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GENERALIZED RATIONAL CONVEXITY IN BANACH ALGEBRAS

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Let A be a complex commutative Banach algebra with identity and let F be a closed subset of its spectrum $X(A)$. There are several hulls, associated with F, which are useful in the study of approximation, interpolation and separation problems: the polynomially and rationally convex hulls are the most popular, the A-convex hull has also been considered and there are also holomorphically convex hulls. In this paper we introduce a family of hulls, denoted by $R_n(F)$, and we study some relations between these hulls and several known objects and invariants in commutative Banach algebras.

We study the relations between $R_n(F)$ and the generalized Shilov boundaries introduced by Basener [**1**] and Sibony [**17**], the topological stable rank, introduced by Rieffel [**15**], the dimension of $X(A)$, the minimal number of generators of A, etc. We describe briefly the contents of the paper. In §1 we introduce the hulls $R_n(F)$ and prove several basic properties of them. In §2 we consider those F's such that $R_n(F) = X(A)$; the intersection of all those F's, denoted by $\Gamma_n(A)$, seems to play a role similar to that of the generalized boundaries $S_n(A)$ of Basener and Sibony; we prove that $\Gamma_0(A) = S_0(A) \subset \Gamma_1(A) \subset S_1(A) \cdots$ but, in general, $\Gamma_n(A)$ is strictly contained in $S_n(A)$. In §3 we introduce the invariant $r(A) = \min\{n \geq 0 : R_n(F) = F$ for every $F\}$, which we call the *rationality* of A, and we relate it with the sets $\Gamma_n(A)$ and $S_n(A)$. In §4 we study the relationship between the topological stable rank of A with the new notions. In §5 we prove that $d(A) \leq r(A) + \gamma(A) \leq 2\gamma(A)$ where $d(A)$ is the (covering) dimension of $X(A)$ and $\gamma(A)$ is the minimal number of generators of A; the proof of this result uses known facts on Čech cohomology groups of some compact subsets of $\mathbb{C}^n$, due to Andreotti and Narasimhan and Duchamp and Stout (see [**7**]).

Section 6 contains a generalization of a result of Forelli [**11**] and §7 contains some results about the hulls R_n in the particular case of the algebra H^∞. Finally, in §8 we collect several open problems. There are several direct predecessors of this paper: one is a work of Csordas and Reiter [**5**], who introduced the hulls R_1 (which they call *L-sets*) and the condition $r(A) = 1$ (this is a *separating algebra* in their terminology); another ancestor is a paper of Eifler **[9]**, who studied F's such that $R_1(F) = X(A)$ (these are his *inverting sets* for A); finally we have taken several facts about hulls from Stout's book **[19]**. We thank the referee for his valuable comments, in particular for correcting a mistake in Theorem 7.1.

Preliminaries. In this paper a *Banach algebra* A means a complex commutative Banach algebra with identity. The *spectrum* of A is the set $X(A)$ of all (identity preserving) homomorphisms $A \to \mathbb{C}$ (called *characters*). Every character is continuous; so we can consider on $X(A)$ the weak $*$ topology of the dual A^*. With this topology $X(A)$ is a compact Hausdorff space. The Gelfand map of A is the homomorphism $g: A \to C(X(A))$ defined by $g(a) = \widehat{a}$, $\widehat{a}(h) = h(a)$ $(h \in X(A))$. The letter F denotes a non-void closed subset of $X(A)$ and A_F is the closed subalgebra of $C(F)$ generated by the restrictions $\widehat{a}|F$ $(a \in A)$. In general, for a space X, $C(X)$ is the algebra of all complex continuous functions on X. For $a \in A$, $\|\widehat{a}\|_F = \|\widehat{a}|F\|_\infty$. Given $a \in A^n$ we set $\sigma(A) = \{h(z): h \in X(A)\} = \widehat{a}(X(A))$. The zero set of $a \in A^n$ is $Z_a = \{h \in X(A): h(a) = 0\}$. It is known that $X(A_{Z_a})$ is homeomorphic to Z_a. If A is a Banach algebra and $n \geq 1$ we use in A^n the norm $\|a\| = \|(a_1, \dots, a_n)\| = (\sum_{k=1}^n \|a_i\|^2)^{1/2}$. This holds, in particular, for $z \in C^n$, where we write $|z|$ instead of $\|z\|$. In §6 we denote by H^k the kth Čech cohomology functor with integer coefficients.

1. Generalized rational hulls.

1.1. DEFINITIONS. Given a closed subset F of $X(A)$ define $R_0(F) = \{h \in X(A): |h(a)| \leq \|\widehat{a}\|_F \ (a \in A)\}$. For $n \geq 1$ and $a \in A^n$ we put $\Delta(F, a) = \widehat{a}^{-1}(\widehat{a}(F)) = \{h \in X(A): h(a) \in \widehat{a}(F)\}$ and $R_n(F) = \bigcap\{\Delta(F, a): a \in A^n\}$.

1.2. REMARKS. (i) $R_0(F)$ is usually denoted by $\widehat{F}$ and referred to as the *A-convex hull* of F

(ii) The sets $R_1(F)$ were first considered in **[5]**; see also **[19**, p. 369].

(iii) Given a compact subset K of $\mathbb{C}^\Lambda$ (where Λ is an arbitrary set and $\mathbb{C}^\Lambda$ is provided with the product topology) and a set of functions

$\mathcal{S}$ defined on $\mathbb{C}^\Lambda$ we can define the hulls

$$R_0(K,\mathcal{S}) = \{z \in \mathbb{C}^\Lambda : |f(z)| \leq \|f\|_K \text{ for all } f \in \mathcal{S}\},$$
$$R_n(K,\mathcal{S}) = \{z \in \mathbb{C}^\Lambda : f(z) \in f(K) \text{ for all } f = (f_1, \dots, f_n) \in \mathcal{S}^n\} \quad (n \geq 1).$$

If $\mathcal{S} = \mathcal{P}$ is the algebra of all complex polynomials in the variables X_λ $(\lambda \in \Lambda)$, $R_0(K,\mathcal{P})$ is the usual polynomial hull of K (see **[19]**). If $\mathcal{S} = \mathcal{R}_K$ the algebra of all rational functions with poles off K, $R_0(K,\mathcal{P}_K)$ is the rational hull of K. It is well known that $R_0(K,\mathcal{R}_K) = R_1(K,\mathcal{P})$. These classical hulls motivate the name of *generalized rational hulls* given to the sets $R_n(F)$, which should be denoted by $R_n(F,\widehat{A})$.

We collect some facts about $R_n(F)$ in the next proposition.

1.3. Proposition. (i) $X(A) \supset \cdots \supset R_n(F) \supset R_{n+1}(R) \supset \cdots \supset F$ $(n \leq 0)$.

(ii) *If $R_n(F) = F$ then $R_k(F) = F$ for all $k \geq n$.*

(iii) *If $R_n(F) = X(A)$ then $R_k(F) = X(A)$ for all $k \leq n$.*

(iv) *For $h \in X(A)$, $h \notin fR_0(F)$ if and only if for every $\varepsilon > 0$ there exists $a \in A$ such that $h(a) = 1$ and $\|\widehat{a}\|_F < \varepsilon$.*

(v) *If $n \geq 1$ and $h \in X(A)$, then $h \notin R_n(F)$ if and only if there exists $a \in A^n$ such that $h(a) = 0$ and $0 \notin \widehat{a}(F)$.*

(vi) *If $n \geq 1$ and D is a dense subset of A^n then*

$$R_n(F) = \bigcap \{\Delta(a,F) : a \in D\}.$$

(vii) *For every $k \geq 0$ it holds*

$$\bigcap \{R_n(F) : n \geq 0\} = \bigcap \{R_n(F) : n \geq k\} = F.$$

Proof. (i) Obviously $\Delta(F,a) \supset F$ for every $a \in A^n$, so $R_n(F) \supset F$ for $n \geq 1$. For $a \in A^n$, $\Delta(F,a) = \Delta(F,(a,0))$, so $R_{n+1}(F) \subset R_n(F)$ for $n \geq 1$. The case $R_1(F) \subset R_0(F)$ is clear.

(ii) and (iii) follow easily from (i).

(iv) If $h \in X(A) \backslash R_0(F)$, there is $a \in A$ such that $|h(a)| > \|\widehat{a}\|_F$. Multiplying a by an appropriate constant we can suppose that $1 = h(a) > \|\widehat{a}\|_F$. Finally, replacing a by a^n with n large enough we get $h(a) = 1$ and $\|\widehat{a}\|_F < \varepsilon$. The converse is obvious.

(v) If $h \in X(A) \backslash R_n(F)$, there is $a \in A^n$ such that $h(a) \notin \widehat{a}(F)$. If $b = a - h(a)$, then $h(b) = 0$ and $0 \notin \widehat{b}(F)$.

(vi) It suffices to prove that $R_n(F) \supset \bigcap\{\Delta(F,a): a \in D\}$. Let $h \in X(A)\backslash R_n(F)$. By (v) there is $a \in A^n$ such that $h(a) = 0 \notin \widehat{a}(F)$. Then $\delta = \inf\{|k(a)|: k \in F\} > 0$ and we can choose $b \in D$ such that $\|a-b\| < \delta/2$. We shall show that $|h(b)| < \inf\{|k(b)|: k \in F\}$, which will imply that $h(b) \notin \widehat{b}(F)$, i.e. that $h \notin \bigcap\{\Delta(F,d): d \in D\}$. Now, $\|a-b\| < \delta/2$ implies that $|h(b)| = |h(b) - h(a)| \leq \|a-b\| < \delta/2$, so it suffices to prove that $|k(b)| \geq \delta/2$ for every $k \in F$. If $k \in F$ then $||k(a)| - |k(b)|| \leq |k(a) - k(b)| \leq \|a-b\| < \delta/2$ so we get $|k(b)| \geq |k(a)| - \delta/2 \geq \delta - \delta/2 = \delta/2$.

(vii) Part (i) implies that, for each $k \geq 0$, $\bigcap\{R_n(F): n \geq k\} = \bigcap\{R_n(F): n \geq 0\} \supset F$. If $h \notin F$, for every $k \in F$ there is $a_k \in A$ such that $\widehat{a}_k(k) \neq 0$ and $\widehat{a}_k(h) = 0$.

Let U_k be an open neighborhood of k such that $\widehat{a}_k(l) \neq 0$ for all $l \in U_k$. By compactness, there are $k_1, \ldots, k_n \in F$ such that $F \subset U_{k_1} \cup \cdots \cup U_{k_n}$. If $a = (a_{k_1}, \ldots, a_{k_n})$ then $h(a) = 0$ and $0 \notin \widehat{a}(F)$. This means, by (v), that $h \notin R_n(F)$, so $h \notin \bigcap R_n(F)$ □

1.4. Remarks. (i) The equality $\bigcap R_n(F) = F$ can also be proved by means of the axiomatic joint spectra theory of Zelazko [**25**] (see also Curto [**6**], Eschmeier [**10**] and Vasilescu [**23**]). A *spectral system* on a Banach algebra A is a rule $\widetilde{\sigma}$ which assigns to each $a \in A^n$ $(n \geq 1)$ a compact subset $\widetilde{\sigma}(a)$ of $\mathbb{C}^n$ such that $\widetilde{\sigma}(a) \subset \sigma(a)$, $p_1(\widetilde{\sigma}(a,b)) = \widetilde{\sigma}(a)$ and $p_2(\widetilde{\sigma}(a,b)) = \widetilde{\sigma}(b)$, where $p_1: \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n$, $p_2: \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^m$ are the usual projections. Given a spectral system $\widetilde{\sigma}$ on A, $\Delta = \bigcap\{\widehat{a}^{-1}(\widetilde{\sigma}(a)): a \in A^n, n \geq 1\}$ is the only closed subset of $X(A)$ such that $\widetilde{\sigma}(a) = \widehat{a}(\Delta)$ (see [**10**, 1.1], for instance). It is easily seen that, for every F, $\widetilde{\sigma}: a \to \widehat{a}(F)$ is a spectral system with $\Delta = \bigcap\{R_n(F): n \geq 1\}$ and $\widetilde{\sigma}(a) = \widehat{a}(F)$ by definition. By the uniqueness of Δ it follows that $\Delta = F$.

(ii) Observe that each R_n is an involution operator in the sense that $R_n(F_n(F)) = R_n(F)$.

2. The condition $R_n(F) = X(A)$.

2.1. Proposition. *The following conditions are equivalent.*

(1) $R_n(F) = X(A)$.

(2) *For every* $a \in A^n$, $\sigma(a) = \widehat{a}(F)$.

(3) *For every* $a \in A^n$, $Z_a \cap F = \varnothing$ *implies that* $Z_a = \varnothing$.

Proof. (1) $\Rightarrow$ (2). Suppose that $\sigma(a) \neq \widehat{a}(F)$ for some $a \in A^n$. Then there is $h \in X(A)$ such that $h(a) \notin \widehat{a}(F)$. This means that $h \notin \Delta(F,a)$

so $h \notin R_n(F)$, which contradicts (1). (2) $\Rightarrow$ (3). Take $a \in A^n$ such that $Z_a \neq \varnothing$. Then $0 \in \sigma(a) = \widehat{a}(F)$ and there exists $k \in F$ such that $k(a) = 0$, or, which is the same, $k \in Z_a \cap F$.

(3) $\Rightarrow$ (1). If $h \notin R_n(F)$, there exists $a \in A^n$ such that $h(a) = 0$ and $0 \notin \widehat{a}(F)$; in other terms, $h \in Z_a$ and $Z_a \cap F = \varnothing$; this contradicts (3). □

2.2. DEFINITION. Let $n \geq 1$. F is *n-inverting* (for A) if $0 \notin \widehat{a}(F) \Rightarrow a \in U_n(A) = \{c \in A^n : \sum_{k=1}^n Ac_k = A\}$, for all $a \in A^n$. For $n = 1$ this is a notion due to Eifler **[9]**.

It is easy to see that $a \in U_n(A)$ if and only if $0 \notin \sigma(a)$. Thus, F is n-inverting if and only if $R_n(F) = X(A)$. We say that F is *0-inverting* if $R_0(F) = X(A)$.

The intersection of all n-inverting subsets of $X(A)$ will be denoted $\Gamma_n(A)$:

$$\Gamma_n(A) = \bigcap\{F : R_n(F) = X(A)\} \qquad (n \geq 0).$$

The proof of the next result follows easily from 1.3 (iii) and the definition of the Shilov boundary (see also the next paragraph):

2.3. PROPOSITION. (i) $\Gamma_0(A)$ *is the Shilov boundary of* A.
(ii) $\Gamma_0(A) \subset \Gamma_1(A) \subset \Gamma_2(A) \subset \cdots$.
Therefore $\Gamma_n(A)$ *is non-void for all* $n \geq 0$.

The next result relates the notion of hulls R_n to that of generalized Shilov boundaries introduced by Basener **[1]** and Sibony **[17]**. Recall the definitions. A *boundary* for A is a closed subset F of $X(A)$ such that $\|\widehat{a}\|_F = \|\widehat{a}\|_{X(A)}$ $(a \in A)$. The Shilov boundary $S_0(A)$ is the intersection of all boundaries and it turns to be itself a boundary. For $n \geq 1$, the nth boundary of A is the closure in $X(A)$ of $\bigcup\{S)_0(A_{Z_a}) : a \in A^n\}$ (this makes sense because $S_0(A_{Z_a}) \subset X(A_{Z_a}) = Z_a \subset X(A)$). We denote it by $S_n(A)$.

An interesting result of Tonev **[21]** is the following: $S_n(A)$ is the intersection of all n-boundaries of A and it is itself an n-boundary, where a closed subset E of $X(A)$ is an *n-boundary* (for A) if $\min\{|k(a)| : k \in E\} = \min\{|k(a)| : k \in X(A)\}$ for all $a \in U_{n+1}(A)$.

For more information on these generalized boundaries the reader is referred to **[1, 17, 21]**.

2.4. PROPOSITION. *For* $n \geq 0$ $R_n(S_n(A)) = X(A)$.

Proof. The case $n = 0$ is trivial. For $n \geq 1$ suppose, on the contrary, that there is $h \in X(A) \backslash R_n(S_n(A))$. Then, by 1.3 (v), there exists $a \in A^n$

such that $h(a) = 0$ and $0 \notin \widehat{a}(S_n(A))$. This means that $S_n(A) \cap Z_a = \varnothing$. Now $Z_a \neq \varnothing$ and $S_0(A_{Z_a}) \neq \varnothing$ but, by definition of $S_n(A)$, $S_0(A_{Z_a}) \subset S_n(A) \cap Z_a$, contradiction. Thus $R_n(S_n(A)) = X(A)$, as claimed. □

2.5. COROLLARY. $\Gamma_n(A) \subset S_n(A)$ $(n \geq 1)$.

2.6. PROPOSITION. *For every* $n \geq 1$ $\Gamma_n(A) \supset S_{n-1}(A)$.

Proof. Suppose, on the contrary, that there exist $h \in S_{n-1}(A)$ and a closed subset F of $X(A)$ not containing h such that $R_n(F) = X(A)$. Consider an open neighborhood U of h which does not meet F. From Tonev's characterization of $S_{n-1}(A)$ **[21]** there is $a \in U_n(A)$ such that

$$\begin{aligned}\min\{|k(a)|: k \in X(A)\} &= \min\{|k(a)|: k \in k \in U\} \\ &< \min\{|k(a)|: k \in X(A)\backslash U\} \leq \min\{|k(a)|: k \in F\}.\end{aligned}$$

Let $h_0 \in U$ such that $|h_0(a)| = \min\{|k(a)|: k \in U\}$. Then $|h_0(a)| < \min\{|k(a)|: k \in F\}$, by the inequalities above, so that $h_0(a) \notin \widehat{a}(F)$, which means that $h_0 \notin R_n(F)$, contradiction. □

2.7. COROLLARY. $\Gamma_0(A) = S_0(A) \subset \Gamma_1(A) \subset S_1(A) \subset \cdots$. *Therefore* $\bigcup_{n\geq 0}^{\infty} S_n(a) = \bigcup_{n\geq 0} \Gamma_n(A)$. □

We prove now that the unions considered at 2.7 are dense in $X(A)$.

2.8. PROPOSITION. $\bigcup\{S_n(A): n \geq 0\}$ *is dense in* $X(A)$.

Proof. Suppose on the contrary that there exists $h \in X(A)\backslash F$, where F is the closure of $\bigcup S_n(A)$.

Let $U = X(A)\backslash F$. By 1.3 (v) there exist $n \in \mathbb{N}$ and $a \in A^n$ such that $h(a) = 0$ and $\widehat{a}(F) \not\ni 0$. Then $F \cap Z_a = \varnothing$ so $S_0(A_{Z_a}) \subset Z_a \subset U$ and $S_n(A) \cap U$ is non-void which is absurd by the definition of U. Thus $F = X(A)$ as claimed. □

2.9. REMARK. It is worth noting that, in general, $\Gamma_n(A)$ is not an n-inverting set. For this, let $A = A(\mathbb{D}^n) =$ the n-polydisc algebra, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ with $|\alpha_k| < 1$ $(k = 1, \ldots, n)$ and $\varphi_\alpha^{(k)} \in A$ defined by $\varphi_\alpha^{(k)}(z) = (z - \alpha_k)/(1 - z\overline{\alpha}_k)$. By a result of Rudin **[16]**, Theorem 4.7.2 and the definition of R_k, $R_1(K_\alpha \cup \mathbb{T}^n) = \mathbb{D}^n = X(A)$, where K_α is the image of $\varphi_\alpha = (\varphi_\alpha^{(1)}, \ldots, \varphi_\alpha^{(n)})$ and $\mathbb{T} = \{z \in \mathbb{D}: |z| = 1\}$. Then, by its definition, $\Gamma_1(A)$ is contained in $\bigcap(K_\alpha \cup \mathbb{T}^n)$ where the intersection is taken over all $\alpha \in \mathbb{C}^n$ with $|\alpha_k| < 1$ for all k, which is obviously $\mathbb{T}^n$. Moreover, by 2.3, $\Gamma_1(A) \supset \Gamma_0(A) = S_0(A) = \mathbb{T}^n$ so that $\Gamma_1(A) = \mathbb{T}^n$.

However $R_1(\mathbb{T}^n) = \mathbb{T}^n$, for if $w \in \mathbb{D}^n \backslash \mathbb{T}^n$ then $|w_k| < 1$ for some $k = 1, \dots, n$ and the polynomial $p(z_1, \dots, z_n) = z_i - w_i$ vanishes at w and has no zero at $\mathbb{T}^n$. □

3. The condition $R_n(F) = F$. Let $r(A) = \min\{n \geq 0 : R_n(F) = F\ \forall F\}$ and $r(A) = +\infty$ if there is no such n; we call $r(A)$ the *rationality* of A.

3.1. PROPOSITION. (i) *$R_0(F) = F$ if and only if for every $\varepsilon > 0$ and $h \in X(A) \backslash F$ there exists $a \in A$ such that $h(a) = 1$ and $\|\widehat{a}\|_F < \varepsilon$.*

(ii) *If A is regular then $r(A) = 0$.*

(iii) *Suppose that there exists $a \in A^n$ such that $\widehat{a} : X(A) \to \mathbb{C}^n$ is injective. Then $r(A) \leq n$.*

Proof. (i) follows easily from 1.3 (iv).

(ii) If A is regular and $h \in X(A) \backslash F$ there exists $a \in A$ such that $\widehat{a}|F = 0$ and $h(a) = 1$; apply (i).

(iii) If $\widehat{a}$ is injective, for every F it holds $\Delta(a, F) = \widehat{a}^{-1}(\widehat{a}(F)) = F$, so $R_n(F) = \bigcap \Delta(a, F)$ and $r(A) \leq n$. □

3.2. REMARKS. (i) In **[24]** Wilken says that A is *approximately regular* (on $X(A)$) if for each $h \in X(A)$, each closed set F in $X(A)$ not containing h and each $\varepsilon > 0$ there is $a \in A$ with $h(a) = 1$ and $\|\widehat{a}\|_F < \varepsilon$. By 3.1 (i) A is approximately regular if and only if $r(A) = 0$.

(ii) Let A be n-generated, in the sense that there are $a_1, \dots, a_n$ in A such that the subalgebra generated by $a_1, \dots, a_n$ is dense in A. Then 3.1 (iii) implies that $r(A) \leq n$, because if $a = (a_1, \dots, a_n)$ then $\widehat{a} : X(A) \to \sigma(a) \subset \mathbb{C}^n$ is a homeomorphism. Thus, $r(A) \leq \gamma(A)$ if $\gamma(A)$ is the minimum number of generators of A. However, this inequality is not sharp. In fact, if X is an infinite dimensional compact space, e.g. $X = [0, 1]^{\mathbb{N}}$, then $A = C(X)$ is regular, so $r(A) = 0$ by 3.1 (ii), and $\gamma(A) = +\infty$. We will see later a more precise relationship between $r(A)$, $\gamma(A)$ and the dimension of $X(A)$.

(iii) Suppose that there exist $a_1, \dots, a_n$ in A such that the closed *full* subalgebra generated by them is A. Then $\widehat{a}$ is injective and $r(A) \leq n$.

(iv) The condition $r(A) \leq 1$ has been introduced by Csordas and Reiter **[5]**, who called such A's "separating algebras". Most of their results can be generalized to our setting.

3.3. PROPOSITION. *If $r(A) \leq n$ then $\Gamma_n(A) = S_n(A) = X(A)$.*

Proof. By 2.5 it suffices to see that $\Gamma_n(A) = X(A)$. But if $r(A) \leq n$ the only closed set F in $X(A)$ which satisfies $R_n(F) = X(A)$ is $X(A)$. □

3.4. EXAMPLES. Given a compact subset X of $\mathbb{C}^n$ let $A(X)$ be the algebra of all maps $X \to \mathbb{C}$ which are holomorphic in the interior of X. Then $r(A(X)) \leq n$ if X is polynomially convex, for in this case the coordinate functions $z_1, \ldots, z_n$ generate $A(X)$. In particular, if $\mathbb{D}$ is the closed unit disc $r(A(\mathbb{D})) = 1$ because $R_0(F) \neq F$ for every F with $\mathbb{C}\backslash F$ disconnected, so $0 < r(A(\mathbb{D})) \leq 1$. In general, $r(A(\mathbb{D}^n)) = n$. We will return later to this example.

3.5. PROPOSITION. *If $R_n(F) = F$ then F is the intersection of a family of sets $P_{a,\delta} = \{h \in X(A): |h(a)| > \delta\}$ for some elements $a \in A^n$ and $\delta > 0$.*

Proof. Let $h_0 \in X(A)\backslash F$. By 1.3 (v) there is $a \in A^n$ such that $h_0(a) = 0 \notin \widehat{a}(F)$. Let $\delta = \min\{|h(a)|: h \in F\}$. Then $\delta > 0$ and $h_0 \notin P_{a,\delta/2}$. Thus, $F = \bigcap P_{a,\delta}$ for a family $\{(a,\delta)\}$ in $A^n \times \mathbb{R}^+$. □

4. The topological stable rank, R_n and S_n. The *topological stable rank* of A is the least n such that $U_n(A)$ is dense in A^n; we denote it by $\operatorname{tsr}(A)$. This is a notion introduced by Rieffel **[15]** which replaced efficiently, in the case of Banach algebras, the more algebraic stable range conditions of Hyman Bass **[2]**.

4.1. PROPOSITION. *Suppose that* $\operatorname{tsr}(A) \leq n$. *Then* $R_n(S_{n-1}(A)) = X(A)$.

Proof. By **[4**, 1.10**]** there is a dense subset D of A^n such that $\sigma(a)$ has no interior points, for every $a \in D$. Then $\sigma(a)$ coincides with its boundary $\partial\sigma(a)$. But, by a result of Tonev **[21]** $\partial\sigma(a) \subset \widehat{a}(S_{n-1}(A))$ $(a \in A^n)$. Thus, for every $a \in D$, $\sigma(a) = \widehat{a}(S_{n-1}(A))$, so that by 1.3 (vi),

$$\begin{aligned} R_n(S_{n-1}(A)) &= \bigcap\{\Delta(a, S_{n-1}(A)): a \in D\} \\ &= \bigcap\{\widehat{a}^{-1}(\sigma(a)): a \in D\} = X(A), \end{aligned}$$

because $\widehat{a}^{-1}(\sigma(a)) = X(A)$ $(a \in A^n)$. □

4.2. COROLLARY. *Suppose that* $\operatorname{tsr}(A) \leq n$ *and* $r(A) \leq n$. *Then* $S_{n-1}(A) = X(A)$. □

4.3. PROPOSITION. *Suppose that* $\operatorname{tsr}(A) = 1$. *Then* $R_0(F) = R_1(F)$.

Proof. It always holds $R_0(F) \supset R_1(F)$, (1.3 (i)). Let $h \notin R_1(F)$. By 1.3 (v) there is $a \in A$ such that $h(a) = 0$ and $\min\{|k(a)| : k \in F\} > 1$. By hypothesis, the invertible elements form a dense subset of A, so there is an invertible $u \in A$ with $\|a - u\| < 1/2$. Then $|h(u)| < 1/2$ and $\min\{|k(u)| : k \in F\} > 1/2$, so $|h(u^{-1})| > 2$ and $\|u^{-1}\|_F < 2$, which implies that $h \notin R_0(F)$. □

4.4. COROLLARY. *Suppose that* $\operatorname{tsr}(A) = 1$ *and* $r(A) \leq 1$. *Then* $r(A) = 0$.

It is an open problem if $\operatorname{tsr}(A)$ and $\operatorname{tsr}(C(X(A)))$ coincide. We present two results relating this problem with the invariant $r(A)$.

4.5. PROPOSITION. *Suppose that* $r(A) = 0$. *Then* $\operatorname{tsr}(C(X(A))) \leq \operatorname{tsr}(A)$.

Proof. Suppose that $\operatorname{tsr}(C(X(A))) \geq n + 1$. Then there exist F, a closed subset of $X(A)$, and $\varphi_1, \dots, \varphi_n \in C(X(A))$ such that $\varphi(F) = (\varphi_1(F), \dots, \varphi_n(F)) \subset \mathbb{C}_*^n \setminus \{0\}$ and $\varphi|F$ admits no extension to $C(X(A), \mathbb{C}_*^n)$ [22]. By the fact that $R_0(F) = F$, there is $\theta \in A_F^n$ lying in the component of $\varphi|F$ in $C(F, C_*^n)$: in fact, $X(A_F) = R_0(F) = F$ and it is well known that, for a Banach algebra B, $U_n(B)$ and $C(X(B), \mathbb{C}_*^n)$ are homotopy equivalent [20]. Now, by definition of A_F, $\widehat{A}|F$ is dense in A_F so that there is $a \in A^n$ such that $\widehat{a}|F$ belongs to the component of θ in $C(F, \mathbb{C}_*^n)$. In particular, $\widehat{a}|F$ admits no extension to $C(X(A), \mathbb{C}_*^n)$. By the results of Vaserstein [22] or Rieffel [15], this means precisely that $\widehat{a}$ does not belong to the closure of $U_n(C(X(A)))$ (in $C(X(A))^n$), which implies that a does not belong to the closure of $U_n(A)$ (in A^n). Thus $\operatorname{tsr}(A) \geq n + 1$. This proves that $\operatorname{tsr}(A) \geq \operatorname{tsr}(C(X(A)))$. □

4.6. COROLLARY. *If* $r(A) \leq 1$ *and* $\operatorname{tsr}(A) = 1$ *then* $\operatorname{tsr}(C(X(A))) = 1$.

Proof. Combine 4.5 and 4.4. □

4.7 *Problem.* It is not known if $\operatorname{tsr}(A) = \operatorname{tsr}(C(X(A))$, in general, even for regular algebras.

5. The main result. This section contains the main result of the paper, which states that the topological dimension of $X(A)$ is at most $\gamma(A) + r(A)$. We need some results about the cohomology of certain compact subsets of $\mathbb{C}^n$.

5.1. LEMMA. *Let $R_1, \ldots, R_m$ be rational convex compact subsets of $\mathbb{C}^n$ and $J = \bigcup_{k=1}^m R_k$. Then $H^{n+i}(J) = 0$ for all $i \geq m$.*

Proof. We proceed by induction in m. Let $m = 1$. If $K \subset \mathbb{C}^n$ is polynomially convex then $H^{n+i}(K) = 0$ for all $i \geq 0$ [7]. If $K \subset \mathbb{C}^n$ is rationally convex, then K is homeomorphic to some polynomially convex $K' \subset \mathbb{C}^{n+1}$ [**19**, p. 373] so that $H^{n+i}(K) = H^{n+i}(K') = 0$ for all $i \geq 1$. Suppose that the assertion holds for $1, 2, \ldots, m$ and let $J = \bigcup_1^{m+1} R_k$, where each R_k is rationally convex, $R_k \subset \mathbb{C}^n$. Consider the Mayer-Vietoris exact sequence

$$\cdots \to H^{n+i-1}\left(\left(\bigcup_1^m R_k\right) \cap R_{m+1}\right) \to H^{n+i}\left(\bigcup_1^{m+1} R_k\right)$$
$$\to H^{n+i}\left(\bigcup_1^m R_k\right) \oplus H^{n+i}(R_{m+1}) \to \cdots .$$

Then, by inductive hypothesis, if $i \geq m+1$,

$$H^{n+i}\left(\bigcup_1^m R_k\right) = 0, \qquad H^{n+i}(R_{m+1}) = 0$$

and

$$H^{n+i-1}\left(\left(\bigcup_1^m R_k\right) \cap R_{m+1}\right) = H^{n+i-1}\left(\bigcup_1^m R_k \cap R_{m+1}\right) = 0$$

because each $R_k \cap R_{m+1}$ is rationally convex. Then, by the exactness, $H^{n+i}(J) = H^{n+i}(\bigcup_1^{m+1} R_k) = 0$. □

We say that a compact subset K of $\mathbb{C}^n$ is *k-rationally convex* if $R_k(K) = K$, where

$$R_k(K) = \bigcap \left\{p^{-1}(p(K)) : p = (p_1, \ldots, p_k), p_i \in \mathbb{C}[t_1, \ldots, t_n]\right\}.$$

This set is denoted $h_k^r(K)$ in Słodkowski's paper [**18**, p. 323].

5.2. COROLLARY. *Let K be an $(n-1)$-rationally convex subset of $\mathbb{C}^n$. Then K does not separate $\mathbb{C}^n$ (i.e., $\mathbb{C}^n \backslash K$ is connected).*

Proof. Suppose, on the contrary, that $\mathbb{C}^n \backslash K$ is disconnected, and assume that $K \subset B = \{z \in C^n : |z| = (\sum_{i=1}^n |z_i|^2)^{1/2} \leq M\}$. Then there are polynomials $p_1, \ldots, p_{n-1}$ such that $p(z_0) = 0 \notin p(K)$, where $p = (p_1, \ldots, p_{n-1})$. Then there is $\delta > 0$ such that $K \subset \bigcup_{k=1}^{n-1} R_k$, where $R_k = \{z \in B : |p_k(z)| \geq \delta\}$. Observe that each R_k is rationally convex, so that, by 6.1, $H^{n+i}(\bigcup_{k=1}^{n-1} R_k) = 0$ for all $i \geq n-1$; in particular $H^{2n-1}(\bigcup_{k=1}^{n-1} R_k) = 0$, so that $\bigcup_{k=1}^{n-1} R_k$ does not separate $\mathbb{C}^n$ [**14**, p. 100]. But $z_0 \in B \setminus \bigcup_{k=1}^{n-1} R_k$, $K \subset \bigcup_{k=1}^{n-1} R_k$ and K separates $\mathbb{C}^n$, so $\bigcup_{k=1}^{n-1} R_k$ should separate $\mathbb{C}^n$, contradiction. □

5.3. COROLLARY. *For every compact subset K of $\mathbb{C}^n$, $R_{n-1}(K)$ contains all bounded components of $\mathbb{C}^n \setminus K$.*

5.4. REMARK. It is false, in general, that $R_{n-1}(K)$ consists, exactly, of K and all bounded components of $\mathbb{C}^n \setminus K$. For instance, at 2.9 with $n = 2$, any $\alpha = (\alpha_1, \alpha_2)$ and $K = K_\alpha \cup \mathbb{T}^2$ it holds $R_1(K) = \mathbb{D}^2$ and K does not separate $\mathbb{C}^2$.

5.5. PROPOSITION. *If A is generated by $a_1, \ldots, a_n$ then $R_k(\widehat{a}(F)) = \widehat{a}(R_k(F))$ for every closed subset F of $X(A)$.*

Proof. The inclusion $\widehat{a}(F_k(F)) \subset R_k(\widehat{a}(F))$ holds in general: if $h \in R_k(F)$ and $p = (p_1, \ldots, p_k)$ is a polynomial map $p(h(a)) = h(p(a)) \in p(\widehat{a})(F) = p(\widehat{a}(F))$, so that $h(a) \in R_k(\widehat{a}X(A)(F))$. To see the converse inclusion let $z \in R_k(\widehat{a}(F))$. Observe that $R_k(\widehat{a}(F)) \subset R_k(\sigma(a)) = \sigma(a)$ because the joint spectrum of a system of generators is always polynomially convex, a fortiori k-rationally convex for every $k \geq 1$. Then $z = h(a)$ for some $h \in X(A)$, and $p(z) \in p(\widehat{a}(F))$ for all p, that is $h(p(a)) \in p(\widehat{a})(F)$ for all p. Now, using the fact that the set of all $p(a)$ is dense in A^k and applying 1.3 (vi) we conclude that $h \in R_k(F)$. Thus, $R_k(\widehat{a}(F)) \subset \widehat{a}(R_k(F))$. □

5.6. THEOREM. *If A is a complex commutative Banach algebra with identity and $d(A)$ is the dimension of its spectrum $X(A)$ then*

$$d(A) \leq \gamma(A) + r(A) \leq 2\gamma(A).$$

Proof. By 3.2 (ii) $r(A) \leq \gamma(A)$ and it suffices to prove the first inequality. Let $n = \gamma(A)$ and $k = r(A)$. By [**14**, Ch. VIII] it suffices to prove that $H^{n+i}(F) = 0$ for all closed subsets F of $X(A)$ and $i \geq k$. For $i \geq n+1$, $H^{n+i}(F) = 0$ because F is homeomorphic to some compact subset of $\mathbb{C}^n$. For $k \leq i \leq n$ we prove the assertion by a reverse induction.

Consider a (fixed) homeomorphism $\widehat{a}: X(A) \to \sigma(a) \subset \mathbb{C}^n$, and let $T = \sigma(a)$.

(+) Suppose that $H^{n+j}(F) = 0$ for $j \geq i > k$ and for all closed subsets F of $X(A)$.

(a) Consider first a compact set of the form

$$P = \bigcup_{j=1}^{k} \{z \in T : |p_j(z)| \geq \delta_j\}$$

for some polynomials $p_1, \ldots, p_k$, and positive numbers $\delta_1, \ldots, \delta_k$. Let $R_j = \{z \in T : |p_j(z)| \geq \delta_j\}$. Then

$$H^{n+i-1}(P) = H^{n+i-1}\left(\bigcup_{j=1}^{k} R_j\right) = 0 \qquad \text{by 6.1.}$$

(b) Let $\mathscr{P}$ be the collection of all P like in (a). If $J = \bigcap_{l=1}^{m} P_l$ ($P_l \in \mathscr{P}$) then $H^{n+i-1}(J) = 0$. We prove this assertion by induction in m. For $m = 1$, it has been proved at (a). Suppose that it holds for $1, \ldots, m$ and let $P_1, \ldots, P_{m+1} \in \mathscr{P}$, $E = \bigcap_{l=1}^{m} P_l$, $F = P_{m+1}$ and $r = n + i - 1$. Consider the portion of the Mayer-Vietoris exact sequence

$$\cdots \to H^r(E) \oplus H^r(F) \to H^r(E \cap F) \to H^{r+1}(E \cup F) \to \cdots.$$

Then $H^r(E) = 0$ by inductive hypothesis, $H^r(F) = 0$ by (a) and $H^{r+1}(E \cup F) = 0$ by (+) (that is, by our first inductive hypothesis), because $E \cup F$ is homeomorphic to a closed subset of $X(A)$. Thus

$$H^{n+i-1}\left(\bigcap_{l=1}^{m+1} P\right) = H^r(E \cap F) = 0,$$

which proves the assertion of (b).

Finally, we consider a closed subset F of $X(A)$ and its image $F' \subset \mathbb{C}^n$ by $\widehat{a}: X(A) \to T$.

F' is k-rationally convex by 5.5, so that $F' = \bigcap Q_\alpha$ for some (possibly infinite collection of) $Q_\alpha \in P$. For each Q_α it holds $H^{n+i-1}(Q_\alpha) = 0$ by (b). Thus, by the continuity of Čech cohomology we get $H^{n+i-1}(F) = H^{n+i-1}(F') = 0$, which finishes the proof. □

5.7. Corollary. *If $d(A) = 2n$ and $\gamma(A) = n$ then $r(A) = n$.* □

5.8. Examples. (i) Let $X \subset \mathbb{C}^n$ be the polydisc $\mathbb{D}^n$ or the ball B_n or any polynomially convex compact subset of C^n with interior. Then $r(A(X)) = n$.

(ii) Let $A(\mathbb{D}^\infty)$ be the closure in $C(\mathbb{D}^\infty)$ of all polynomials in a finite number of variables.

Observe that every $A(\mathbb{D}^n)$ is a quotient of $A(\mathbb{D}^\infty)$.

It is easy to prove, in general, that $r(A/I) \leq r(A)$ for all ideals I of A.

Then $r(A(\mathbb{D}^\infty)) \geq r(A(\mathbb{D}^n)) = n$ for all n, by 5.7.

Thus, $r(A(D^\infty)) = \infty$.

(iii) This is an example of a k-rationally convex compact subset of $\mathbb{C}^n$ which is not the union of two 1-rationally convex sets. Consider

$$K_1 = \{z \in D^4 : |z_1| \geq 1/2 \text{ or } |z_2| \geq 1/2\},$$
$$K_2 = \{z \in D^4 : |z_3| \geq 1/2 \text{ or } |z_4| \geq 1/2\}.$$

It is clear that K_1, K_2 and $K_1 \cap K_2$ are 2-rationally convex sets. We consider the exact sequence

$$\cdots \to H^6(K_1) \oplus H^6(K_2) \to H^6(K_1 \cap K_2) \to H^7(K_1 \cup K_2)$$
$$\to H^7(K_1) \oplus H^7(K_2) \cdots .$$

Now $H^6(K_1) = H^6(K_2) = 0$ because K_1 and K_2 are unions of two 1-rationally convex sets. By the same reason, $H^7(K_1) = H^7(K_2) = 0$. Thus, $H^6(K_1 \cap K_2) \cong H^7(K_1 \cup K_2) \neq 0$, because $K_1 \cup K_2$ separates $\mathbb{C}^4$, so that by 5.1 $K_1 \cup K_2$ is not the union of two 1-rationally convex sets.

6. A result of Forelli. Our generalized rational hulls give a notion of separation associated to n-tuples of elements of A. In this section we consider n-tuples of elements of an ideal I of A. In **[11]** Forelli proves that, if A is the disc algebra, I is an ideal of A and F is a compact subset of $\mathbb{D}$ such that $F \cap \text{hull}(I) = \varnothing$ (where $\text{hull}(I) = \{z \in \mathbb{D} : f(z) = 0 \ \forall f \in I\}$) then there exists $f \in I$ such that $f|F$ does not vanish. Here we generalize Forelli's theorem in the following sense:

6.1. THEOREM. *Let A be n-generated, F a closed subset of $X(A)$ such that $F \cap \text{hull}(I) = \varnothing$. Then there exist $a = (a_1, \ldots, a_n) \in I^n$ with $0 \notin \widehat{a}(F)$.*

Proof. First we suppose that A is uniform. Then A is $A(X)$ for some polynomially convex compact $X \subset \mathbb{C}^n$. Now, if V is an open neighborhood of hull(I) such that $V \cap F = \varnothing$, then by the polynomial convexity of hull(I) and a result of Gunning and Rossi **[13**, p. 218**]** there exist n polynomials $p_1, \ldots, p_n$ in $\mathbb{C}[t_1, \ldots, t_n]$ such that

$$V \supset \{z \in X : |p_1(z)| < 1, \ldots, |p_n(z)| < 1\} \supset \text{hull}(I).$$

Define $C_k = \{z \in X : |p_k(z)| \geq 1\}$. Then $\text{hull}(I) \cap C_k = \varnothing$, C_k is rationally convex ($k = 1, \ldots, n$) and $F \subset \bigcup_{k=1}^n C_k$. Fix k. By compactness, there exists $g = (g_1, \ldots, g_m) \in I^m$ such that $0 \notin g(C_k)$.

The spectrum of $R(C_k)$ (the closed subalgebra generated by the rational functions with poles off C_k) is C_k, is rationally convex. Thus, $0 \notin g(C_k)$ means that $g \in U_m(R(C_k))$ and, by the definition of $R(C_k)$, there exist polynomials r_j, s_j ($j = 1, \dots, m$) such that s_j does not vanish on C_k and $h = \sum_{j=1}^m r_j g_j / s_j$ does not vanish on C_k. Thus $f_k = (\prod_{j=1}^m s_j) h \in I$ and f_k does not vanish on C_k. Now, k being arbitrary, we get $f = (f_1, \dots, f_n) \in I^n$ such that $0 \notin f(\bigcup_{k=1}^n C_k)$ and, a fortiori, $0 \notin f(F)$, as claimed.

For A not necessarily uniform, we only need to observe that, in the proof above, we work with elements in $\widehat{A}$ so the result holds in general. □

6.2. COROLLARY. *Under the same hypothesis if $V \subset X(A)$ is a neighborhood of* hull(I) *then there exists an n-generated ideal $J \subset I$ such that* hull$(I) \subset$ hull$(J) \subset V$.

Proof. It suffices to take $F = X(A) \backslash V$ and $J = \langle f_1, \dots, f_n \rangle$. □

7. Some results on H^∞. The famous "corona theorem" of Carleson [3] says that if $f_1, \dots, f_n$ are bounded holomorphic functions on the open disc Δ (in symbols, $f_1, \dots, f_n \in H^\infty$) such that $|f_1| + \dots + |f_n| \geq \delta$ for some $\delta > 0$ then there exist $g_1, \dots, g_n \in H^\infty$ such that $\sum_{k=1}^n f_k g_k = 1$; if we suppose Δ imbedded in $X(H^\infty)$, Carleson's result says that Δ is dense. The rationality of H^∞ is closely related to the corona theorem. More precisely, if we knew that $r(H^\infty) \leq m$ then it would suffice to prove Carleson's theorem for m-tuples. In fact, if there exists $\varphi \in H(H) \backslash \overline{\Delta}$ (here $\overline{\Delta}$ is the closure of Δ in $X(H^\infty)$), then there is an $f = (f_1, \dots, f_r) \in (H^\infty)^r$ with $r = r(H^\infty)$ such that $\widehat{f}(\varphi) = 0$ and $\widehat{f}(x) \neq 0$ for all $x \in \overline{\Delta}$. We may assume that $f \in (H^\infty)^m$, filling the last coordinates with zeros if it is necessary. Thus, by compactness of $\overline{\Delta}$, $|f(z)| = |f_1(x)| + \dots + |f_m(z)| \geq \delta > 0$ for some δ and all $z \in F$, so f verifies the corona hypothesis and then, by our assumption, $f \in U_m(H^\infty)$, which contradicts the existence of φ.

In particular, if we knew that $r(H^\infty) = 1$ this would imply trivially Carleson's theorem. We collect in this section a few results about the hull R_n of H^∞.

We recall that $\{z_n\} \subset \Delta$ is an *interpolating sequence* if for every bounded sequence $\{b_n\}$ there exists $f \in H^\infty$ such that $f(z_n) = b_n$ ($n \geq 1$). By identifying the points of Δ with some characters of H^∞, we define $G = \{x \in X(H^\infty) : x$ is a cluster point of an interpolating sequence $\{z_n\}\}$.

Obviously $G \supset \Delta$ and it can be shown that $G \cap S_0(H^\infty) = \emptyset$ but $G \cup S_0(H^\infty)$ is strictly contained in $X(H^\infty)$.

7.1. THEOREM. *Let F be a compact subset of $X(H^\infty)$.*
(1) *If $F \subset S_0(H^\infty)$ then $R_1(F) = F$.*
(2) *If $F \subset X(H^\infty)$ then*

$$R_2(F)\backslash F \subset X(H^\infty)\backslash G.$$

Proof. (1) Let $x \in X(H^\infty)\backslash S_0(H^\infty)$. By a theorem of Newman [**12**, p. 194] there is a Blaschke product B such that $B(x) = 0$ and $|B|F| \equiv 1$. Then $x \notin R_1(F)$. This proves that $R_1(F) \subset S_0(H^\infty)$. Let $x \in S_0(H^\infty)\backslash F$. Recall that $X(L^\infty) \approx S_0(H^\infty)$ and $\widehat{L}^\infty = C(X(L^\infty))$. Then, there is $u \in L^\infty$ such that $u(F) = 1$, $|u| \equiv 1$ and $u(x) = -1$. By a theorem of Douglas and Rudin [**12**, p. 192] there exist Blaschke products B_1 and B_2 such that $||B_1/B_2 - u||_{S_0(H^\infty)} < \varepsilon < 1$, if $\varepsilon > 0$. Let $h = B_1 - B_2$. Then, if $z \in F$

$$|h(z)| = |B_1(z) - u(z)B_2(z)| < \varepsilon|B_2(z)| = \varepsilon$$

and

$$\begin{aligned} |h(x)| &= |B_1(x) + B_2(x) - 2B_2(x)| \\ &\geq ||2B_2(x)| - |B_1(x) + B_2(x)|| \\ &= 2 - |B_1(x) + B_2(x)| \\ &= 2 - |B_1(x) - u(x)B_2(x)| > 2 - \varepsilon. \end{aligned}$$

This proves that $|h(x)| > ||h||_F$, a fortiori $x \notin R_1(F)$.

(2) Let $x \in G\backslash F$. Then there is an interpolating sequence $\{z_n\}$ such $x \in L =$ closure of $\{z_n : n \in \mathbb{N}\}$. Let $Q: H^\infty \to l^\infty$ be defined by $Q(f) = \{f(z_n)\}_{n\in\mathbb{N}}$. Then Q is onto and $\widehat{Q}: \widehat{H}^\infty \to \widehat{l}^\infty = C(X(l^\infty)) = C(L)$ is onto, too, because $Q^*: X(l^\infty) \to L$ is a homeomorphism.

Let $g \in C(L)$ such that $Z_g \cap (F \cap L) = \emptyset$. Then there is $f \in H^\infty$ such that $f|L = g$. Thus, if B is a Blaschke product whose zeros a_n ($n \in \mathbb{N}$), then (B, f) separates x from F, for $(B, f)(x) = 0$ and if $z \in F$ and $B(z) = 0$ by [**15**, pp. 379] $z \in L$ so that $f(z) = g(z) \neq 0$. □

8. Some open problems. There are several questions about the subjects considered in this paper that we have not been able to answer. We collect in this section those that we consider the most relevant.

8.1. H^∞. As usual in uniform algebra theory, this algebra is a source of many interesting questions. We do not know neither the dimension of its spectrum nor its rationalilty $r(H^\infty)$. It is possible that $r(H^\infty) = +\infty$. We also ignore its Bass' stable rank and its topological stable rank.

8.2. $r(A)$ vs. $\operatorname{tsr}(A)$. We suspect that $r(a) \leq \operatorname{tsr}(A) - 1$, but we do not have a proof even for the case $\operatorname{tsr}(A) = 1$.

8.3. $\operatorname{tsr}(A)$ vs. $\operatorname{tsr}(C(X(A)))$. These two invariants should be related, but we only have partial answers (see §4).

8.4. Is it true that $S_{n-1}(A) = X(A)$ if $\operatorname{tsr}(A) \leq n$?

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