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APÉRY BASIS AND POLAR INVARIANTS OF PLANE CURVE SINGULARITIES

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Let C be an irreducible plane algebroid curve singularity over an algebraically closed field K, defined by a power series $f \in K[[X, Y]]$. In this paper, we study those power series $h \in K[[X, Y]]$ for which the intersection multiplicity $(f \cdot h) = \dim_K(K[[X, Y]]/(f, y))$ is an element of the Apéry basis of the value semigroup for C. We prove a factorization theorem for these power series, obtaining strong properties of their irreducible factors. In particular we show that some results by M. Merle and R. Ephraim are a special case of this theorem.

Introduction. In this paper we denote by K an algebraically closed field of arbitrary characteristic.

Let C be an irreducible plane algebroid curve over K (i.e. $C = \operatorname{Spec}(R)$, where $R = K[[X, Y]]/(f)$, with f irreducible). We will suppose $f \notin YK[[X, Y]]$ and we will write $n = \operatorname{Ord}_x(f(X, 0))$.

We will denote by $S(C)$ the semigroup of values of C (see **[2]**, 11.0.1 and **[3]**, 4.3.1), by $A_n = \{0 = a_0 < a_1 < \cdots < a_{n-1}\} = \{\min(S(C)n(k + n\mathbf{Z}_+); 0 \le k \le n-1\}$ the Apéry basis of $S(C)$ relative to n (see **[2]**, 1.1.1) and by $\{v_0, \dots, v_r\}$ the n-sequence in $S(C)$, where $v_0 = n$, and $v_i = \min\{v \in S(C);\ \gcd(v_0, v_1, \dots, v_{i-1}) > \gcd(v_0, v_1, \dots, v_{i-1}, v)\}$, $1 \le i \le r$ (see **[1]**, 6.6, **[2]**, 1.3.2 and **[6]**). (Note that $\gcd(v_0, \dots, v_r) = 1$.)

The main objective of this work is the proof of the following theorem.

FACTORIZATION THEOREM. *Let $h \in K[[X, Y]]$ be such that $0 \le k = \operatorname{Ord}_x(h(X, 0)) \le n-1$. Then $(f \cdot h) \le a_k$. Suppose $(f \cdot h) = a_k$. If $k = \sum_{0 \le q \le r} s_q(n/d_{q-1})$, where $d_q = \gcd(v_0, \dots, v_q)$, $(d_0 = v_0 = n, d_r = 1)$, $0 \le s_q \le r$ and $0 \le s_q \le d_{q-1}/d_q$, then*

$$h = \prod_{1 \le i \le r} h_i \quad \text{and} \quad h_i = \prod_{1 \le j \le m_i} h_{ij},$$

with h_{ij} either irreducible or unit in $K[[X, Y]]$, $1 \le j \le m_i$, $1 \le i \le r$, and

(1) $\sum_{1 \le j \le m_i} \operatorname{Ord}_x(h_j(X, 0)) = s_i(n/d_{i-1})$, $1 \le i \le r$.

(2) $(f \cdot h_{ij}(X,0)) = d_{i-1}v_i/n$ *if* $s_i \neq 0$ *and* h_{ij} *is a unit in* $K[[X,Y]]$ *if* $s_i = 0$, $1 \leq j \leq m$, $1 \leq i \leq r$.

Here $(f \cdot h)$ denotes, for two power series f and h, the intersection multiplicity of the algebroid cycles defined, respectively, by f and h.

In the fourth section we see that the polars of an irreducible complex analytic germ of a plane curve singularity satisfy the hypotheses of the above theorem for $k = n-1$. Thus, the Theorem 3.1 of **[5]** and Lemma 1.6 of **[4]** follow from the above Factorization Theorem.

1. Apéry basis and the n-sequence. In this section we will summarize some properties of the Apéry basis. For other properties you can see **[2]** and **[6]**.

PROPOSITION 1. *If* $M_j = K[[Y]] + K[[Y]]X + \cdots + K[[Y]]X^j$, $0 \leq j \leq n-1$, *then*:
(1) $\{a_j\} = v(M_{j-1} + X^j) - v(M_{j-1})$, $1 \leq j \leq n-1$,
(2) $v(M_j) = \bigcup_{0 \leq i \leq j}(a_i + n\mathbf{Z}_+)$, $0 \leq j \leq n-1$,
(3) $a_i + a_j \leq a_{i+j}$, $0 \leq i+j \leq n-1$,
where $v(M_i) = \{(f \cdot g);\ g \in M_i - \{0\}\}$, $0 \leq i \leq n-1$ *and* $v(M_{i-1} + X^i) = \{(f \cdot (g + X^i));\ g \in M_{i-1}\}$, $1 \leq i \leq n-1$.

Proof. See **[2]**, Satz 3 and **[6]**, Proposition 2.

REMARK 2. Note that in the above proposition $a_j \geq (f \cdot (g + X^j))$ for each $g \in M_{j-1}$, $1 \leq j \leq n-1$. (If $(f \cdot (g + X^j)) > a_j$, then there exists $g_{j-1} \in M_{j-1}$ such that $(f \cdot (g_{j-1} + X^j)) = a_j$, so $a_j = (f \cdot (g - g_{j-1}))$ and we get a contradiction.)

PROPOSITION 3. *One has*

$$a_{s_1(d/d_0)+\cdots+s_j(d/d_{j-1})} = s_1v_1 + \cdots + s_jv_j,$$

and $v_{j+1} > (d_{j-1}/d_j)v_j$, $0 \leq j \leq r-1$, *with* $0 \leq s_i \leq (d_{i-1}/d_i)$, $1 \leq i \leq r$.

Proof. See **[2]**, Satz 2 and **[6]**, Proposition 1.

REMARK 4. Note that $v_j = a_{d/d_j}$, $1 < j < r$ and

$$A_n = \{a_{s_1(d/d_0)+\cdots+s_r(d/d_{r-1})};\ 0 \leq s_i < (d_{i-1}/d_i), 1 < i < r\}.$$

EXAMPLE 5. Here we give some examples of different possibilities for the Apéry basis and n-sequences. Let us consider the curves

$C_i = \mathrm{Spec}(K[[X,Y]]/(f_i))$, $1 \le i \le 3$, where $f_1 = X^2 + Y^5$, $f_2 = (Y + X^2)^2 + X^5$ and $f_3 = Y^2 + X^5$. It is easy to check that

$$S(C_1) = S(C_2) = S(C_3) = \{0, 2, 4, 5, 6, 7, 8, \dots\},$$

and one has $f_i \notin YK[[X,Y]]$, $1 \le i \le 3$, and $\mathrm{Ord}_X(f_1(X,0)) = 2$, $\mathrm{Ord}_X(f_2(X,0)) = 4$ and $\mathrm{Ord}_X(f_3(X,0)) = 5$. So $A_2 = \{0 = a_0, a_1 = 5\}$. The 2-sequence is $\{v_0 = 2, v_1 = 5\}$, $a_1 = (f_1 \cdot X)$, $d_0 = d = 2$ and $d_1 = 1$. $A_4 = \{0 = a_0, a_1 = 2, a_2 = 5, a_3 = 7\}$. The 4-sequence is $\{v_0 = 4, v_1 = 2, v_3 = 5\}$, $a_1 = (f_2 \cdot X)$, $a_2 = (f_2 \cdot (Y + X^2))$, $a_3 = (f_2 \cdot (Y + X^2)X)$, $d_0 = d = 4$, $d_1 = 2$ and $d_2 = 1$. And $A_5 = \{0 = a_0,\ a_1 = 2,\ a_2 = 4,\ a_3 = 6,\ a_4 = 8\}$. The 5-sequence is $\{v_0 = 5, v_1 = 2\}$, $a_i = (f_3 \cdot X^i)$, $1 \le i \le 4$, $d_0 = d = 5$ and $d_1 = 1$.

2. *n*-sequences and Hamburger-Noether expansions. Let x and y be, respectively, the residue classes of X and Y in R. Assume that $n_0 = (f \cdot X) \le (f \cdot Y) = n$, that is, X is a generic coordinate (or x is a transversal parameter of C, see **[3]**) and Y could be generic, or have maximal contact with f, or any thing in between. In this form, we can study all of these possibilities for Y simultaneously. This is the point of taking the Apéry basis with respect to a general n, rather than $n = n_0$. If $n = n_0$ then Y should be generic.

Let

$$\begin{aligned}
y &= a_{01}x + \cdots + a_{0h_0}x^{h_0} + x^{h_0}z_1,\\
x &= z_1^{h_1}z_2,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
z_{s_1-1} &= a_{s_1k_1}z_{s_1}^{k_1} + \cdots + a_{s_1h_{s_1}}z_{s_1}^{h_{s_1}} + z_{s_1}^{h_{s_1}}z_{s_1+1},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
z_{s_g-1} &= a_{s_gk_g}z_{s_g}^{k_g} + \cdots
\end{aligned}$$

be the Hamburger-Noether expansion of C in the basis (x,y) (see **[3]**, 2.2.2 and 3.3.4), and let $n_i = \mathrm{Ord}_{z_{s_g}}(z_i)$, $0 \le i \le s_g$ $(z_0 = x)$, $(1 = n_{s_g} < n_{s_g-1} < \cdots < n_0 \le n = \mathrm{Ord}_{z_{s_g}}(y)$, see **[3]**, 2.2.5).

Note that the Hamburger-Noether expansion is nothing but an explicit description of the minimal resolution of singularities $\overline{C}$ of C by a sequence of point blowing-ups. z_i, z_{i-1} are the regular parameters of the ambient plane at the $h_0 + \cdots + h_i$th blowing up. z_{s_g} is a regular parameter of C. In particular, for any $h \in K[[X,Y]]$ such that f does not divide h

$$(f \cdot h) = \mathrm{Ord}_{z_{s_g}}(h).$$

The following proposition is an easy consequence of the Hamburger-Noether expansion and the formula for Zariski exponents of a plane curve (see [3] 4.2.7 and 4.3.10).

Proposition 6. *With the above notations one has:*
(1) $n_0 = \min(S(C) - \{0\})$,
(2) $n_0 \leq n = v_0 \leq h_0 n_0 + n_1$,
(3)(i) *If* $v_0 \leq v_1$, *then* $r = g$, $v_0 = n_0$ *and*

$$v_{i+1} = (1/n_{s_i}) \sum_{0 \leq j \leq s_i} h_j n_j^2 + n_{s_i+1},$$

$0 \leq i \leq r-1$, $(s_0 = 0)$. *Moreover* $a_{01} \neq 0$.
(ii) *If* $v_0 > v_1$ *and* $d_1 = v_1$, *then* $r = g+1$, $v_0 = k_0 v_1$, $k_0 \geq 2$, $v_1 = n_0$ *and*

$$v_{i+2} = (1/n_{s_i}) \sum_{0 \leq j \leq s_i} h_j n_j^2 + n_{s_i+1},$$

$0 \leq i \leq r-1$, $(s_0 = 0)$. *Moreover* $a_{0j} = 0$, $1 \leq j < k_0$ *and* $a_{1k_0} \neq 0$.
(iii) *If* $v_0 > v_1$ *and* $d_1 < v_1$, *then* $r = g$, $v_1 = n_0$, $v_0 = h_0 n_0 + n_1$ *and*

$$v_{i+1} = (1/n_{s_i}) \sum_{0 \leq j \leq s_i} h_j n_j^2 + n_{s_i+1},$$

$0 \leq i \leq r-1$, $(s_0 = 0)$. *Moreover* $a_{0j} = 0$, $1 \leq j \leq h_0$.

Proof. (1) and (2) are obvious from the Hamburger-Noether expansions. We must only prove (3).

For this, if one writes $\overline{\beta}_0 = n_0$ and

$$\overline{\beta}_i = (1/n_{s_i}) \sum_{0 \leq j \leq s_i} h_j n_j^2 + n_{s_i+1},$$

$0 \leq i \leq g-1$, then one has

(I) $\overline{\beta}_0 = \min(S(C) - \{0\})$ and $\overline{\beta}_i = \min\{\overline{\beta} \in S(C); \gcd(\overline{\beta}_0, \ldots, \overline{\beta}_{i-1}) > \gcd(\overline{\beta}_0, \ldots, \overline{\beta}_{i-1}, \overline{\beta})\}$, $1 \leq i \leq g$ (see [3], 4.2.7 and 4.3.10).

On the other hand, note that one has the equalities

(II) $v_0 = n$ and $v_i = \min\{v \in S(C); \gcd(v_0, \cdots, v_{i-1}) > \gcd(v_0, \ldots, v_{i-1}, v)\}$, $1 \leq i \leq r$.

We distinguish the following three possibilities:

(i) $n_0 = n < h_0 n_0 + n_1$. In that case $a_{01} \neq 0$, $v_0 = n_0$ and it follows from (I) and (II) that $r = g$ and $v_i = \overline{\beta}_i$, $1 \leq i \leq g$.

(ii) $n_0 < n = k_0 n_0 < h_0 n_0 + n_1$. Then $a_{0j} = 0$, $1 \leq j \leq k_0$, $a_{0k_0} \neq 0$, $v_0 = k_0 n_0$, $v_1 = n_0$ and it follows from (I) and (II) that $r = g+1$ and $v_{i+1} = \overline{\beta}_i$, $1 \leq i \leq r-1$.

(iii) $n_0 < n = h_0 n_0 + n_1$. Now $a_{0j} = 0$, $1 \leq j \leq h_0$, $v_0 = h_0 n_0 + n_1$, $v_1 = n_0$ and it follows from (I) and (II) that $r = g$ and $v_i = \overline{\beta}_i$, $2 \leq i \leq r$.

3. Infinitely near points and intersection multiplicity. Now consider another irreducible plane algebroid curve over K, $C' = \mathrm{Spec}(R')$, with $R' = K[[X,Y]]/(f')$, $C' \neq C$ and $f' \notin YK[[X,Y]]$. Let x' and y' be the residue classes of X and Y, respectively, in R'. We denote by

$$y' = a'_{01}x' + \cdots + a'_{0h'_0}x'^{h'_0} + x'^{h'_0}z'_1,$$

$$x' = z_1'^{h'_1} z'_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$z'_{s'_1-1} = a'_{s'_1k'_1}z'^{k'_1}_{s'_1} + \cdots + a'_{s'_1h'_{s'_1}}z'^{h'_{s'_1}}_{s'_1} + z'^{h'_{s'_1}}_{s'_1}z'_{s'_1+1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$z'_{s'_{g'}-1} = a'_{s'_{g'}k'_{g'}}z'^{k'_{g'}}_{s'_{g'}} + \cdots$$

the Hamburger-Noether expansion of C in the basis (x', y'). We also put $n'_i = \mathrm{Ord}_{z'_{s'_{g'}}}(z'_i)$, $0 \leq i \leq s'_{g'}$, $(x' = z'_0)$ and $n' = \mathrm{Ord}_X(f'(X,0)) = \mathrm{Ord}_{z'_{s'_{g'}}}(y')$.

Let N be the number of infinitely near points that C and C' have in common (i.e. $N = h_0 + h_1 + \cdots + h_{s-1} + i - 1$, s being the largest integer for which $h_q = h'_q$, $0 \leq q \leq s-1$, and $a_{jk} = a'_{jk}$, $i \leq k \leq h_j$, $0 \leq j \leq s-1$, and i being the least index such that $a_{si} \neq a'_{si}$ $(i \leq h_s + 1, i \leq h'_s + 1)$) (see [3] 2.3.2).

PROPOSITION 7. *If*

$$\sum_{0 \leq q \leq s_{i-1}-1} h_q + k_{i-1} - 1 < N \leq \sum_{0 \leq q \leq s_i - 1} h_q + k_i - 1,$$

$1 \leq i \leq g$, $(s_0 = 0)$, *then* $(f \cdot f') \leq n'd_{j-1}v_j/n$, *where* $j = i$ *if* $v_0 < v_1$ *or* $v_0 > v_1$, $d_1 < v_1$, *and* $j = i + 1$ *if* $v_0 > v_1$, $d_1 = v_1$. *Furthermore, if* $(f \cdot f') < n'd_{j-1}v_j/n$, *then* d_{j-1} *divides* $(f \cdot f')$.

Proof. One has $n = h_{q+1}n_{q+1} + n_{q+2}$, $s_j \leq q \leq s_{j+1} - 2$, $n_{s_{j+1}-1} = k_{j+1}n_{s_{j+1}}$, $0 < j \leq g - 1$, and $n'_p = h'_{p+1}n'_{p+1} + n'_{p+2}$, $s'_j \leq p \leq s'_{j+1} - 2$, $n'_{s'_{j+1}-1} = k'_{j+1}n'_{s'_{j+1}}$, $0 < j \leq g' - 1$.

So n_{s_i} divides n_i, and $n'_{s'_j}$ divides n'_k for $i < s_j$ and $k < s'_j$. On the other hand, since

$$\sum_{0\le q\le s_{i-1}-1} h_q + k_{i-1} \le N$$

then $h_q = h'_q$, $0 \le q \le s_{i-1} - 1$ and $k_{i-1} = k'_{i-1}$, so

(III) $n/n_{s_{i-1}}$, $n_q/n_{s_{i-1}} = n'_q/n'_{s_{i-1}}$, $0 \le q \le s_{i-1}$.

From Proposition 5 we see that

(IV) $d_{j-1} = n_{s_{i-1}}$.

Thus, one can compute $(f \cdot f')$ in terms of the possible values of N (see [3], 2.3.2 and 2.3.3). Namely, one has the following possibilities:

(A) $N = \sum_{0\le q\le s_{i-1}-1} h_q + k_{i-1}$, with $k_{i-1} < k < \min(h_{s_{i-1}}, h'_{s_{i-1}})$.

In that case one has

$$\begin{aligned}(f \cdot f') &= \sum_{0\le q< s_{i-1}-1} h_q n_q n'_q + k n_{s_{i-1}} n'_{s_{i-1}} \\ &< \sum_{0\le q\le s_{i-1}} h_q n_q n'_q + n_{s_{i-1}+1} n'_{s_{i-1}} = \alpha,\end{aligned}$$

so d_{j-1} divides $(f \cdot f')$ by (IV), and $\alpha = n' d_{j-1} v_j / n$, by (III), (IV) and Proposition 6.

(B) $N = \sum_{0\le q\le s} h_q$, with $s_{i-1} \le s < \min(s_i, s'_i)$ and $h_s < h'_s$.

Now one has

$$\begin{aligned}(f \cdot f') &= \sum_{0\le q\le s} h_q n_q n'_q + n_{s+1} n'_s \\ &< \sum_{0\le q\le s-1} h_q n_q n'_q + h'_s n_s n'_s + n_s n'_{s+1} = \beta.\end{aligned}$$

(Note that $h_s < h'_s$, so $n_{s-1} n'_s = h_s n_s n'_s + n_{s+1} n'_s < (h_s + 1) n_s n'_s \le h'_s n_s n' < h'_s n_s n'_s + n_s n'_{s+1}$.) By (III), (IV) and Proposition 6, it follows that

$$(f \cdot f') = \sum_{0\le q< s_{i-1}} h_q n_q n'_q + n_{s_{i-1}+1} n_{s_{i-1}} = n' d_{j-1} v_j / n, \quad \text{or}$$

$$(f \cdot f') = \sum_{0\le q< s_{i-1}} h_q n_q n'_q + n_{s_{i-1}} n'_{s_{i-1}+1} < \beta = n' d_{j-1} v_j / n,$$

and d_{j-1} divides $(f \cdot f')$.

The other cases can be proved in a similar way:

(B′) $N = \sum_{0\le q\le s-1} h_q + h'_s$, with $s_{i-1} \le s < \min(s_i, s'_i)$ and $h'_s < h_s$.

(C.1) $N = \sum_{0\le q\le s_i-1} h_q + k_i - 1$, with $s_i < s'_i$ and $k_i < h'_{s_i}$.

(C.2) $N = \sum_{0\le q\le s_i-1} h_q + h'_{s_i}$, with $s_i < s'_i$ and $h'_{s_i} < k_i$.

(C′.1) $N = \sum_{0\le q\le s'_i-1} h_q + k'_i - 1$, with $s'_i < s_i$ and $k'_i < h_{s'_i}$.
(C′.2) $N = \sum_{0\le q\le s'_i-1} h_q + h_{s'_i}$, with $s'_i < s_i$ and $h'_{s'_i} < k'_i$.
(D) $N = \sum_{0\le q<s_i-1} h_q + k_i - 1$, with $s_i = s'_i$ and $k_i < k'_i$.
(D′) $N = \sum_{0\le q\le s_i-1} h_q + k_i - 1$, with $s_i = s'_i$ and $k'_i < k_i$.
(E) $N = \sum_{0\le q<s_i-1} h_q + k_i - 1$, with $s_i = s'_i$, $k_i = k'_i$ and $a_{s_ik_i} \neq a'_{s_ik_i}$.

COROLLARY 8. *For each nonnegative integer j, $1 \le j \le r$, the following statements are equivalent*:

(1) $$(f\cdot f') > n'd_{j-1}v_j/n,$$

(2) $$N = \sum_{0\le q<s_i-1} h_q + k_i - 1,$$

where $i = j$ if $v_0 < v_1$ or $v_0 > v_1$ and $d_1 < v_1$, and $i = j-1$, $k_0 = v_0/v_1$ if $v_0 > v_1$ and $d_1 = v_1$. In particular, if either (1) *or* (2) *is true then $n' = n'_{s_i}n/d_j$.*

Proof. (1) $\Rightarrow$ (2). If $v_0 > v_1$, $d_1 = v_1$ and $(f\cdot f') > n'v_1$ then $N > k_0 - 1$. Indeed, suppose $N \le k_0 - 1$. Then $a_{0q} = a'_{0q}$, for $q \le N$ and $a_{0N+1} \neq a'_{0N+1}$. If $a'_{0N+1} \neq 0$ then $(N+1)n_0 = n'$ and if $a'_{0N+1} = 0$ then $N+1 = k_0$ and $(N+1)n'_0 \le n'$, so in any case $(f\cdot f') = (N+1)n_0n'_0 \le n'v_1$ and we get a contradiction.

Now suppose $(f\cdot f') > n'd_{j-1}v_j/n$ and

$$\sum_{0\le q\le s_i-1} h_q + k_i - 1 < N$$

with $j \ge 1$ if $v_0 < v_1$ or $v_0 > v_1$ and $d_1 < v_1$, and with $j \ge 2$ if $v_0 > v_1$ and $d_1 = v_1$. Then we can assume

$$\sum_{0\le q\le s_{p-1}-1} h_q + k_{p-1} < N \le \sum_{0\le q\le s_{p-1}} h_q + k_p - 1,$$

with $1 \le i \le p$. It follows from Proposition 7 that $(f\cdot f') \le n'd_{s-1}v_s/n$, with $s \le j$ and $d_{s-1}v_s \le d_{j-1}v_j$ (see [2], Satz 2) which is a contradiction.

(2) $\Rightarrow$ (1). If $v_0 > v_1$, $d_1 = v_1$ and $N > k_0 - 1$, then $(f\cdot f') > k_0n_0n'_0$, and $n' = k_0n'_0$, $(a_{0k_0} = a'_{0k_0})$, so one has $(f\cdot f') > n'v_1$ $(n_0 = v_1)$.

Now if

$$\sum_{0\le q\le s_i-1} h_q + k_i - 1 < N$$

with $i \geq 1$ then $n/n_{s_i} = n'/n'_{s_i}$, $n_q/n_{s_i} = n'_q/n'_{s_i}$, $0 \leq q \leq s_i$ and

$$(f \cdot f') = \sum_{0 \leq q \leq s_i - 1} h_q n_q n'_q + k_i n_{s_i} n'_{s_i} = \gamma.$$

By Proposition 6

$$(n'/n) d_{j-1} v_j = (n'_{s_{i-1}}/n_{s_{i-1}}) \left(\sum_{0 \leq q \leq s_{i-1}} h_q n_q^2 + n_{s_{i-1}+1} n_{s_{i-1}} \right).$$

Now

$$\gamma = \sum_{0 \leq q \leq s_{i-1}} h_q n_q n'_q + k_i n_{s_i} n'_{s_i} = (n_{s_{i-1}}/n_{s_{i-1}}) \left(\sum_{0 \leq q \leq s_{i-1}} h_q n_q^2 + k_i n_{s_i}^2 \right).$$

Thus we have to show that

$$\sum_{0 \leq q \leq s_{i-1}} h_q n_q^2 + n_{s_{i-1}+1} n_{s_{i-1}} = \sum_{0 \leq q \leq s_i - 1} h_q n_q^2 + k_i n_{s_i}^2.$$

But this follows by repeated application of the identities $n_{q-1} = h_q n_q + n_{q+1}$, since $k_i n_{s_i} = n_{s_i - 1}$.

COROLLARY 9. *For $1 \leq j \leq r$, if $(f \cdot f') < n' d_{j-1} v_j / n$, then d_{j-1} divides $(f \cdot f')$.*

Proof. If $v_0 > v_1$, $d_1 = v_1$ and $(f \cdot f') < n' v_1$ then $N \leq k_0 - 1$ (Corollary 8). Thus, if $a_{0q} = a'_{0q}$, $1 \leq q \leq N$, and $a_{0N+1} \neq a'_{0N+1}$ then $N + 1 = k_0$ and $(f \cdot f') = (N+1) n_0 n'_0 = n'_0 v_0$. (For if $N + 1 < k_0$ then $(f \cdot f') = n' v_1$ which is a contradiction.)

Now we can assume $(f \cdot f') < n' d_{j-1} v_j / n$, with $j \geq 1$ if $v_0 < v_1$ or $v_0 > v_1$ and $d_1 < v_1$, and $j \geq 2$ if $v_0 > v_1$ and $d_1 = v_1$. By Corollary 8 one has

$$\sum_{0 \leq q \leq s_i - 1} h_q + k_i - 1 \geq N$$

with $i = j$ if $v_0 < v_1$ or $v_0 > v_1$ and $d_1 < v_1$, and with $i = j - 1$ if $v_0 > v_1$ and $d_1 = v_1$. So, by Proposition 7, d_{j-1} divides $(f \cdot f')$.

4. Proof of the Factorization Theorem. As $\mathrm{Ord}_X(h(X, 0)) = k$ we can write $h = uh'$, with $h' \in M_{k-1} + X^k$ and $u \in K[[X, Y]]$ being a unit. So $(f \cdot h) = (f \cdot h') \leq a_k$.

Also, we can write $a_k = \sum_{0 \leq q \leq e} s_q v_q$ and $k = \sum_{0 \leq q \leq r} s_q (d/d_q)$, with $0 \leq s_q < d_{q-1}/d_q$ (see Remark 4). Let q be the greatest index such that $s_q \neq 0$ and let

$$h = \prod_{0 \leq j \leq m} h_j$$

be the factorization of h as a product of irreducible elements in $K[[X,Y]]$.

If for any j

$$(f \cdot h_j)/\operatorname{Ord}_X(h_j(X,0)) > d_{q-1}v_q/n$$

then, by Corollary 8, $\operatorname{Ord}_X(h_j(X,0)) = an/d_q$ $(a \neq 0)$, but $k < n/d_q$ which is a contradiction. (Note that $s_p = 0$ for $p > q$ and

$$k \leq \sum_{1\leq p\leq q} ((d_{p-1}/d_p) - 1) = (d/d_q) - 1 < d/d_q = n/d_q.)$$

On the other hand, if for $1 \leq j \leq m$

$$(f \cdot h_j)/\operatorname{Ord}_X(h_j(X,0)) < d_{q-1}v_q/n$$

then d_{q-1} divides $(f \cdot h)$ by Corollary 9. So d_{q-1}/d_q divides s_q, and hence $s_q = 0$ since $0 \leq s_q < d_{q-1}/d_q$, and we get a contradiction.

Thus, there exists h_{j_0} such that

$$(f \cdot h_{j_0})/\operatorname{Ord}_X(h_{j_0}(X,0)) = d_{q-1}v_q/n.$$

Moreover, if $q \geq 2$ then $\operatorname{Ord}_X(h_{j_0}(X,0)) = an/d_{q-1}$ by Corollary 8, as $d_{q-1}v_q > d_q v_{q-1}$ (see Proposition 3). If $q = 1$ then $(f \cdot h_{j_0}) = \operatorname{Ord}_X(h_{j_0}(X,0)) = an/d_{q-1}$. In any case $\operatorname{Ord}_X(h_{j_0}(X,0)) = an/d_{q-1}$ with $0 \leq a \leq s_q$.

(Note that $k \leq \sum_{1\leq p\leq q-1}((d_{p-1} - 1) - 1)(d/d_{p-1}) + s_q d/d_{q-1} < (d/d_{q-1}) + s_q d/d_{q-1} = (s_q + 1)d/d_{q-1} = (s_q + 1)n/d_{q-1}.$)

So $h' = h/h_{j_0}$ satisfies $\operatorname{Ord}_X(h'(X,0)) = k' = k - an/d_{q-1}$ and $(f \cdot h') = a_k - a(n/d_{q-1})d_{q-1}v_q/n = a_k - av_q = a_{k'}$; hence the Theorem follows by iterating the above reasoning using h' instead of h in the next step.

5. The complex analytic case. In this section, C is assumed to be an irreducible complex analytic germ at $0 \in C^2$ of a plane curve singularity.

Let n be the multiplicity of C and let $P(C)$ be a general polar of C (i.e. $P(C)$ is defined by a reduced element $h = \lambda(\partial f/\partial X) - \mu(\partial f/\partial Y)$ of $C\{X,Y\}$, and $n - 1$ is the multiplicity of $P(C)$). M. Merle in **[5]** has proved that $P(C)$ descomposes into g curves $\Gamma_{(1)}, \ldots, \Gamma_{(g)}$, where $\Gamma_{(g)}$ $(1 \leq q \leq g)$ is such that

(1) its multiplicity is $(n/e_{q-1})((e_{q-1}/e_q) - 1)$,

(2) every irreducible component of $\Gamma_{(q)}$, $\Gamma_{(q)i}$ has a contact of order β_q with C and $(\Gamma_{(q)i} \cdot C)/m(\Gamma_{(q)i}) = \overline{\beta}_q/(n/e)$.

Here $\{\overline{\beta}_0, \dots, \overline{\beta}_g\}$ is the minimal system of generators of $S(C)$, $e_q = \gcd(\overline{\beta}_0, \dots, \overline{\beta}_q)$, $0 \leq q \leq g$, $\beta_0 < \beta_1 < \cdots < \beta_g$ are the Puiseux exponents and $m(\Gamma_{(q)i})$ denotes the multiplicity of $\Gamma_{(q)i}$.

Without loss of generality, we may assume that $n = \mathrm{Ord}_X(f(X,0))$, and therefore $n-1 = \mathrm{Ord}_X(h(X,0))$.

On the other hand,

$$(f \cdot h) = \sum_{0 \leq q \leq g} ((e_{q-1}/e_q) - 1)\overline{\beta}_q.$$

and hence $(f \cdot h) = a_{n-1}$, since $\{\overline{\beta}_0, \dots, \overline{\beta}_g\}$ is the n-sequence in $S(C)$ (see **[2]**, Satz 2 and **[5]**, Prop. 1.1).

Thus, h satisfies the hypotheses of the Factorization Theorem for $k = n-1$, and the above Theorem 3.1 of **[5]** is a special case of ours. (Note that $\Gamma_{(q)i}$ has a contact of order β_q with C if and only if $(\Gamma_{(q)i} \cdot C)/m(\Gamma_{(q)i}) = \overline{\beta}_q/(n/e_{q-1})$, see **[5]**, Prop. 2.4.)

In general, if M is a smooth germ of a plane curve singularity defined by $z \in C\{X, Y\}$, then the polar of C with respect to M is the (possibly nonreduced) germ whose defining ideal is generated by the Jacobian $J(f, z) = \partial(f, z)/\partial(X, Y)$ (see **[4]**). In particular, a general polar $P(C)$ of C is defined by $h = J(f, \lambda X + \mu Y)$ with (λ, μ) general.

Thus, without loss of generality, we may assume that $z = Y$ (since M is smooth) and $J(f, z) = \partial f/\partial X$.

PROPOSITION 10. *Keeping the above notations, one has*
(a) $\mathrm{Ord}_X((\partial f/\partial X)(X,0)) = \mathrm{Ord}_X(f(X,0)) - 1 = n-1$.
(b) $(f(\partial f/\partial X)) = a_{n-1}$.

Proof. (a) It is obvious.

(b) If $n = \mathrm{Ord}_X(f(X,0)) \geq \mathrm{Ord}_Y(f(0,y)) = m$ then one has a Puiseux type parametrization of C

$$X = t^m, \qquad Y = \Psi(t)$$

and we can write (up to multiplication by a unit)

$$f(X,Y) = \prod_{0 \leq q \leq m} (X - \Psi(W^q X^{1/m}))),$$

Thus,

$$\begin{aligned}(f \cdot (\partial f/\partial X)) &= \mathrm{Ord}_t((\partial f/\partial X)(t^m, \Psi(t))) \\ &= \mathrm{Ord}_t(\Psi^1(t^m)) + \mathrm{Ord}_t\left(\prod_{1 \leq q \leq m-1} (\Psi(t) - \Psi(W^q t)\right).\end{aligned}$$

where $\Psi^1(X^{1/m}) = \partial/\partial X(\Psi(X^{1/m}))$.

On the other hand, we can write

$$\Psi(X^{1/m}) = \sum_{1\le j\le i_0} a_{0j}X^{jn/m} + \sum_{0\le j\le i_1} a_{1j}X^{(\beta_1+je_1)/m} + \cdots + \sum_{0\le j} a_{gj}X^{(\beta_g+je_g)/m},$$

where $m = \beta_0 < \beta_1 < \cdots < \beta_g$ are the Puiseux exponents of C and $e_i = \gcd(\beta_0, \ldots, \beta_i)$, $1 \le i \le g$.

Then we have $\mathrm{Ord}_t \Psi^1(X^{1/n}) = n - m$, and

$$\mathrm{Ord}\left(\prod_{1\le q\le m-1} (\Psi(t) - \Psi(w^q t))\right) = \sum_{1\le q\le g} (e_{i-1} - e_i)\beta_i.$$

(Note that $\mathrm{Ord}_t(\Psi(t) - \Psi(w^q t)) = \beta_j$, if

$$q \in \{k(e_{j-2}/e_{j-1});\ 1 \le k < e_{j-1}\} - \{k(e_{j-1}/e_j); 1 \le k < e_j\},\quad 1 \le j \le g \quad (e_{-1} = e_0 = m).)$$

Now

$$\sum_{1\le i\le g} (e_{i-1} - e_i)\beta_i = c + m - 1,$$

where c is the conductor of $S(C)$ (i.e. $c = \min\{d \in S(C); d + \mathbf{Z}_+ \subset S(C)\}$, see [3], 4.4) and $c + n - 1 = a_{n-1}$, since

$$A_n = \{\min(S(C) \cap (j + n\mathbf{Z}_+)); 0 \le j \le n - 1\}.$$

Finally, a similar argument shows that $(f \cdot \partial f/\partial X) = c + n - 1$, if $n = \mathrm{Ord}_X(f(X, 0)) < \mathrm{Ord}_Y(f(0, Y))$.

REMARK 11. Proposition 10 shows that if h defines the polar of C with respect to M then h satisfies the hypotheses in the Factorization Theorem for $k = n - 1$, so Lemma 1.6 of **[4]** is also a special case of (2) in the Factorization Theorem.

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