and say that it is *eventually increasing* if there is a number k such that $k \leq x < y$ implies $g(x) \leq g(y)$.

The following theorem combines two well-known results which are proved in [**1**] and [**2**] respectively.

THEOREM T1. *Let f be a recursive and eventually increasing function. Then f_Λ maps Λ_R into Λ_R, and if α is the range of f, then $\alpha_R = f_\Lambda(\Lambda_R)$.*

The primary motivation for this paper is an attempt to understand the consequences of removing the clause "eventually increasing" from the above theorem, and the main techniques applied to investigate those consequences are infinite series of isols, [**4**], and some of the fundamental work of E. Ellentuck in [**7**].

2. Preliminaries. We would like to assume that the reader is familiar with topics and terminology in the theory of isols. In particular, we will assume a familiarity with techniques involving infinite series of isols. Important to our study is a fundamental representation theorem of Ellentuck in [7], and also the method of [7] for representing an extension to the isols of an increasing function as an infinite series of isols. These topics we shall now review.

It is a well-known result that if the extension to the isols of an increasing recursive function is evaluated at a regressive isol, then this value can be expressed as an infinite series of isols. We wish now to discuss a similar representation due to Ellentuck, but first we must define the notion $\leq^*$ due to Dekker, [5], and the difference function Δr of a function r. For functions, t, u, define $t \leq^* u$ if there is a partial recursive function p with $\rho t \subseteq \delta p$ and $p(t_n) = u_n$ for $n \in \omega$. For Y a regressive isol and u a function, define $Y \leq^* u$ if Y is finite or t is a regressive function with $\rho t \in Y$ and $t \leq^* u$. For a function r, define $\Delta r(n) = r(n+1) - r(n)$. We can now state Ellentuck's representation, [7], for extending functions that are increasing but not necessarily recursive. Let r be any increasing function, and let Y be a regressive isol. Then the extension of r, r_Λ, will be a partial function on the isols defined as follows: if Y is finite, then $r_\Lambda(Y) = r(Y)$; if Y is infinite and $Y \leq^* \Delta r$, then

$$r_\Lambda(Y) = r(0) + \sum_Y \Delta r.$$

In [7] Ellentuck showed that the value of $r_\Lambda(Y)$ is equivalent to the value obtained by taking the Nerode extension, [**10**], of r to isols. We note that if r is a recursive increasing function, then Δr is also recursive, and therefore $Y \leq^* \Delta r$ for every regressive isol Y. Also, let us note that when r is an increasing function and Y is a regressive isol with $Y \leq^* \Delta r$ or$Y$ finite, then the value of $r_\Lambda(Y)$ is also a regressive isol. This property is well known; it may be obtained from the infinite series representation of $r_\Lambda(Y)$ when Y is infinite, and from knowledge that $r_\Lambda(Y)$ is finite when Y is finite.

The following theorem is a basic result for our paper, and it is proved in [7].

THEOREM T2 (*Ellentuck*). *Let f be a recursive function, and let r be any increasing function such that the composition $f \circ r$, $(f \circ r)(n) = f(r(n))$, is also increasing. Let $A \in \Lambda_R$ with $A \leq^* \Delta r$. Then $A \leq^* \Delta(f \circ r)$ and $(f \circ r)_\Lambda(A) = f_\Lambda(r_\Lambda(A))$, and therefore also the isol $r_\Lambda(A)$ belongs to* Dom f_Λ.

It turns out that the representation given in this theorem leads to a characterization of the form of the regressive isols that are in the domain of f_Λ.

3. On the isols in Dom f_Λ. Let $f: \omega \to \omega$ denote any recursive function. It is known, from results in **[1]**, that if f is not eventually increasing, then there will be some infinite regressive isols that f_Λ does not map into the isols. However, in contrast to the above, there will always be some infinite regressive isols that are in the domain of f_Λ, i.e., that are mapped by f_Λ into the isols. Actually much more is known with respect to the existence of isols that lie within, and without, the domain of f_Λ. In **[11]** A. Nerode showed that there are infinite regressive isols that belong to Dom f_Λ for every recursive function f. In [7] E. Ellentuck called such isols *recursively strongly torre* and studied some of their properties. Ellentuck, in [7], also introduced a class of regressive isols called *recursively strongly universal.* If such an isol belongs to Dom f_Λ for a recursive function f, then f is eventually increasing.

It is well known that if A is a regressive isol and $f_\Lambda(A) \in \Lambda$, then $f_\Lambda(A) \in \Lambda_R$. It follows from this property that Dom f_Λ may be expressed as:

$$\operatorname{Dom} f_\Lambda = (A \in \Lambda_R : f_\Lambda(A) \in \Lambda) = (A \in \Lambda_R : f_\Lambda(A) \in \Lambda_R).$$

If $d: \omega \to \omega$ is any function, then we shall sometimes write d_n for $d(n)$. We would now like to state a property about certain infinite series of isols.

Let $u: \omega \to \omega$ and $v: \omega \to \omega$ be any functions, and let A be any infinite regressive isol. Assume that $A \leq^* u$, $A \leq^* v$, and also that

$$\sum_A u = \sum_A v. \tag{1}$$

Because A is an infinite regressive isol, it is then possible to verify that there will be infinitely many numbers d such that

$$u(0) + \cdots + u(d) = v(0) + \cdots + v(d).$$

Furthermore, there is a variety of other properties that follow from (1), and we shall state these properties in the following lemma.

LEMMA L1. *Let $u: \omega \to \omega$ and $v: \omega \to \omega$ be any functions, and let A be any infinite regressive isol. Assume that $A \leq^* u$, $A \leq^* v$, and also that*

$$\sum_A u = \sum_A v.$$

Then there is a strictly increasing function $d\colon \omega \to \omega$ with

$$u(0) + \cdots + u(d_i) = v(0) + \cdots + v(d_i)$$

for every number $i \in \omega$. Moreover, the function d has the following additional properties. (i) *Let $a\colon \omega \to \omega$ be any regressive function that ranges over a set in A. Set $b(n) = a(d_n)$, for each number $n \in \omega$. Then b will be a regressive function.* (ii) *Let $\beta = (b_0, b_1, \ldots)$ and let B be the regressive isol with $\beta \in B$. Then $B \leq A$.* (iii) *Define the function p by*

$$p(0) = 1 + d(0),$$
$$p(n+1) = d(n+1) - d(n).$$

Then $B \leq^* p$ and

$$A = \sum_B p.$$

Proof. Let a be a regressive function with $\rho a \in A$. Then

(1) $\quad (j(a_0, 0), \ldots, j(a_0, u_0 - 1), j(a_1, 0), \ldots, j(a_1, u_1 - 1), \ldots)$

is a representative of $\sum_A u$ with the understanding that the sequence $j(a_i, 0), \ldots, j(a_i, u_i - 1)$ is empty if $u_i = 0$. Likewise, a representative of $\sum_A v$ is

(2) $\quad (j(a_0, 0), \ldots, j(a_0, v_0 - 1), j(a_1, 0), \ldots, j(a_1, v_1 - 1), \ldots)$.

Denote the set (1) by α_u and the set (2) by α_v. Since $\sum_A u = \sum_A v$, there is a one-one partial recursive function p with $\alpha_u \subseteq \delta p$, $\alpha_v \subseteq \rho p$, and $p(\alpha_u) = \alpha_v$. We wish to show the existence of a strictly increasing function $d\colon \omega \to \omega$ with $u(0) + \cdots + u(d_i) = v(0) + \cdots + v(d_i)$ for every $i \in \omega$. To accomplish this, we begin by considering the set

$$\gamma = (x\colon u(0) + \cdots + u(x) = v(0) + \cdots + v(x)).$$

Suppose γ is finite. Then there is a number k such that

(3) $\quad x \geq k \to u(0) + \cdots + u(x) \neq v(0) + \cdots + v(x).$

Let $a_0, a_1, \ldots, a_k$ be given. Since $\rho a \in A$ and $a \leq^* u$, we can effectively find $u_0, u_1, \ldots, u_k$, and hence we can form the set

$$\alpha_u^k = (j(a_0, 0), \ldots, j(a_0, u_0 - 1), \ldots, j(a_k, 0), \ldots, j(a_k, u_k - 1)).$$

Since $A \leq^* v$, we can likewise form the set

$$\alpha_v^k = (j(a_0, 0), \ldots, j(a_0, v_0 - 1), \ldots, j(a_k, 0), \ldots, j(a_k, v_k - 1)).$$

However, since $\operatorname{card} \alpha_u^k = u(0) + \cdots + u(k)$ and $\operatorname{card} \alpha_v^k = v(0) + \cdots + v(k)$, by (3) we have $\operatorname{card} \alpha_u^k \neq \operatorname{card} \alpha_v^k$. If $\operatorname{card} \alpha_u^k > \operatorname{card} \alpha_v^k$, we can use p to produce an element $j(a_x, y) \in \alpha_v$ with $x > k$. If $\operatorname{card} \alpha_u^k < \operatorname{card} \alpha_v^k$, we can use p^{-1} to produce an element $j(a_x, y) \in \alpha_u$ with $x > k$. In either case, we can effectively find an $a_{k(1)}$ with $k(1) > k$. As above, since a is a regressive function, $A \leq^* u$, and $A \leq^* v$, we can effectively form the sets

$$\alpha_u^{k(1)} = (j(a_0, 0), \ldots, j(a_0, u_0 - 1), \ldots, j(a_{k(1)}, 0), \ldots, j(a_{k(1)}, u_{k(1)} - 1))$$

and

$$\alpha_v^{k(1)} = (j(a_0, 0), \ldots, j(a_0, v_0 - 1), \ldots, j(a_{k(1)}, 0), \ldots, j(a_{k(1)}, v_{k(1)} - 1)).$$

Again, using (3) and $k(1) > k$, we have $\operatorname{card} \alpha_u^{k(1)} \neq \operatorname{card} \alpha_v^{k(1)}$ and by repeating the above argument, we can effectively find an $a_{k(2)}$ with $k(2) > k(1)$. By iterating the above construction, we effectively produce an infinite recursively enumerable subset of ρa. But this contradicts $\rho a \in A$ being an isol. Thus, $\gamma = (x\colon u(0) + \cdots + u(x) = v(0) + \cdots + v(x))$ is infinite. If we now let d_0 = least member of $\gamma = (\mu x)\ (x \in \gamma)$ and $d_{i+1} = (\mu x)\ (x \in \gamma \text{ and } x > d_i)$, then d will be a strictly increasing function with

$$u(0) + \cdots + u(d_i) = v(0) + \cdots + v(d_i)$$

for every $i \in \omega$.

Turning our attention to (i), we let $a\colon \omega \to \omega$ and $d\colon \omega \to \omega$ be as above, and define $b_n = a(d_n)$. To establish (i), we need to show that from b_{n+1}, we can effectively find b_n. Let $b_{n+1} = a(d_{n+1})$ be given. Since a is a regressive function, we can effectively find

$$a(d_{n+1} - 1), \ldots, a(d_n), \ldots, a_0 \tag{4}$$

and

$$d_{n+1} - 1, \ldots, d_n, \ldots, 0. \tag{5}$$

From (4), $A \leq^* u$, and $A \leq^* v$, we can find

$$u(d_n - 1), \ldots, u(d_n), \ldots, u_0 \tag{6}$$

and

$$v(d_n - 1), \ldots, v(d_n), \ldots, v_0. \tag{7}$$

From (6) and (7), we can find the largest number x from (5) for which $u(0) + \cdots + u(x) = v(0) + \cdots + v(x)$. By the definition of d, we have

$d_n = x$. From d_n and (4), we can find $a(d_n) = b_n$, and therefore, (i) is established.

If we let $\beta = (b_0, b_1, b_2, \dots)$, then to establish (ii), we need to show that β and $\rho a - \beta$ are separated. To this end, let $x \in \rho a$. Since a is a regressive function, we can effectively find the j such that $x = a_j$ and in the process we can produce the list

$$(8) \qquad a_j, a_{j-1}, \dots, a_0.$$

From (8), $A \leq^* u$, and $A \leq^* v$ we can effectively find $u_0, \dots, u_j$, and $v_0, \dots, v_j$. Then by the definition of β, we have $x \in \beta$ if and only if $u_0 + \cdots + u_j = v_0 + \cdots + v_j$.

Our first step in establishing (iii) is to show that $B \leq^* p$. Thus, from $b_n = a(d_n)$, we need to effectively find p_n. Recall that $a, b \colon \omega \to \omega$ are regressive functions. Now, given b_n, we can effectively find n. If $n = 0$, then from $b_0 = a(d_0)$ we can effectively obtain d_0, and then $p_0 = 1 + d_0$. If $n > 0$, we can effectively find b_{n-1}. From b_n, b_{n-1}, we can obtain d_n, d_{n-1} in an effective manner, and then $p_n = d_n - d_{n-1}$.

To finish the proof of (iii), and of the lemma, we need to show that $A = \sum_B p$. That is, we need to show that a representative of A:

$$(9) \qquad (a_0, a_1, a_2, \dots),$$

and a representative of $\sum_B p$:

$$(10) \qquad (j(b_0, 0), \dots, j(b_0, p_0 - 1), \dots, j(b_n, 0), \dots, j(b_n, p_n - 1), \dots),$$

are recursively equivalent. To accomplish this it suffices to show that the mappings $a_n \to t_n$ and $t_n \to a_n$ have effective extensions where $t \colon \omega \to \omega$ is the function ranging over the set in (10). We thus need an explicit definition of t. We define:

$$\begin{aligned} t_n &= j(b_0, n) \quad \text{for } 0 \leq n \leq p_0 - 1, \\ &= j(b_{k+1}, n - (p_0 + \cdots + p_k)) \\ &\qquad\qquad \text{for } p_0 + \cdots + p_k \leq n \leq p_0 + \cdots + p_k + p_{k+1} - 1. \end{aligned}$$

Since b_n is one-to-one and $p_k > 0$ for every k, t is a well-defined one-one function from ω onto the set in (10).

Let a_n be given. Since d_x is a strictly increasing function, either $n \leq d_0$ or $d_k < n \leq d_{k+1}$ for a unique $k \in \omega$. Since a_x is a regressive function, we can effectively find $a_0, a_1, \dots, a_n$. In addition, since d_x ranges over

$$\gamma = (x \colon u_0 + \cdots + u_x = v_0 + \cdots + v_x),$$

$A \leq^* u$, $A \leq^* v$, and $\sum_A u = \sum_A v$, we can use the technique employed at the beginning of this proof to effectively produce

$$a_n, \ldots, a_{d(0)} \quad \text{or} \quad a_n, \ldots a_{d(k+1)}$$

depending on whether $n \leq d_0$ or $d_k < n \leq d_{k+1}$ respectively.

First, suppose $n \leq d_0$. Then, as noted above, we can effectively find $a_{d(0)} = b_0$, and hence also d_0 and $p_0 = 1 + d_0$. Since $n \leq d_0 = p_0 - 1$, we have effectively found $t_n = j(b_0, n)$.

Now, suppose $d_k < n \leq d_{k+1}$. Again, we can effectively find $a_{d(k+1)} = b_{k+1}$. Since b_x is a regressive function, we effectively obtain

$$b_{k+1} = a_{d(k+1)}, \ldots, b_0 = a_{d(0)}$$

from b_{k+1}, and hence also

$$d_{k+1}, \ldots, d_0. \tag{11}$$

From (11), we can calculate

$$p_{k+1} = d_{k+1} - d_k, \ldots, \quad p_1 = d_1 - d_0, \quad p_0 = d_0 + 1. \tag{12}$$

Using (12) and our assumption that $d_k < n \leq d_{k+1}$, we obtain

$$p_k + \cdots + p_0 - 1 < n \leq p_{k+1} + \cdots + p_0 - 1.$$

Hence, we have effectively found

$$t_n = j(b_{k+1}, n - (p_0 + \cdots + p_k)),$$

so that the mapping $a_n \to t_n$ has an effective extension.

We have left to show that from t_n we can effectively produce a_n. Thus, let t_n be given. Now, $t_n = j(x, y)$ with $x = b_{k'}$.

If $k' = 0$ then $x = b_0 = a_{d(0)} = a_{p(0)-1}$, and therefore we can effectively find

$$a_{p(0)-1}, \ldots, a_0, \tag{13}$$

since a_z is a regressive function. Also, since $x = b_0$, $n = y$ and $0 \leq y \leq p_0 - 1$. Thus, from (13) we can find a_n.

If $k' > 0$, then $x = b_{k'} = b_{k+1}$ with $k \geq 0$ and $y = n-(p_0+\cdots+p_k) = n - (d_k + 1)$. Since b_z is a regressive function, we can effectively find

$$b_{k+1} = a_{d(k+1)}, \quad b_k = a_{d(k)}, \ldots, b_0 = a_{d(0)}. \tag{14}$$

Since a_z is a regressive function, we can, from (14), effectively find d_k and

$$a_{d(k+1)}, a_{d(k+1)-1}, \ldots, a_{d(k)}.$$

Thus, since y and d_k are known, we have found $a_n = a_{d(k)+y+1}$. Thus, from t_n we can effectively produce a_n thereby completing the proof of the lemma.

THEOREM T3. *Let $f: \omega \to \omega$ be any recursive function, and let A be any regressive isol. Then $A \in \operatorname{Dom} f_\Lambda$ if and only if there is an increasing function r and a regressive isol E such that $f \circ r$ is increasing, $E \leq^* \Delta r$, and $A = r_\Lambda(E)$.*

Proof. In view of Theorem T2 one only need verify one direction of the theorem. Let us assume $A \in \operatorname{Dom} f_\Lambda$. Then we wish to show that there is a function r and a regressive isol E with the desired properties. Assume that $f_\Lambda(A) = Y$.

If A is finite, then we may set $r(n) = A$, for all numbers n, and let $E = 0$. Then $r_\Lambda(E) = r(0) = A$ and $f \circ r$ is a constant function. Also $E \leq^* \Delta r$, since E is a finite number. This gives the desired result in the case A is finite.

Assume now that A is infinite. Let f^+ and f^- be recursive increasing functions such that $f(n) = f^+(n) - f^-(n)$, for all numbers n. One may select f^+ and f^- as the *positive* and *negative* combinatorial parts that are associated with f. We will write f_Λ^+ for $(f^+)_\Lambda$, and f_Λ^- for $(f^-)_\Lambda$. Then, it follows that $f_\Lambda(A) = f_\Lambda^+(A) - f_\Lambda^-(A)$, and therefore also,

$$f_\Lambda^+(A) = f_\Lambda^-(A) + Y. \tag{1}$$

If we express each of the extensions $f_\Lambda^+(A)$ and $f_\Lambda^-(A)$ in (1) as an infinite series, we may then, from (1), also obtain

$$f^+(0) + \sum_A \Delta f^+ = f^-(0) + \sum_A \Delta f^- + Y. \tag{2}$$

In (2) we would like to attach the value of $f^+(0)$ to the infinite series next to it, and similarly with the value $f^-(0)$ to the series next to it. Let us modify Δf^+ by making $\Delta f^+(0)$ be $f^+(0) + \Delta f^+(0)$, and making $\Delta f^-(0)$ be $f^-(0) + \Delta f^-(0)$. Then (2) may be written as

$$\sum_A \Delta f^+ = \sum_A \Delta f^- + Y. \tag{3}$$

We recall that Y will be a regressive isol. Also, by the work of J. Gersting in **[8]**, it follows from (3) that Y can also be expressed as an infinite series. In particular, as

$$Y = \sum_A c,$$

where $A \leq^* c$, and $0 \leq c(n) \leq \Delta f^+(n)$, for each number n. Then, by [**9**, Lemma 17.18], the summands of the series representations of the right side of (3) may be combined so as to obtain,

$$\sum_A \Delta f^+ = \sum_A (\Delta f^- + c), \tag{4}$$

where $A \leq^* \Delta f^+$ since Δf^+ is recursive, and $A \leq^* (\Delta f^- + c)$ since Δf^- is recursive and $A \leq^* c$. To equation (4) we now apply Lemma L1, treating A as itself, Δf^+ as u, and $(\Delta f^- + c)$ as v. Let the functions d and p, and the regressive isol B, be as introduced in Lemma L1.

We note that, from the definition of d, it follows that for all numbers n,

$$\sum_0^{d(n)} \Delta f^+(i) = \sum_0^{d(n)} (\Delta f^- + c)(i).$$

Also,

$$f^+(d(n)+1) = \sum_0^{d(n)} \Delta f^+(i), \quad \text{and}$$

$$f^-(d(n)+1) = \sum_0^{d(n)} \Delta f^-(i),$$

for each number n. It therefore also follows,

$$f(d(n)+1) = \sum_0^{d(n)} c(i), \tag{5}$$

for each number n. Let the function r be defined by, $r(n) = d(n) + 1$, for $n \in \omega$. In view of (5), we see that the composition function $f \circ r$ is increasing. Also, we note that $r(0) = p(0)$ and $\Delta r(n) = p(n+1)$, where p is the function obtained from Lemma L1. In addition, also by Lemma L1, one has $B \leq^* p$ and

$$A = \sum_B p. \tag{6}$$

Because $B \leq^* p$ and $\Delta r(n) = p(n+1)$, it follows that $B - 1 \leq^* \Delta r$. Combining this property with $r(0) = p(0)$, we may then obtain from (6) the following representations for the isol A,

$$\begin{aligned} A &= p(0) + \sum_{B-1} \Delta r \\ &= r(0) + \sum_{B-1} \Delta r = r_\Lambda(B-1). \end{aligned}$$

If we now set $E = B - 1$ then the desired representation for the isol A is obtained. This completes our proof.

4. On the extension of recursively enumerable sets to the isols. Let α be a non-empty recursively enumerable (RE) set, and let α_Λ be the extension of α to the isols. Let $\alpha_R = \Lambda_R \cap \alpha_\Lambda$. Let f be any recursive function that has range equal to α. In this section we wish to study some properties related to the two collections $f_\Lambda(\mathrm{Dom}\, f_\Lambda)$ and α_R.

To begin, we will introduce some terminology to enable us to describe the representation of α_R that is obtained in [3]. Assume $r\colon \omega \to \omega$ is a strictly increasing function. We will say that r is the *principal* function for the set of numbers that is the range of r. We also define

$$\mathrm{Dom}\, r_\Lambda = (A \in \Lambda_R\colon A \leq^* \Delta r).$$

Let δ be any set of numbers. We define a collection of isols δ_Σ in the following way. If δ is a finite set, then $\delta_\Sigma = \delta$. If δ is an infinite set, then

$$\delta_\Sigma = r_\Lambda(\mathrm{Dom}\, r_\Lambda),$$

where r is the principal function that has range δ. Based on our earlier comments we note that all the elements of δ_Σ are regressive isols.

REMARK R1. The collections δ_Σ were first introduced in [3]. In [3] their definition is made a little differently than here, but the two are readily shown to be equivalent. We would like to verify that fact here. In [3], δ_Σ is defined in the following way. If δ is finite then $\delta_\Sigma = \delta$. If δ is infinite, let r be its principal function, and let the function d be defined by, $d(0) = r(0)$, and $d(n+1) = r(n+1) - r(n)$. In [3], d is called the *e-difference* function of r, and δ_Σ is defined by

$$\delta_\Sigma = \left(\sum_A d\colon A \in \Lambda_R,\ A \geq 1 \quad \text{and} \quad A \leq^* d \right).$$

To show that the two definitions of δ_Σ are equivalent, we may assume that δ is an infinite set. Let the functions r and d be as introduced in the preceding paragraph, and note that, $d(0) = r(0)$, and $d(n+1) = \Delta r(n)$. Then, from the characterization of δ_Σ given in [3] and noted above, we have

$$\begin{aligned}
\delta_\Sigma &= \left(\sum_A d \colon A \in \Lambda_R,\ A \leq 1,\ A \leq^* d\right) \\
&= \left(r(0) + \sum_{A-1} \Delta r \colon A \in \Lambda_R,\ A \geq 1,\ A-1 \leq^* \Delta r\right) \\
&= (r_\Lambda(A-1) \colon A \in \Lambda_R,\ A \geq 1,\ A-1 \leq^* \Delta r) \\
&= (r_\Lambda(B) \colon B \in \Lambda_R,\ B \leq^* \Delta r) = r_\Lambda(\mathrm{Dom}\, r_\Lambda),
\end{aligned}$$

and the desired equivalence follows.

To represent a union among sets we shall write $\sum$. The following result is proved in [3].

Proposition P1. *Let α be any recursively enumerable set. Then*

$$\alpha_R = \sum_{\delta \subset \alpha} \delta_\Sigma.$$

From Proposition P1 it follows that for any RE set α one has $\alpha \subset \alpha_R$. Also if both α and β are RE sets, then $\alpha \subset \beta$ implies $\alpha_R \subset \beta_R$. These properties are actually true for the extension to the isols for any sets of numbers, and can be obtained from the earlier work of A. Nerode in [**10**]. We also want to observe two additional properties, and they will be presented in the following lemma.

Lemma L2. *Let α and β be any* RE *sets, and let f and g be any recursive functions. Then,*

(a) *If α and β differ by only a finite set, and Y is an infinite isol, then $Y \in \alpha_R$ if and only if $Y \in \beta_R$.*

(b) *If k is a finite number, and if $f(x) = g(x)$ for all numbers $x \geq k$, then $f_\Lambda(A) = g_\Lambda(A)$ for all isols A with $A \geq k$.*

Proof. Property (a) follows from Lemma 1 in [**3**]. Property (b) was already used in the proof of Theorem T3. It is a well-known property and it may be verified in the following way. Assume $k \in \omega$ and that $x \geq k$ implies $f(x) = g(x)$. Then the equation $f(x+k) = g(x+k)$ is valid for all $x \in \omega$, and therefore, by the Myhill-Nerode metatheorem, the equation $f_\Lambda(X+k) = g_\Lambda(X+k)$ is valid for all isols X. This gives the desired result, and completes the proof.

Let us now consider the two sets $f_\Lambda(\mathrm{Dom}\, f_\Lambda)$ and α_R, when f is a recursive function and α is its range. We note that by Theorem T1 it follows that if f is an increasing function, then $f_\Lambda(\mathrm{Dom}\, f_\Lambda) = \alpha_R$. In general, we have $f_\Lambda(\mathrm{Dom}\, f_\Lambda) \subset \alpha_R$, yet the inclusion can be proper, even when α is a recursive set. Also, if α is any non-empty RE set, then there exist recursive functions f that have range α, and with $f_\Lambda(\mathrm{Dom}\, f_\Lambda) = \alpha_R$. Some recursive functions that are related to α in this fashion are called *complete for* α, and are introduced below. We would now like to verify these properties. It is helpful for the following presentation to first reflect on a special feature of some infinite series of isols.

Let $u\colon \omega \to \omega$ be any function and let A be any infinite regressive isol with $A \leq^* u$. Let

$$Y = \sum_A u,$$

and assume Y is infinite. It follows that there will be infinitely many numbers m for which $u(m)$ is positive. Let $\sigma = (x\colon u(x) > 0)$, and let s be the principal function of σ. Let a be any regressive function that ranges over a set in A. Set $b(n) = a(s(n))$, and let B be the isol that contains the range of b. Since a is a regressive function and $A \leq^* u$, it is readily seen that b is also a regressive function and $B \leq^* u \circ s$. In addition, $B \leq A$ and

$$Y = \sum_B u \circ s.$$

It is this representation of Y that is useful for us to observe. It is applied in the proof of the following theorem.

THEOREM T4. *Let α be a non-empty recursively enumerable set and let f be a recursive function that has range α. Then, $f_\Lambda(\mathrm{Dom}\, f_\Lambda) \subset \alpha_R$.*

Proof. Let $Y \in f_\Lambda(\mathrm{Dom}\, f_\Lambda)$. Then $Y = f_\Lambda(A)$, and by Theorem T3, we may write $A = r_\Lambda(B)$ where r and $f \circ r$ are increasing functions, and $B \leq^* \Delta r$. Applying Theorem T2 also yields $Y = (f \circ r)_\Lambda(B)$, with $B \leq^* \Delta(f \circ r)$, and

$$Y = (f \circ r)(0) + \sum_B \Delta(f \circ r). \tag{1}$$

We will now consider three cases.

Case 1. B is finite. From (1) it then follows that Y is also finite and $Y = (f \circ r)(B) = f(r(B))$, and therefore $Y \in \alpha$. Hence also $Y \in \alpha_R$.

Case 2. B is infinite and Y is finite. Then from (1) it follows that there will be a number k such that $\Delta(f \circ r)(m) = 0$ for all numbers $m \geq k$. Then

$$Y = (f \circ r)(0) + \sum_{i=0}^{k} \Delta(f \circ r)(i) = (f \circ r)(k+1).$$

In this case then $Y = f(r(k+1)) \in \alpha$, and therefore $Y \in \alpha_R$.

Case 3. B is infinite and Y is infinite. Then A is infinite. Let g be defined by, $g(x) = 0$ for $x \leq r(0)$, and $g(x) = f(x)$ for $x > r(0)$. Then $Y = f_\Lambda(A) = g_\Lambda(A)$ by Lemma L2(b). Let λ be the range of the function g. Then λ and α differ by only a finite set, and hence $Y \in \alpha_R$ if and only if $Y \in \lambda_R$, by Lemma L2(a). It therefore suffices for us to show $Y \in \lambda_R$. Let us note that one also has that $g \circ r$ is an increasing function, and $B \leq^* \Delta(g \circ r)$. We then have

$$Y = \sum_B \Delta(g \circ r), \tag{2}$$

since $(g \circ r)(0) = 0$. Combining (2) and the fact that Y is infinite, it follows by our earlier observation that there will exist a strictly increasing function s and an infinite regressive isol U such that, s ranges over the set $(x\colon \Delta(g \circ r)(x) > 0)$, and $U \leq^* \Delta(g \circ r) \circ s$, and

$$Y = \sum_U (\Delta(g \circ r)) \circ s. \tag{3}$$

Let the function p be defined by,

$$\begin{aligned} p(0) &= 0, \quad \text{and} \\ p(m+1) &= (\Delta(g \circ r) \circ s)(0) + \cdots + (\Delta(g \circ r) \circ s)(m). \end{aligned}$$

Then p is strictly increasing, and $\Delta p = (\Delta(g \circ r)) \circ s$. Since $U \leq^* \Delta(g \circ r) \circ s$ then also $U \in \operatorname{Dom} p_\Lambda$, and therefore from (3) it follows that

$$Y = p_\Lambda(U). \tag{4}$$

Let us note that $p(0) = 0$, and

$$\begin{aligned} p(m+1) &= \sum_{i=0}^{m} (\Delta(g \circ r))(s(i)) = \sum_{i=0}^{m} \Delta(g \circ r)(i) \\ &= (g \circ r)(s(m)+1) = g(r(s(m)+1)), \end{aligned}$$

where the second equality follows from the fact $\Delta(g \circ r)(t) = 0$ for any t with $t < s(m)$ and $t \notin (s(0), \ldots, s(m))$, and the third equality follows because $(g \circ r)(0) = 0$. Thus it follows that p is a strictly increasing function that ranges over a subset of the range of g. If p ranges over δ, then $\delta \subset \lambda$, and, in view of (4), $Y \in \delta_\Sigma$. Therefore $Y \in \lambda_R$ and then, as previously noted, $Y \in \alpha_R$. This completes our proof.

We would like to illustrate in the following example a setting where the inclusion obtained in Theorem T4 may be proper. This may even be true when the range of the function f is a recursive set.

EXAMPLE E1. Let f be the recursive permutation defined by, $f(2n) = 2n+1$ and $f(2n+1) = 2n$. Then the range of f is ω, and it is known that $\omega_R = \Lambda_R$. We would like to show that $f_\Lambda(\mathrm{Dom}\, f_\Lambda)$ is a proper subset of Λ_R. Because f is not eventually increasing, it follows from the main result of [1] that there exists some $Y \in \Lambda_R$ with $f_\Lambda(Y) \notin \Lambda$. Equivalently, this means $Y \in \Lambda_R - \mathrm{Dom}\, f_\Lambda$. But then it will also follow that $Y \notin f_\Lambda(\mathrm{Dom}\, f_\Lambda)$. For if there is some $A \in \Lambda_R$ with $Y = f_\Lambda(A)$, then, since $f(f(x)) = x$ is always true in ω, one would have

$$A = f_\Lambda(f_\Lambda(A)) = f_\Lambda(Y),$$

which would imply Y is in the domain of f_Λ. Hence $Y \in \Lambda_R - f_\Lambda(\mathrm{Dom}\, f_\Lambda)$.

Let α be any non-empty RE set. While for some recursive functions f that range over α one may have, as in Example E1, $f_\Lambda(\mathrm{Dom}\, f_\Lambda) \neq \alpha_R$, it turns out to always be possible to choose a recursive function f with range α for which the two corresponding sets are the same.

DEFINITION D1. Let α be a non-empty recursively enumerable set. A recursive function f is called *complete for* α, if (1) the range of f is α, and (2) every infinite subset of α is the range of f upon a set of numbers where f is strictly increasing.

It is easy to see that every non-empty RE set has a recursive function that is complete for it; for example, any recursive function that ranges over the set and takes on every value in the range infinitely often will be complete for the set. We note that if α is infinite and recursive, then the strictly increasing recursive function that has range α will be complete for α.

THEOREM T5. *Let α be a non-empty recursively enumerable set, and let f be a recursive function that is complete for α. Then $f_\Lambda(\mathrm{Dom}\, f_\Lambda) = \alpha_R$.*

Proof. In view of Theorem T4, we need only verify the property $\alpha_R \subset f_\Lambda(\mathrm{Dom}\, f_\Lambda)$.

Let $Y \in \alpha_R$. If Y is finite, then $Y \in \alpha$ so that there is a finite number a with $Y = f(a) = f_\Lambda(a)$. Thus, since $\omega \subset \operatorname{Dom} f_\Lambda$, $Y \in f_\Lambda(\operatorname{Dom} f_\Lambda)$.

If α is a finite set, then $\alpha_R = \alpha$ so that every member of α_R is finite. Thus, $\alpha_R \subset f_\Lambda(\operatorname{Dom} f_\Lambda)$ follows from the preceding paragraph.

Let us now assume that α is an infinite set, and that Y is an infinite isol with $Y \in \alpha_R$. By Proposition P1, it follows that there will be a subset δ of α such that $Y \in \delta_\Sigma$. Since Y is an infinite isol, δ will be an infinite set. Let r be the principal function of δ. Then there will be an infinite regressive isol A with $A \leq^* \Delta r$ and $Y = r_\Lambda(A)$. Let the function t be defined by

$$t(0) = (\mu s)(f(s) = r(0)),$$
$$t(m+1) = (\mu s)(s > t(m) \text{ and } f(s) = r(m+1)).$$

Because r is a strictly increasing function that ranges over a subset of α, and f is a recursive function which is complete for α, it follows that t is a well-defined and increasing function, with $f(t(n)) = r(n)$, for all numbers $n \in \omega$. In addition, because $A \leq^* \Delta r$, it will also be true that $A \leq^* \Delta(f \circ t)$ and $A \leq^* \Delta t$. By Theorem T2 one may then obtain

$$f_\Lambda(t_\Lambda(A)) = (f \circ t)_\Lambda(A) = r_\Lambda(A) = Y.$$

Let $B = t_\Lambda(A)$, and note that $B \in \Lambda_R$. Then also $B \in \operatorname{Dom} f_\Lambda$, since $f_\Lambda(B) = Y \in \Lambda_R$. Hence, $Y \in f_\Lambda(\operatorname{Dom} f_\Lambda)$, and this completes our proof.

COROLLARY C1. *Let α be a recursively enumerable set and let f be an increasing recursive function. Then $f_\Lambda(\alpha_R) = f(\alpha)_R$.*

Proof. If $\alpha = \varnothing$, then the result is close at hand, because all of the sets involved are also empty. Let us assume then that α is non-empty. Let g be a recursive function that is complete for α. Let us observe that the composition function $f \circ g$ is complete for the recursively enumerable set $f(\alpha)$. Furthermore, since f is an increasing function, $\operatorname{Dom}(f \circ g)_\Lambda = \operatorname{Dom} g_\Lambda$. Now, since $(f \circ g)_\Lambda = f_\Lambda \circ g_\Lambda$ for recursive functions f and g, one may use Theorem T5 to obtain:

$$\begin{aligned} f_\Lambda(\alpha_R) &= f_\Lambda(g_\Lambda(\operatorname{Dom} g_\Lambda)) \\ &= f_\Lambda \circ g_\Lambda(\operatorname{Dom} g_\Lambda) = (f \circ g)_\Lambda(\operatorname{Dom} g_\Lambda) \\ &= (f \circ g)_\Lambda(\operatorname{Dom}(f \circ g)_\Lambda) = (f \circ g(\omega))_R = f(\alpha)_R, \end{aligned}$$

and this is the desired result.

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