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COHOMOLOGY OPERATIONS FROM HIGHER PRODUCTS IN THE DERHAM COMPLEX

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We give a new construction of all Steenrood cyclic reduced powers $\mathscr{P}^i$ and of Pontrjagin-Thomas pth powers $\mathscr{B}_p$ for each prime p. The cohomology operations are indued by operations, analogous to the p-fold cup-i products, defined in the deRham complex of Cartan-Miller. These operations form a basis of all the cohomology operations derived from cyclic groups. This extends the construction of the Steenrod squares based on the analogue of the cup-i product in the deRham complex. From the construction of these new operations in the deRham complex it follows that the commutative cochain problem does not have a solution over the integers.

A key feature of cohomology is the existence of a commutative multiplication. This multiplication is induced by the cup product on cochains. While the product gives the cohomology a structure of a commutative ring, the cup product on the cochains is not commutative for arbitrary coefficients. The large number of cochains on any given space together with non-commutativity of the cup product makes any effective computation with cochains difficult.

Motivated by the rational deRham complex and its success in the rational homotopy theory, attempts were made to construct, for any space, a cochain complex with a commutative multiplication whose cohomology would be the cohomology of the space for any coefficient ring. One such construction was given by Cartan and Miller ([**1**], [**4**]). The commutative product in their complex induces the multiplication on cohomology with integer coefficients up to an additional factor, depending on the representing cochains. Another commutative cochain complex was constructed by Cenkl and Porter ("DeRham theorem with cubical forms," Pacific J. Math. **112** (1984), 35–48). In that paper we used a ring system for coefficients to solve the commutative cochain problem (as formulated in §5) for any space in terms of polynomial differential forms on simplices. The solution is the best possible in the sense of Theorem 2. The multiplication induced on cohomology by the commutative product of forms in that complex is exactly the usual one. A construction of the cohomology operations could be also done by defining an appropriate generalized product in that complex.

The deRham complexes of Cartan-Miller and Cenkl-Porter have been used for the computations of the cohomology of the fundamental groups of nilmanifolds. The deRham complex of Cenkl-Porter also turned out to be suitable for the construction of a free model in the tame homotopy theory.

The constructions of the cohomology operations presented in this paper constitute part of the program to study the role of commutative algebras in homotopy theory. In fact it seems to be the first construction of the cohomology operations based on the operations in a commutative cochain algebra.

1. Products of cochains, $\mathscr{P}^i$ and $\mathscr{B}_p$. Let R be a commutative ring with a unit and $\mathscr{S}$ the category of simplicial sets. Denote by $\mathscr{M}_R$, $d\mathscr{M}_R^*$ and $d\mathscr{M}_R^{*,*}$ the category of R-modules, differential graded R-modules and differential bigraded R-modules respectively.

Denote by $A = A^{*,*} = A_R^{*,*}\colon \mathscr{S} \to d\mathscr{M}_R^{*,*}$ the deRham functor of Cartan and Miller ([**1**], [**4**]), and by $\Gamma_R = \Gamma_R(t)$ the divided power algebra over R on a single generator t. $\Gamma_R = \bigoplus_{q\geq 0}\Gamma_q$, where Γ_q denotes the free R-modules on a single generator $\gamma_q = \gamma_q(t)$. Γ_R is given the structure of a commutative ring with unit $\gamma_0 = 1$ by the pairing

$$\gamma\colon \Gamma_p \otimes \Gamma_q \to \Gamma_{p+q}$$

defined by

$$\gamma_p\gamma_q = \frac{(p+q)!}{p!q!}\gamma_{p+q}.$$

In general, the associativity of this pairing gives the map

$$\gamma\colon \Gamma_{q_1} \otimes \cdots \otimes \Gamma_{q_k} \to \Gamma_{q_1+\cdots+q_k},$$
$$\gamma_{q_1}\cdots\gamma_{q_k} = \frac{(q_1+\cdots+q_k)!}{q_1!\cdots q_k!}\gamma_{q_1+\cdots+q_k}.$$

Denote by $C_*\colon \mathscr{S} \to d\mathscr{M}_R^*$ the functor $C_p(X) =$ the free R-module of p-chains generated by the p-simplices. If $p < 0$, $C_p(X) = 0$ for any $X \in \mathscr{S}$. The boundary operator is denoted by $\partial\colon C_p \to C_{p-1}$. Let $C = C^{*,*}\colon \mathscr{S} \to d\mathscr{M}_R^{*,*}$ be the functor defined by $C^{p,q}(X) = C^p(X;\Gamma_q) =$ the normalized p-cochains with coefficients in Γ_q. If $p < 0$ or $q < 0$ then $C^{p,q}(X) = 0$ for any $X \in \mathscr{S}$. The coboundary operator is denoted by $d\colon C^{p,q} \to C^{p+1,q}$. The index, $\mathrm{In}(c)$, of a 0-chain $c \in C_0(X)$ is the sum of its coefficients.

In his original construction of the reduced powers $\mathscr{P}^i$ in **[5]** Steenrod uses a family of chain operations. Let us recall the relevant points of Steenrod's construction.

Let π be the cyclic permutation group of degree p and order p for any integer p and $\mathbb{Z}[\pi]$ be the group ring. Denote by $s\colon \mathbb{Z}[\pi] \to \mathbb{Z}$ the homomorphism $s(\sum x_i g_i) = \sum x_i$, $x_i \in \mathbb{Z}$, $g_i \in \pi$. An infinite sequence $\alpha_1, \alpha_2, \ldots,$ of elements of $\mathbb{Z}[\pi]$ is called the 0-sequence if $s(\alpha_1) = 0$, and $\alpha_{i+1}\alpha_i = 0$ for $i = 1, 2, \ldots$. Denote by T the generator of π,

$$T(a_1, a_2, \ldots, a_p) = (a_p, a_1, a_2, \ldots, a_{p-1}),$$

where $a_1, \ldots, a_p$ is a set of p letters, and by I the identity permutation. Then the elements of $\mathbb{Z}[\pi]$

$$\begin{aligned} \alpha_{2j-1} &= T - I, \\ \alpha_{2j} &= I + T + T^2 + \cdots + T^{p-1}, \end{aligned}$$

$j = 1, 2, \ldots,$ form a 0-sequence. From **[5]**, Lemma 5.5, it follows that there exists a sequence of homomorphisms

$$\begin{aligned} D_i\colon C_n(X) \to {} & [\underbrace{C_*(X) \otimes \cdots \otimes C_*(X)}_{p\text{-times}}]_{n+i} \\ = {} & \bigoplus_{\substack{n_1+\cdots+n_p=n+i \\ n_j \geq 0}} \{C_{n_1}(X) \otimes \cdots \otimes C_{n_p}(X)\}, \end{aligned}$$

$i = 0, 1, 2, \ldots,$ such that:

(i) $\mathrm{In}(D_0 C) = \mathrm{In}(c)$ for any $c \in C_0(X)$,

(ii) $\partial D_i + (-1)^{i+1} D_i \partial - \alpha_i D_{i-1}$, $i = 1, 2, \ldots$.

Definition. The p-fold cup-i product

$$\bigcup_i^p = \phi^p_{\cup_i}\colon C^{n_1,q_1}(X) \otimes \cdots \otimes C^{n_p,q_p}(X) \to C^{n-i,q}(X),$$

$$n = n_1 + \cdots + n_p, \qquad q = q_1 + \cdots + q_p,$$

is the composition of the map

$$\psi_i^p : C^{n_1,q_1}(X) \otimes \cdots \otimes C^{n_p,q_p}(X) \to C^{n-i}(X; \Gamma_{q_1} \otimes \cdots \otimes \Gamma_{q_p}),$$

$$\psi_i^p(c_1 \otimes \cdots \otimes c_p)(\sigma) = (c_1 \otimes \cdots \otimes c_p) D_i(\sigma)$$

with the p-bold product $\gamma\colon \Gamma_{q_1} \otimes \cdots \otimes \Gamma_{q_p} \to \Gamma_q$; i.e.,

$$\phi^p_{\cup_i} = \gamma \circ \psi_i^p.$$

$\cup_i^p = \phi^p_{\cup_i} = 0$ in dimensions $< i$.

From the identity (ii) it follows that

$$d\phi^p_{\cup_i} + (-1)^{i+1}\phi^p_{\cup_i} d = (-1)^{i+1}\phi^p_{\cup_{i-1}} \circ \alpha_i,$$

$i = 1, 2, \ldots, p \geq 2$.

In particular $\phi^2_{\cup_i} = \phi_{\cup_i} = \bigcup_i$ is the cup-i product and $\phi^p_{\cup_0}$ is the p-fold cup product.

Cohomology operations. Here we assume that p is a prime. Let $e_q\colon C^*(X;\Gamma_q) \to C^*(X;\mathbb{Z}_p)$ be the map induced by the coefficient map which sends γ_q to the generator 1 of $\mathbb{Z}_p$. Define

$$\tilde{\phi}^p_i\colon C^{n_1}(X;\mathbb{Z}_p) \otimes \cdots \otimes C^{n_p}(X;\mathbb{Z}_p) \to C^{n-i}(X;\mathbb{Z}_p),$$
$$\frac{(q_1 + \cdots + q_p)!}{q_1! \cdots q_p!}\tilde{\phi}^p_i \circ (c_{q_1} \otimes \cdots \otimes e_{q_p}) = e_q \circ \phi^p_{\cup_i}.$$

Let $\hat{\phi}^p_i$ be the map induced by $\tilde{\phi}^p_i$ on the cohomology restricted to the diagonal. Then the pth cyclic reduced power, $\mathscr{P}^i$, of Steenrod [7] is the operation

$$\mathscr{P}^i\colon H^n(X;\mathbb{Z}_p) \to H^{n+2i(p-1)}(X;\mathbb{Z}_p), \qquad i = 0, 1, \ldots.$$

For any $u \in H^n(X;\mathbb{Z}_p)$

$$\mathscr{P}^i u = (-1)^{mi+nm(n-1)/2}(m!)^{2i-n}\hat{\phi}^p_i(u^p), \qquad m = \tfrac{1}{2}(p-1),$$
$$u^p = u \otimes \cdots \otimes u \ (p\text{-times}).$$

Let $e^j_q\colon C^*(X;\Gamma_q) \to C^*(X;\mathbb{Z}/p^j\mathbb{Z})$ be the map induced by the coefficient map $\Gamma_q \to \mathbb{Z}/p^j\mathbb{Z}$, $\gamma_q \to 1$. Furthermore, let $\Sigma^* = \sum_{k=1}^{p-1} kT^{p-k} \in \mathbb{Z}[\pi]$. Then we define the map

$$\tilde{\phi}^p\colon C^{n_1}(X;\mathbb{Z}/p^k\mathbb{Z}) \otimes \cdots \otimes C^{n_p}(X;\mathbb{Z}/p^k\mathbb{Z}) \to C^{pn}(X;\mathbb{Z}/p^{k+1}\mathbb{Z}),$$

$n = n_1 + \cdots + n_p$, by

$$\frac{(q_1 + \cdots + q_p)!}{q_1! \cdots q_p!}\tilde{\phi}^p(e^k_{q_1} \otimes \cdots \otimes e^k_{q_p})$$
$$= e^{k+1}_q \circ \gamma \circ \{\psi^p_0 + (-1)^{n^2-p}\psi^p_1\, d\Sigma^*\}.$$

If $\hat{\phi}^p$ is the restriction to the diagonal of the map induced by $\tilde{\phi}^p$ on the cohomology then the Pontrjagin-Thomas pth power is the operation

$$\mathscr{B}_p\colon H^{2m}(X;\mathbb{Z}_{p^k}) \to H^{2mp}(X;\mathbb{Z}_{p^{k+1}}), \qquad n = 2m,$$

such that

$$\mathscr{B}_p u = \hat{\phi}^p(u^p), \qquad u^p = u \otimes \cdots \otimes u \ (p\text{-times}).$$

2. Main results. Let $A\colon \mathcal{S} \to d\mathcal{M}_R^{*,*}$ be the deRham functor of Cartan-Miller ([**1**],[**4**]), and let

$$\mu\colon A \to C$$

be the linear transformation induced by the integration of forms over simplices. This map induces isomorphism of modules

$$H^p(A^{*,q}(X)) \xrightarrow{\mu^*} H^p(X;\Gamma_q)$$

which commutes with the wedge and cup products.

There exists a family of cochain operations on the deRham complex $A(X)$ inducing the Steenrod cyclic reduced powers and the Pontrjagin-Thomas pth powers for each prime p. The following properties of the cup and wedge products and of the map μ allow us to define a family of such operations:

(i) The wedge product on $A(X)$ is commutative and associative.

(ii) The cup product on $C(X)$ is associative and commutative in dimension zero.

(iii) The map μ is multiplicative in dimension zero.

(iv) There exists a transformation of functors $\tau\colon C \to A$ which is a chain homotopy equivalence such that $\mu\tau =$ identity.

The proof of (i) and (iii) can be found in [**1**] and [**4**]; (ii) is a standard fact; (iv) is proved by acyclic model argument.

THEOREM 1. *There are natural maps $\phi^p_{\underset{\wedge}{i}} = \overset{i}{\wedge}_p$, called p-fold wedge-i products, such that the diagram*

$$\begin{tikzcd}
A_R^{n_1,q_1}(X) \otimes \cdots \otimes A_R^{n_p,q_p}(X) \arrow[r, "\phi^p_{\underset{\wedge}{i}}"] \arrow[d, "\mu\otimes\cdots\otimes\mu"'] & A_R^{n-i,q}(X) \arrow[d, "\mu"'] \\
C^{n_1}(X;\Gamma_{q_1}) \otimes \cdots \otimes C^{n_p}(X;\Gamma_{q_p}) \arrow[r, "\phi^p_{\cup_i}"] & C^{n-i}(X;\Gamma_q),
\end{tikzcd}$$

$n = n_1+\cdots+n_p$, $q = q_1+\cdots+q_p$, is homotopy commutative. The p-fold cup-i product $\phi^p_{\cup_i}$ is defined via the cochain operations of Steenrod [**5**].

COROLLARY 1. *Let $a_j \in A_{\mathbb{Z}_p}^{n_j,q_j}(X)$, $j = 1,2,\ldots,p$, be cocycles such that $e_{q_i}\mu(a_i) = e_{q_j}\mu(a_j)$ in $C^*(X;\mathbb{Z}_p)$, for all $i,j = 1,2,\ldots,p$. Then, for $i \le p$, $\phi^p_{\underset{\wedge}{i}}(a_1^p)$, $a_1^p = a_1\otimes\cdots\otimes a_1$ (p-times) is a cocycle in $A_{\mathbb{Z}_p}^{np-i,q}(X)$,*

$n = n_1 = \cdots = n_p$, $q = q_1 + \cdots + q_p$, whose class is related to the cyclic reduced power of Steenrod

$$\mathscr{P}^i : H^n(X; \mathbb{Z}_p) \to H^{n+2i(p-1)}(X; \mathbb{Z}_p)$$

by the following formula:

$$\begin{aligned} &\frac{(q_1 + \cdots + q_n)!}{q_1! \cdots q_n!} \mathscr{P}^i [e_{q_1}(\mu(a_1))] \\ &\quad = (-1)^{mi+mn(n-1)/2} (m!)^{2i-n} [\phi^p_{\overset{i}{\wedge}} (a^p_1)], \qquad m = (p-1)/2. \end{aligned}$$

[$*$] *stands for the cohomology class determined by the cocycle* $*$.

COROLLARY 2. *Let $a_j \in A^{n,a_j}_{\mathbb{Z}_{p^k}}(X)$, $j = 1, 2, \ldots, p$ be cocycles such that $e_{q_i}\mu(a_i) = e_{q_j}\mu(a_j)$ in $C^*(X; \mathbb{Z}_{p^k})$ for $i, j = 1, 2, \ldots, p$. Then*

$$\mathscr{B}_p(a_1) = (\phi^p_{\overset{0}{\wedge}} + (-1)^{n^2-p} \phi^p_{\overset{1}{\wedge}} \, d\Sigma^*)(a^p_1)$$

is a cocycle in $A^{np}_{\mathbb{Z}_{p^{k+1}}}(X)$ and

$$\frac{(q_1 + \cdots + q_p)!}{q_1! \cdots q_p!} \mathscr{B}_p[a_{q_1}\mu(a_1)] = [\mathscr{B}_p(a_1)].$$

COROLLARY 3 ([2]). *In particular the Steenrod squares are constructed from the 2-fold wedge-i products, called simply wedge-i products and denoted by $\overset{i}{\wedge}$, as follows:*

Let a_1, a_2 be cocycles in $A^{p,q_1}_{\mathbb{Z}_2}(X)$, $A^{p,q_2}_{\mathbb{Z}_2}(X)$ respectively with $e_{q_1}\mu(a_1) = e_{q_2}\mu(a_2)$ in $C^p(X; \mathbb{Z}_2)$. For $i \le p$, $a_1 \overset{i}{\wedge} a_2$ is a coocyle in $A^{2p-i, q_1+q_2}_{\mathbb{Z}_2}(X)$ whose class is related to the Steenrod squares by the identity

$$[e_{q_1+q_2}\mu(a_1 \overset{i}{\wedge} a_2)] = \frac{(q_1+q_2)!}{(q_1)!(q_2)!} S^{p-1}_q [\mu(a_1)].$$

3. p-fold wedge-i product on forms. In this part we use the following notation

$$\begin{aligned} A_i &= A \otimes \cdots \otimes A \qquad (i\text{-times}), \\ A^{k,q}_i &= \bigoplus_{\substack{k_1 + \cdots + k_i = k \\ q_1 + \cdots q_i = q}} (A^{k_1,q_1} \otimes \cdots \otimes A^{k_i,q_i}), \\ A^k_i &= A^{k,*}_i, \end{aligned}$$

and similarly

$$C_i = C \otimes \cdots \otimes C \qquad (i\text{-times}), \text{ etc.}$$

The functors A, C and the transformation

$$\mu \colon A \to C$$

have the following important properties: The functors $A_i^{k,q}$ are acyclic on models in the dimensions bounded by the filtration q from above ([**2**], [**3**]); $C^{k,q}$ is corepresentable; μ commutes with products in dimension zero. Furthermore, both the wedge product on A and the cup product on C are associative. Therefore the p-fold wedge product

$$\phi_\wedge^p \colon A_p \to A$$

and the p-fold cup product

$$\phi_\cup^p \colon C_p \to C$$

are well-defined.

Hence we can prove, by the method of acyclic models that:

PROPOSITION. *There exist transformations of functors*

$$\mu_{p,1} = \{\mu_{p,1}^k\},$$
$$\mu_{p,1}^k \colon A^{k_1,q_1} \otimes \cdots \otimes A^{k_p,q_p} \to C^{k-1,q},$$
$$k \geq 1, n \geq 2, k = k_1 + \cdots + k_p, q = q_1 + \cdots + q_p,$$

such that

$$d\mu_{p,1}^{k-1} + \mu_{p,1}^k d = \mu\phi_\wedge^p = \phi_\cup^p(\mu^p), \qquad \mu^p = \mu \otimes \cdots \otimes \mu \ (p\text{-}times),$$

where $\mu_{p,1}^0 = 0$.

REMARK. $\mu_{p,1}$ is a generalization of the transformation $\mu_{2,1}$ used by Gugenheim [**3**] to prove that the map of the bar constructions $B(\mu) \colon B(A) \to B(C)$, induced by μ, is a map of coalgebras.

The transformation $\mu_{2,1}$ of [**3**] extends also in another direction.

LEMMA. *Let $\mu \colon A \to C$ be the transformation of functors induced by integration and let*

$$\phi_{\cup_i}^p \colon C^{k_1,q_1} \otimes \cdots \otimes C^{k_p,q_p} \to C^{k-i,q},$$
$$k = k_1 + \cdots + k_p, \quad q = q_1 + \cdots + q_p,$$

be the p-fold cup-i product and let

$$\phi^p_\wedge\colon A^{k_1,q_1}\otimes\cdots\otimes A^{k_p,q_p}\to A^{k,q},$$
$$k=k_1+\cdots+k_p,\quad q=q_1+\cdots+q_p,$$

be the p-fold wedge product.

Then there exist natural transformations

$$\mu_{p,i}=\{\mu^k_{p,i}\},$$
$$\mu^k_{p,i}\colon A^{k_1,q_1}\otimes\cdots\otimes A^{k_p,q_p}\to C^{k-i,q},\qquad i\geq 1,$$

such that

$$d\mu^{k-1}_{p,i}+(-1)^{i+1}\mu^k_{p,i}d=(-1)^{i+1}\mu^{k-1}_{p,i-1}\circ\alpha^{i-1}-\phi^p_{u_{i-1}}(\mu^p),$$
$$\mu^p=\mu\otimes\cdots\otimes\mu\ (p\text{-}times),$$

for $i\geq 2$, where the existence and properties of the transformations $\mu^k_{p,1}$ are given by the Proposition.

Suppose that $\alpha_1,\alpha_2,\ldots$ is the 0-sequence used in the definition of the p-fold cup-i product and that $\tau\colon C^{k,q}\to A^{k,q}$ is a transformation of functors with the properties: $d\tau=\tau d$, $\mu\tau=I$, and that $\tau\mu$ is chain homotopic to the identity ([**2**]).

DEFINITION. The p-fold wedge-i product ($\phi^p_{\underset{\wedge}{i}}$-product) is the map

$$\phi^p_{\underset{\wedge}{i}}\colon A^{k_1,q_1}\otimes\cdots\otimes A^{k_p,q_p}\to A^{k-i,q},$$
$$k=k_1+\cdots+k_p,\quad q=q_1+\cdots+q_p,$$
$$\phi^p_{\underset{\wedge}{i}}=(-1)^i\tau\circ\mu_{p,i}\circ\alpha_i\quad\text{for } i\geq 1,$$
$$\phi^p_{\underset{\wedge}{0}}=\phi^p_\wedge=\text{ the p-fold wedge product.}$$

4. Proofs.

Proof of the Proposition. The statement is proved by the method of acyclic models. Denote by $\hat{A}^k_i$, $\hat{C}_i$ the functors associated with A^k_i, C_i respectively and by $\phi\colon A^k_i\to\hat{A}^k_i$, $\phi\colon C_i\to\hat{C}_i$ the transformations which composed with the canonical maps $\psi\colon\hat{A}^k_i\to A^k_i$, $\psi\colon\hat{C}_i\to C_i$ give the identity. See [**2**] or [**3**] for the definitions.

Suppose that p is an integer $p \geq 2$. Consider the diagram

$$\begin{tikzcd}
& \hat{A}^1_p \arrow[r, "\hat{d}"] \arrow[r, bend right, "\hat{h}"'] & \hat{A}^2_p \arrow[r] & \cdots \arrow[r] & \hat{A}^{k-1}_p \arrow[r, "\hat{d}"] & A^k_p \arrow[l, bend left, "\hat{h}"] \arrow[r] & \cdots \\
A^0_p \arrow[r, "d"] & A^1_p \arrow[u, "\phi"] \arrow[d, "K^1"] \arrow[dl, "\mu^1_{p,1}"'] \arrow[r, "d"] & A^2_p \arrow[u, "\phi"] \arrow[dl, "\mu^2_{p,1}"'] \arrow[r] & \cdots \arrow[r, "d"] & A^{k-1}_p \arrow[u, "\phi"] \arrow[d, "K^{k-1}"] \arrow[r, "d"] & A^k_p \arrow[u, "\phi"] \arrow[dl, "\mu^k_{p,1}"'] \arrow[r] & \cdots \\
C^0 \arrow[r, "d"] \arrow[uur, "\hat{K}^1"] & C^1 \arrow[r, "d"] \arrow[d, "\phi"] & C^2 \arrow[r] & \cdots \arrow[r, "d"] \arrow[uur, "\hat{K}^{k-1}"] & C^{k-1} \arrow[r, "d"] \arrow[d, "\phi"] & C^k \arrow[r] & \cdots \\
& \hat{C}^1 \arrow[r, "d"] & \cdots & {} \arrow[r] & \hat{C}^{k-1} \arrow[r, "d"] & \cdots &
\end{tikzcd}$$

where $\hat{h}$ is the chain homotopy whose existence follows from the acyclicity of A^k_p on models.

Let $K = \mu\phi^p_\wedge - \phi^p_\cup(\mu^p)$, $\mu^p = \mu \otimes \cdots \otimes \mu$ (p-times), $K^k = K|A^k_p$ for $k \geq 1$, $K^0 = 0$. Define $\mu_{p,1} = \{\mu^k_{p,1}\}$ inductively by k. Set $\mu^0_{p,1} = 0$ and define $\mu^1_{p,1}$ so that $\mu^1_{p,1}\, d = 0$ and then extend arbitrarily to A^1_p. Since $K^0 = 0$ we have $\mu^1_{p,2}\, d = K^0$ on the elements of degree zero. Define

$$\mu^2_{p,1} = \psi \hat{K}^1 \hat{h} \phi - d\mu^1_{p,1}.$$

Since $K^1\, d = 0$ on the elements of degree zero we get

$$\mu^2_{p,1}\, d = K^1,$$

or equivalently

$$d\mu^1_{p,1} + \mu^2_{p,1}\, d = \mu\phi^p_\wedge - \phi^p_\cup(\mu^p).$$

Inductively, assume that $\mu^l_{p,1}$ has been defined and satisfies the identities

$$d\mu^{l-1}_{p,1} + \mu^l_{p,1}\, d = \mu\phi^p_\wedge - \phi^p_\cup(\mu^p) \quad \text{for } l \leq k-1.$$

Then for

$$W^l = K^l - d\mu^l_{p,1},$$
$$W^l\, d = 0 \quad \text{for } l \leq k-1.$$

If we define

$$\mu^k_{p,1} = \psi \hat{W}^{k-1} \hat{h} \phi,$$

then

$$\mu^k_{p,1}\, d = W^{k-1},$$

which completes the proof.

Proof of the Lemma. We assume that the formula

$$\begin{aligned}(1)\quad & d\mu^{l-1}_{p,j} + (-1)^{j+1}\mu^l_{p,j}\, d \\ &= (-1)^{j+1}\mu^{l-1}_{p,j-1}\circ\alpha_{j-1} - \phi^p_{\mathrm{U}_{j-1}}(\mu^p),\ \mu^p = \mu\otimes\cdots\otimes\mu \\ &\hspace{20em}(p\text{-times}),\end{aligned}$$

has been proved for $l = 1, 2, \dots$ and $j \le i-1$. We want to show that it is valid also for $j = i$ and for $l = 1, 2, \dots$.

Define $\mu^k_{p,i} = 0$ for $k = 0, 1, \dots, i-1$. Consider the diagram

$$\begin{tikzcd}
\hat{A}^{i-1}_p \arrow[r, "\hat{d}"] \arrow[ddd, bend right=40, "\hat{R}^{i-1}_i"'] & \hat{A}^i_p \arrow[r] \arrow[l, bend left, "\hat{h}"] & \cdots \arrow[r] & \hat{A}^{k-1}_p \arrow[r, "\hat{d}"] \arrow[ddd, bend right=40, "\hat{V}^{k-1}_i"'] & \hat{A}^k_p \\
A^{i-1}_p \arrow[r, "d"] \arrow[u, "\phi"] \arrow[d, "R^{i-1}_i"'] & A^i_p \arrow[r] \arrow[u, "\phi"] \arrow[dl, "\mu^i_{p,i}"'] & \cdots \arrow[r] & A^{k-1}_p \arrow[r, "d"] \arrow[u, "\phi"] \arrow[d, "V^{k-1}_i"'] & A^k_p \arrow[u, "\phi"] \arrow[dl, "\mu^k_{p,i}"'] \\
C^0 \arrow[r, "d"] \arrow[d, shift right, "\phi"'] & C^1 \arrow[r] & \cdots \arrow[r] & C^{k-i} \arrow[r, "d"] \arrow[d, shift right, "\phi"'] & C^{k-i+1} \\
\hat{C}^0 \arrow[r, "\hat{d}"] \arrow[u, shift right, "\psi"'] & {} & \cdots \arrow[r] & \hat{C}^{k-i} \arrow[r, "\hat{d}"] \arrow[u, shift right, "\psi"'] & {}
\end{tikzcd},$$

where R^{i-1}_i is defined by

$$R^{i-1}_i = (-1)^{i+1}\mu^{i-1}_{p,i-1}\circ\alpha_{i-1} - \phi^p_{\mathrm{U}_{i-1}}(\mu^p).$$

Since $\alpha_{i-2}\circ\alpha_{i-1} = 0$, $\phi^p_{\mathrm{U}_{i-1}}\, d = (-1)^{i+1}\, d\phi^p_{\mathrm{U}_{i-1}} + \phi^p_{\mathrm{U}_{i-2}}\alpha_{i-1}$ and $\phi^p_{\mathrm{U}_{i-1}}(\mu^p)$ is zero on A^{i-2}_p we get

$$R^{i-1}_i\, d = 0.$$

Define $\mu^i_{p,i}$ by

$$\mu^i_{p,i} = (-1)^{i+1}\psi\hat{R}^{i-1}_i\hat{h}\phi.$$

Then

$$(-1)^{i+1}\mu^i_{p,i}\, d = R^{i-1}_i.$$

Now we assume that the formula (1) is proved for $l < k$ and $j \leq i$. Set

$$V_i^{k-1} = (-1)^{i+1}\mu_{p,i-1}^{k-1} \circ \alpha_{i-1} - \phi_{\cup_{i-1}}^p(\mu^p) - d\mu_{p,i}^{k-1}.$$

Then

$$\begin{aligned} V_i^{k-1}\, d &= (-1)^{i+1}\mu_{p,i-1}^{k-1}\, d\alpha_{i-1} - d\mu_{p,i-1}^{k-2}\alpha_{i-1} \\ &\quad + (-1)^{i+1}\, d\phi_{\cup_{i-1}}^p(\mu^p) - \phi_{\cup_{i-1}}^p\, d(\mu^p) \\ &= (-1)^{i+1}\mu_{p,i-2}^{k-2}\alpha_{i-2} \circ \alpha_{i-1} = 0. \end{aligned}$$

Hence we define $\mu_{p,i}^k$ by

$$\mu_{p,2}^k = (-1)^{i+1}\psi \hat{V}_i^{k-1}\hat{h}\phi.$$

From here we conclude that

$$(-1)^{i+1}\mu_{p,i}^k\, d = V_i^{k-1},$$

which completes the proof.

5. Nonexistence of commutative cochains over the integers. Let R be a commutative ring.

We say that

$$B^*\colon \mathscr{S} \to d\mathscr{M}_R^*$$

solves the commutative cochain problem over R if:

(i) B^* is acyclic on models with unit

$$\eta\colon R \to B^*;$$

(ii) there is a commutative transformation

$$\Lambda\colon B^* \otimes B^* \to B^*$$

with the diagram

$$\begin{tikzcd} B^* \otimes B^* \arrow[r, "\Lambda"] & B^* \\ R \otimes R \arrow[u, "\eta\otimes\eta"] \arrow[r] & R \arrow[u, "\eta"'] \end{tikzcd}$$

commutative;

(iii) there is a transformation $\mu\colon B^* \to C^*$ inducing an isomorphism

$$\mu\colon H^p(B(X)) \to H^p(X; R)$$

for all p and all simplicial sets X.

THEOREM 2. *Let R be a commutative ring. If the commutative cochain problem over R has a solution then for each element $a \in R$, the element a^2 in R is divisible by two.*

COROLLARY. *There is no solution of the commutative cochain problem over the integers.*

These results were obtained jointly with R. Porter. They were announced in [2].

Proof of the Theorem. Assume that $B^*: \mathscr{S} \to d\mathscr{M}_R^*$ is acyclic on models with units $\eta_B: R \to B^0$, where R is a commutative ring. Further assume that there is a commutative transformation $\Lambda: B^* \otimes B^* \to B^*$ with the diagram in (ii) commutative. Denote by $C^*: \mathscr{S} \to d\mathscr{M}_R^*$ the normalized cochains with R-coefficients.

From the acyclic model theorem it follows that there is a transformation $\mu: B^* \to C^*$, unique up to homotopy, with

$$\begin{tikzcd} & B^* \arrow[dd, "\mu"] \\ R \arrow[ur, "\eta_B"] \arrow[dr, "\eta_C"'] & \\ & C^* \end{tikzcd}$$

commutative. We assume that μ^* in (iii) is an isomorphism for all p and all simplicial sets X.

Since B^* is acyclic on models and $\wedge$ is commutative, the arguments used in the construction of higher homotopies applied to B^* imply the existence of maps

$$\mu_i: (B \otimes B)^n \to C^{n-i}$$

such that

$$d\mu_1 + \mu_1 d = \mu\phi_\wedge - \phi_\cup(\mu \otimes \mu)$$

and

$$d\mu_i + (-1)^{i+1}\mu_i d = \mu_{i-1}(T + (-1)^{i+1}I) - \phi_{\cup_{i-1}}(\mu \otimes \mu) \quad \text{for } i \geq 2.$$

In particular there are maps μ_1 and μ_2 with

$$(*) \qquad d\mu_2 - \mu_2 d = \mu_1(T - I) - \phi_{\cup_1}(\mu \otimes \mu).$$

Let $a \in R$. We want to show that a^2 is divisible by 2. Choose a cocycle $\alpha \in B^1(S^1)$ such that the element $[\mu(\alpha)] \in H^1(S^1; R) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, R)$ is the homomorphism which sends $1 \in \mathbb{Z}$ to the element a in R. (Here S^1 denotes the circle.)

From $(*)$ we have

$$d\mu_2(\alpha\otimes\alpha) = -2\mu_1(\alpha\otimes\alpha) - \mu(\alpha)\cup_1 \mu(\alpha);$$

hence $\mu(\alpha)\cup_1 \mu(\alpha)$ represents the zero class in $H^1(S^1; R/2R) \simeq R/2R$.

On the other hand a direct computation, using Steenrod definition of $\cup_1$, shows that the class $[\mu(\alpha)\cup_1 \mu(\alpha)]$ in $H^1(S^1; R/2R) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, R/2R)$ is the element which sends $1 \in \mathbb{Z}$ to the element a^2 in $R/2R$. Hence $a^2 = 0$ in $R/2R$.

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