

HOMOTOPY COMPLEX PROJECTIVE SPACES WITH DIVISIBLE SPLITTING INVARIANTS

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A PL homotopy complex projective 4-space with integral splitting invariant σ_2 is smoothable if and only if $\sigma_2 \equiv 0$ or $6 \pmod{14}$. A smooth homotopy complex projective 4-space with $\sigma_2 \neq 0$ does not admit a smooth Z_3 action with a codimension 2 fixed submanifold.

1. Introduction. Let X^m be a closed, simply connected PL m -manifold, where $m \geq 6$. The manifold X^m is said to be smooth if its PL structure is compatible with a differentiable structure and almost smooth if it is smooth in the complement of a point. The purpose of this paper is to study these ideas in the special case where X^m is a PL homotopy complex projective n -space, that is, $m = 2n$, $n \geq 3$, and X^{2n} is homotopy equivalent to CP^n , complex projective n -space. The PL homeomorphism type of these manifolds can be described precisely in terms of numerical invariants. Let P_{2k} denote the simply-connected surgery obstruction group for closed $2k$ -manifolds, $k \geq 2$, that is, $P_{2k} = \mathbf{Z}$ if k is even and $P_{2k} = \mathbf{Z}_2$ if k is odd, [8]. A theorem of Sullivan, [8], asserts that the PL homeomorphism type of X^{2n} is determined by an $(n-2)$ -tuple $(\sigma_2, \sigma_3, \dots, \sigma_{n-1})$ in $\prod_{k=2}^{n-1} P_{2k}$. Sullivan suggested the problem of characterizing the smoothability or almost smoothability of these manifolds in terms of these splitting invariants. Every PL homotopy complex projective 3-space is smoothable, [10]. We obtain a complete answer in the case $n = 4$ and in the cases $n = 5$ or 6 , a complete answer modulo the value of the mod 2 invariant σ_3 .

THEOREM 1.1. *Every PL homotopy complex projective 4-space is almost smooth. A PL homotopy complex projective 4-space is smooth if and only if $\sigma_2 \equiv 0$ or $6 \pmod{14}$.*

THEOREM 1.2. *A PL homotopy complex projective 5-space is almost smooth if and only if $\sigma_4 - 4\sigma_2^2 + 10\sigma_2 \equiv 0 \pmod{28}$. There exists a smooth homotopy complex projective 5-space with splitting invariants σ_2 and σ_4 if and only if σ_2 is even and $\sigma_4 - 4\sigma_2^2 + 10\sigma_2 \equiv 0 \pmod{28}$.*

THEOREM 1.3. *There exists an almost smooth homotopy complex projective 6-space with splitting invariants σ_2 and σ_4 if and only if σ_2 is even and $\sigma_4 - 4\sigma_2^2 + 10\sigma_2 \equiv 0 \pmod{28}$. There exists a smooth homotopy complex projective 6-space with splitting invariants σ_2 and σ_4 if and only if σ_2 is even, $\sigma_4 - 4\sigma_2^2 + 10\sigma_2 \equiv 0 \pmod{28}$, and*

$$-8\sigma_4\sigma_2 + \frac{64}{3}\sigma_2^3 - 4\sigma_4 + 20\sigma_2^2 - \frac{421}{3}\sigma_2 \equiv 0 \pmod{496}.$$

We investigate certain finite group actions on homotopy complex projective spaces. Let X^{2n} be a $2n$ -manifold having the same integral cohomology ring as CP^n and let p be an odd prime. Suppose \mathbf{Z}_p acts on X^{2n} in such a way that the fixed point set of the action, $F(X^{2n})$, contains a codimension 2 submanifold F_0 . It follows that $F(X^{2n}) = F_0 \amalg \{\text{point}\}$, ([3], p. 487). In [3], Dovermann has made a detailed study of the algebraic properties of such actions and has shown that they are often algebraically standard in a certain precise sense (see Definition 5.1). In particular, if X^{2n} admits an algebraically standard action, then the Pontrjagin class of X^{2n} must correspond to the Pontrjagin class of CP^n . Dovermann, ([3], p. 490), raised the question of whether every \mathbf{Z}_3 action is algebraically standard in the case $n = 4$, and we answer this question in the affirmative if X^8 is a PL homotopy complex projective 4-space.

THEOREM 1.4. *If X^{2n} is a PL homotopy complex projective n -space with $n = 4$ or 5 , then every locally linear PL \mathbf{Z}_3 action on X^{2n} with a codimension 2 fixed submanifold is algebraically standard.*

This theorem settles the single open case of Dovermann's Theorem A(ii), ([3], p. 488), and so the two results together imply that every smooth \mathbf{Z}_p action of this type on a smooth homotopy complex projective 4-space is algebraically standard. If $\sigma_2 \neq 0$, the Pontrjagin class of X^8 is not standard (Theorem 3.1), and so it follows that the exotic smooth homotopy complex projective 4-spaces of Theorem 1.1 with $\sigma_2 \neq 0$ do not admit smooth \mathbf{Z}_p actions with a codimension 2 fixed submanifold. Theorem 1.4 implies that Dovermann's Proposition 0.3 ([3], p. 489), is sharp at the prime $p = 3$, and, if $d(p)$ ($d'(p)$) is the smallest integer n such that there exists a smooth (locally linear PL) non-standard action of this type on some smooth (PL) homotopy complex projective n -space, then Theorem 1.4, together with Dovermann's Proposition 0.3

and Theorem 0.3 ([3], p. 489), imply that $d'(3) = 6$, and $6 \leq d(3) \leq 15$.

The paper is organized as follows. In §2, we introduce notation and outline our general method of attack. Sections 3 and 4 contain the technical information necessary to implement the program outlined in §2. Section 3 contains a formula which relates the Pontrjagin class of a PL homotopy complex projective space to its integral splitting invariants. Section 4 contains a review of the results of Adams and Walker, [1], and Brumfiel, [2], on fibre homotopically trivial vector bundles over CP^n and the proofs of Theorems 1.1, 1.2, and 1.3. Section 5 contains the proof of Theorem 1.4.

2. Surgery problems and smoothability. Let $\text{hPL}(CP^n)$ and $\text{hS}(CP^n)$ be the sets of equivalence classes of homotopy triangulations and homotopy smoothings of CP^n , respectively ([8], p. 19). If $\alpha_n: \text{hS}(CP^n) \rightarrow \text{hPL}(CP^n)$ is the natural map and if $h: X^{2n} \rightarrow CP^n$ represents an element of $\text{hPL}(CP^n)$, then X^{2n} is smooth if and only if the equivalence class of h is in the image of α_n . If $\text{hS}(CP_0^n)$ is the set of equivalence classes of homotopy almost smoothings of CP^n ([2], p. 381), there is a natural map $\beta_n: \text{hS}(CP_0^n) \rightarrow \text{hPL}(CP^n)$ ([8], p. 46) defined by the coning construction, and a general position argument at the singular point shows that if $h: X^{2n} \rightarrow CP^n$ represents an element of $\text{hPL}(CP^n)$, then X^{2n} is almost smooth if, and only if, the equivalence class of h is in the image of β_n .

We study the mappings α_n and β_n by studying the homomorphism $\gamma_n: [CP^n, F/O] \rightarrow [CP^n; F/PL]$, which sends a smooth surgery problem with range CP^n into the corresponding PL surgery problem. These maps are in a commutative diagram with exact rows ([8], p. 45).

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{hPL}(CP^n) & \longrightarrow & [CP^n; F/PL] & \longrightarrow & P_{2n} \longrightarrow 0 \\
 (2.1) & & \alpha_n \uparrow & & \gamma_n \uparrow & & 1 \parallel \\
 0 & \longrightarrow & \text{hS}(CP^n) & \longrightarrow & [CP^n; F/O] & \longrightarrow & P_{2n} \longrightarrow 0
 \end{array}$$

The maps β_n and γ_{n-1} are related because there is a bijection $\text{hPL}(X^m) \cong [X^m - \text{point}; F/PL]$ ([8], p. 21), and, if X^m is almost smooth, there is a bijection $\text{hS}(X_0^m) \cong [X_0^m; F/O]$ ([2], p. 382). If

$X^m = \mathbf{C}P^n$, we have the commutative diagram below:

$$(2.2) \quad \begin{array}{ccc} \text{hPL}(\mathbf{C}P^n) & \xrightarrow{\cong} & [\mathbf{C}P^{n-1}; F/\text{PL}] \\ \beta_n \uparrow & & \gamma_{n-1} \uparrow \\ \text{hS}(\mathbf{C}P_0^n) & \xrightarrow{\cong} & [\mathbf{C}P^{n-1}; F/O] \end{array}$$

The group $[\mathbf{C}P^n; F/\text{PL}]$ can be described in terms of the splitting invariants. If $f: X^{2n} \rightarrow \mathbf{C}P^n$ is a PL surgery problem and, if $k < n$, assume that f is in general position with respect $\mathbf{C}P^k \subset \mathbf{C}P^n$. If $Y^{2k} = f^{-1}(\mathbf{C}P^k)$, then the splitting invariant σ_k in P_{2k} is defined to be the simply connected surgery obstruction to making $f|Y^{2k}$ normally cobordant to a homotopy equivalence. Sullivan’s theorem asserts that $[\mathbf{C}P^n; F/\text{PL}] \cong \prod_{k=2}^n P_{2k}$, that is, a PL surgery problem is determined by an $(n-1)$ -tuple $(\sigma_2, \sigma_3, \dots, \sigma_n)$. The map $[\mathbf{C}P^n; F/\text{PL}] \rightarrow P_{2n}$ sends a surgery problem into its obstruction, σ_n .

The homomorphism γ_n is related to the homomorphism $\xi_n: [\mathbf{C}P^n; F/O] \rightarrow k(n)$, where $k(n)$ is the kernel of the J -homomorphism, $J: \tilde{K}O(\mathbf{C}P^n) \rightarrow J(\mathbf{C}P^n)$. This map is defined by Spivak uniqueness ([9], p. 105), which states that if $f: X^{2n} \rightarrow \mathbf{C}P^n$ is a smooth surgery problem, then, in $\tilde{K}O(X^{2n})$, the tangent bundle of X^{2n} , $\tau(X^{2n})$, satisfies the equation $\tau(X^{2n}) = f^*\tau(\mathbf{C}P^n) + f^*\xi$, where ξ is in $k(n)$. An element ξ in $k(n)$ determines a smooth surgery problem, but not uniquely. In §4, we review the results of Adams and Walker, [1], and Brumfiel, [2], concerning $k(n)$, make some inferences concerning the image of γ_n (and the images of α_n and β_n via (2.1) and (2.2)), and prove Theorems 1.1, 1.2, and 1.3. In the next section, §3, we establish a formula relating the rational Pontrjagin class of a PL homotopy complex projective n -space and its integral splitting invariants. We will use this formula in the proof of Theorem 1.4 in §5.

3. The Pontrjagin class of a homotopy complex projective space. If $n \geq 3$, let $h: X^{2n} \rightarrow \mathbf{C}P^n$ be a homotopy equivalence with splitting invariants $(\sigma_2, \sigma_3, \dots, \sigma_{n-1})$. The theorem below concerns the relationship between the Pontrjagin class $P_i(X^{2n})$ in $H^{4i}(X^{2n}; \mathbf{Q})$ and the integral splitting invariants. Let y in $H^2(\mathbf{C}P^n; \mathbf{Q})$ be the generator of the cohomology algebra and let $\mathbf{Q}[x_1, x_2, \dots, x_i]$, $i \geq 0$, be the rational polynomial algebra on abstract generators, x_k , $1 \leq k \leq i$, with the convention that if $i = 0$, the symbol denotes \mathbf{Q} .

THEOREM 3.1. *There exist universal polynomials a_i in*

$$\mathbf{Q}[x_1, x_2, \dots, x_i], \quad i \geq 0,$$

such that $a_0 = 1$, $a_i(0, 0, \dots, 0) = 0$, and $i \geq 1$, and, if $1 \leq i \leq [n/2]$, then

$$(3.2) \quad P_i(X^{2n}) = \sum_{k=0}^i \binom{n+1-k}{i-k} a_k(\sigma_2, \sigma_4, \dots, \sigma_{2k}) h^* y^{2i}.$$

LEMMA 3.3. Suppose that for fixed i , $1 < i \leq [n/2]$, h is in general position with respect to $\mathbf{C}P^{2i}$, and that formula (3.2) is valid for $P_j(X^{2n})$ if $1 \leq j \leq i-1$. Then $P_j(Y^{4i}) = \rho_j(h|Y^{4i})^* y^{2j}$, $1 \leq j \leq i-1$, where

$$(3.4) \quad \rho_j = \sum_{k=0}^j \binom{2i+1-k}{j-k} a_k(\sigma_2, \sigma_4, \dots, \sigma_{2k}).$$

Proof of Lemma 3.3. Let ζ denote the canonical complex line bundle over $\mathbf{C}P^m$, $\mathbf{C}P^m \subseteq \mathbf{C}P^n$, if $m \leq n$. Let $\varepsilon_0 \zeta$ be the real 2-plane bundle underlying ζ , where ε_0 is the operator which assigns to a complex bundle the underlying real bundle ([5], p. 191). Since h is in general position with respect to $\mathbf{C}P^{2i}$, there is a direct sum decomposition of block bundles,

$$(3.5) \quad \tau(X^{2n})|Y^{4i} = \tau(Y^{4i}) \oplus (n-2i)h^*(\varepsilon_0 \zeta),$$

where $\tau(X^{2n})$ and $\tau(Y^{4i})$ denote the tangent block bundles of X^{2n} and Y^{4i} , respectively.

Formula (3.5) and an inductive argument can be used to establish formula (3.4). The initial step, $j = 1$: the hypothesis concerning formula (3.2) and formula (3.5) imply that

$$(n+1+a_1(\sigma_2))(h|Y^{4i})^* y^2 = P_1(Y^{4i}) + (n-2i)(h|Y^{4i})^* y^2$$

since the Whitney sum formula holds over \mathbf{Q} . Therefore, $P_1(Y^{4i}) = \rho_1(h|Y^{4i})^* y^2$, where $\rho_1 = 2i+1+a_1(\sigma_2)$. The inductive step: assume that (3.4) holds for all j such that $1 \leq j < j_0 \leq i-1$. To establish (3.4) at level j_0 , note that the hypothesis about (3.2), (3.5), and the inductive hypothesis imply that $P_{j_0}(Y^{4i}) = \rho_{j_0}(h|Y^{4i})^* y^{2j_0}$, where

$$(3.6) \quad \rho_{j_0} = \sum_{k=0}^{j_0} \binom{n+1-k}{j_0-k} a_k - \sum_{k=0}^{j_0-1} \binom{n-2i}{j_0-k} \rho_k.$$

If c_u is the coefficient of a_u in (3.6), $1 \leq u \leq j_0$, it follows that (3.4) at levels below j_0 implies that

$$c_u = \binom{2i+1-u}{j_0-u}.$$

To see this, note that the identity

$$\binom{a+b}{c} = \sum_{i=0}^c \binom{a}{i} \binom{b}{c-i}$$

can be used to show that the coefficient of a_u in the second summation of (3.6) is

$$\binom{n+1-u}{j_0-u} - \binom{2i+1-u}{j_0-u}, \quad 0 \leq u \leq j_0 - 1.$$

Proof of Theorem 3.1. The proof is by induction and Lemma 3.3 will be used in the inductive step. Before starting the arguments, we remark that if one wishes to compute $P_i(X^{2n})$ as a rational multiple of h^*y^{2i} and h is in general position with respect to CP^{2i} , it is enough to compute $P_i(X^{2n})|Y^{4i}$ as a rational multiple of the cofundamental class of Y^{4i} , since $h|Y^{4i}$ has degree $+1$.

The initial step, $i = 1$: put h into general position with respect to $CP^2 \subset CP^n$ and note that it follows from (3.5) at level $i = 1$ that

$$P_1(X^{2n})|Y^4 = P_1(Y^4) + (n-2)(h|Y^4)^*y^2.$$

Since $8\sigma_2 = L_1(P_1(Y^4))[Y^4] - 1$, where L_1 is the Hirzebruch L -class ([9], p. 162, [7], p. 225), and $(h|Y^4)^*y^2$ is the cofundamental class of Y^4 , it is clear that (3.2) holds at level $i = 1$ with $a_1(\sigma_2) = 24\sigma_2$. In particular, $P_1(X^{2n})$ is integral and determined by σ_2 . The inductive step: assume that (3.2) holds for $P_j(X^{2n})$ for all j such that $1 \leq j \leq i - 1$ and that h is in general position with respect to CP^{2i} . Invoking Lemma 3.3 and using the definition of σ_{2i} , we conclude that $P_i(Y^{4i}) = \varphi_i(h|Y^{4i})^*y^{2i}$, where φ_i is the rational polynomial expression in σ_{2k} , $1 \leq k \leq i$, determined by (3.4) together with

$$(3.7) \quad 8\sigma_{2i} = L_i(P_1(Y^{4i}), P_2(Y^{4i}), \dots, P_i(Y^{4i}))[Y^{4i}] - 1,$$

where L_i is the Hirzebruch L -class. The polynomial a_i is defined by

$$(3.8) \quad a_i = \varphi_i - \sum_{k=0}^{i-1} \binom{2i+1-k}{i-k} a_k.$$

Since h is in general position with respect to CP^{2i} , we use (3.5) and conclude that

$$(3.9) \quad P_i(X^{2n}) = \left[\varphi_i + \sum_{k=0}^{i-1} \binom{n-2i}{i-k} \rho_k \right] h^*y^{2i}.$$

It follows from (3.4) and the identity

$$\binom{a+b}{c} = \sum_{i=0}^c \binom{a}{i} \binom{b}{c-i}$$

that if c_u is the coefficient of a_u in the summation in (3.9), $1 \leq u \leq i-1$, then

$$c_u = \binom{n+1-u}{i-u} - \binom{2i+1-u}{i-u},$$

and so (3.2) holds a level i in view of (3.8).

We complete the proof of Theorem 3.1 by noting that $a_i(0, 0, \dots, 0) = 0$, $i \geq 1$, follows since $a_1(0) = 0$ and, if $\sigma_{2j} = 0$, $1 \leq j \leq i$, then

$$\rho_j = \binom{2i+1}{j}, \quad 1 \leq j \leq i-1,$$

by formula (3.4), if it is assumed, inductively, that $a_j(0, 0, \dots, 0) = 0$, $1 \leq j \leq i-1$. This implies that $\rho_i = \binom{2i+1}{i}$ by naturality of the degree +1 map $h|Y^{4i}$ and the fact $L_i[\mathbb{C}P^{2i}] = 1$. Thus $a_i(0, 0, \dots, 0) = 0$ follows from (3.8).

We remark that if n is even in Theorem 3.1, $n = 2i$, then $\sigma_{2i} = 0$ in formula (3.2). It follows easily from this remark and Theorem 3.1 that the Pontrjagin class of X^{2n} corresponds to the Pontrjagin class of CP^n , that is, $P_*(X^{2n}) = (1 + h^*y^2)^{n+1}$ if and only if all the integral splitting invariants of X^{2n} vanish.

We conclude this section with some examples of the polynomials

$$a_i: a_1(\sigma_2) = 24\sigma_2, \quad a_2(\sigma_2, \sigma_4) = (360\sigma_4 + 576\sigma_2^2 - 432\sigma_2)/7,$$

and

$$a_3(\sigma_2, \sigma_4, \sigma_6) = (3780/31)\sigma_6 + (56160\sigma_2\sigma_4 - 6912\sigma_2^3 - 39420\sigma_4 - 57888\sigma_2^2 + 13284\sigma_2)/217.$$

4. The kernel of the J -homomorphism. The reduced K -theory of CP^n , $\tilde{K}O(CP^n)$, is a free abelian group with generators ω^k , $k \geq 1$, where $\omega = \varepsilon_0(\zeta - 1)$, ζ is the canonical complex line bundle, and ε_0 is the realification operator ([1], Theorem 2.2). There are relations which depend on $n \pmod{4}$: $\tilde{K}O(CP^{2i})$ is free abelian with generators ω^k , $1 \leq k \leq i$, $\tilde{K}O(CP^{4i+3})$ is free abelian with generators ω^k , $1 \leq k \leq 2i+1$, and $\tilde{K}O(CP^{4i+1})$ is free abelian with generators ω^k , $1 \leq k \leq 2i+1$, modulo the relation $2\omega^{2i+1} = 0$.

Let $k(n) = \text{kernel}\{J: \tilde{K}O(CP^n) \rightarrow J(CP^n)\}$. Brumfiel ([2], pp. 400–401) has computed $k(n)$ for $n \leq 7$, and, if $u \in [CP^n; F/O]$, $n \leq 7$,

expressed the integral splitting invariants of u in terms of $\xi_n(u) \in k(n)$. We present Brumfiel's results with three very minor modifications. First, the integral splitting invariants as defined by formula (3.7) differ in sign from the splitting invariants in Brumfiel's paper. Secondly, note that it is not hard to see that if $\xi_n(u) \equiv 24a\omega \pmod{\text{higher powers of } \omega}$, then $a = \sigma_2$. To see this, let $f: X^{2n} \rightarrow \mathbb{C}P^n$ represent u , put f into general position with respect to $\mathbb{C}P^2$ (where higher powers of ω are zero), and then compute σ_2 using the Hirzebruch index formula. Finally, we have done a little arithmetic with Brumfiel's formulas ([2], Lemma I.5), so that the formula for a given integral splitting invariant reflects the formulas for integral invariants of smaller subscript.

LEMMA 4.1 (*Brumfiel, [2], Lemmas I.4 and I.5*). *Let $\psi_1 = 24\omega + 98\omega^2 + 111\omega^3$, $\psi_2 = 240\omega^2 + 380\omega^3$, and $\psi_3 = 504\omega^3$. If $n \leq 7$, then $k(n)$ is generated by ψ_i restricted to $\mathbb{C}P^n$, $i = 1, 2$, and 3, and if $u \in [\mathbb{C}P^n; F/O]$ with $\xi_n(u) = a\psi_1 + b\psi_2 + c\psi_3$, then*

$$(4.2) \quad \sigma_2 = a,$$

$$(4.3) \quad \sigma_4 = 4\sigma_2^2 - 10\sigma_2 - 28b,$$

$$(4.4) \quad \sigma_6 = 8\sigma_4\sigma_2 - \frac{64}{3}\sigma_2^3 + 4\sigma_4 - 20\sigma_2^2 + \frac{421}{3}\sigma_2 + 496(b + c).$$

COROLLARY 4.5. *Suppose that $n \leq 7$ and that $v \in [\mathbb{C}P^n; F/PL]$ is determined by splitting invariants $(\sigma_2, \sigma_3, \dots, \sigma_n)$. There exists an element $v' \in [\mathbb{C}P^n; F/PL]$ such that v and v' have the same integral splitting invariants and $v' \in \text{image } \gamma_n$ if and only if the integral splitting invariants of v satisfy the congruences,*

$$(4.6) \quad \sigma_4 - 4\sigma_2^2 + 10\sigma_2 \equiv 0 \pmod{28},$$

$$(4.7) \quad \sigma_6 - 8\sigma_4\sigma_2 + \frac{64}{3}\sigma_2^3 - 4\sigma_4 + 20\sigma_2^2 - \frac{421}{3}\sigma_2 \equiv 0 \pmod{496}.$$

Proof. Suppose that v and v' have the same integral splitting invariants and $v' = \gamma_n(u)$. If $\xi_n(u) = a\psi_1 + b\psi_2 + c\psi_3$, then formulas (4.2), (4.3), and (4.4) relate the integral splitting invariants of v to a , b , and c , and so the congruences (4.6) and (4.7) hold. Conversely, suppose that the integral splitting invariants of v satisfy the equations $\sigma_4 - 4\sigma_2^2 + 10\sigma_2 = 28d$ and

$$\sigma_6 - 8\sigma_4\sigma_2 + \frac{64}{3}\sigma_2^3 - 4\sigma_4 + 20\sigma_2^2 - \frac{421}{3}\sigma_2 = 496e.$$

Let $u \in [CP^n; F/O]$ be an element such that $\xi_n(u) = \sigma_2\psi_1 - d\psi_2 + (e + d)\psi_3$. It follows from Lemma 4.1 that the integral splitting invariants of u are σ_2, σ_4 , and σ_6 and so we may set $v' = \gamma_n(u)$.

Proof of Theorem 1.1. The space PL/O is 6-connected ([7], p. 250), and so the tangent bundle of X^8 -point lifts in the fibration $PL/O \rightarrow BSO \rightarrow BSPL$. Therefore, every PL homotopy complex projective 4-space is almost smooth. To verify the second statement in the theorem, note that it follows from Corollary 4.5 that if $v = (\sigma_2, \sigma_3, \sigma_4) \in [CP^4; F/PL]$, then there exists $\sigma'_3 \in \mathbf{Z}_2$ such that $v' = (\sigma_2, \sigma'_3, \sigma_4) \in \text{image } \gamma_4$ if and only if $\sigma_4 - 4\sigma_2^2 + 10\sigma_2 \equiv 0 \pmod{28}$. Sullivan has remarked ([8], p. 45) that $(0, 1, 0) \in [CP^4; F/PL]$ is in the image of γ_4 . It follows from the properties of the universal F/PL characteristic classes ([6], p. 80) that if $(\sigma_2, \sigma_3, \sigma_4)$ and $(\sigma'_2, \sigma'_3, \sigma'_4)$ are two elements in the group $[CP^4; F/PL]$, then the group theoretic sum of the two elements is given by

$$(\sigma_2, \sigma_3, \sigma_4) + (\sigma'_2, \sigma'_3, \sigma'_4) = (\sigma_2 + \sigma'_2, \sigma_3 + \sigma'_3, \sigma_4 + \sigma'_4 + 8\sigma_2\sigma'_2).$$

In particular,

$$(\sigma_2, \sigma_3, \sigma_4) + (0, 1, 0) = (\sigma_2, \sigma_3 + 1, \sigma_4).$$

It follows that if $(\sigma_2, \sigma'_3, \sigma_4) \in \text{image } \gamma_4$, then $(\sigma_2, \sigma'_3 + 1, \sigma_4) \in \text{image } \gamma_4$, and hence $(\sigma_2, \sigma_3, \sigma_4) \in \text{image } \gamma_4$ if and only if $\sigma_4 - 4\sigma_2^2 + 10\sigma_2 \equiv 0 \pmod{28}$. It follows that $(\sigma_2, \sigma_3, 0) \in \text{image } \alpha_4$ if and only if $\sigma_2 \equiv 0$ or $6 \pmod{14}$.

Proof of Theorem 1.2. The argument in the proof of Theorem 1.1 shows that if $(\sigma_2, \sigma_3, \sigma_4, 0) \in \text{hPL}(CP^5)$, then $(\sigma_2, \sigma_3, \sigma_4, 0) \in \text{image } \gamma_4$ if and only if $\sigma_4 - 4\sigma_2^2 + 10\sigma_2 \equiv 0 \pmod{28}$. This establishes the first assertion. To establish the second assertion, note that it follows from Corollary 4.5 and the fact that $\sigma_2 \pmod{2}$ determines the Arf invariant in the case $n = 5$ ([2], p. 401, [4], p. 16) that there exists an element in $[CP^5; F/O]$ with splitting invariants σ_2 and σ_4 , and with Arf invariant zero (i.e., $\sigma_5 = 0$), if, and only if, σ_2 is even and $\sigma_4 - 4\sigma_2^2 + 10\sigma_2 \equiv 0 \pmod{28}$. In this situation, we have no control over $\sigma_3 \in \mathbf{Z}_2$ because it is not clear that $(0, 1, 0, 0) \in \text{image } \gamma_5$.

Proof of Theorem 1.3. Theorem 1.3 follows from Corollary 4.5 in the same fashion as Theorems 1.1 and 1.2. We remark that smoothability or almost smoothability forces $\sigma_5 = 0$ ([8], p. 45), and hence σ_2 is even

in the statements in the theorem. We are unable to decide the value of σ_3 in this case just as in the case $n = 5$.

5. Actions of \mathbf{Z}_3 with codimension two fixed point component. Let X^{2n} be a closed $2n$ -manifold with cohomology ring $\mathbf{Z}[y]/(y^{n+1})$, where y is a generator of $H^2(X^{2n}; \mathbf{Z})$. Let p be an odd prime and suppose that \mathbf{Z}_p acts locally smoothly on X^{2n} in such a way that the fixed point set, $F(X^{2n})$, contains a codimension 2 submanifold, F_0 . It follows that $F(X^{2n}) = F_0 \amalg$ point and, if $j: F_0 \subset X^{2n}$ and $y_0 = j^*y$, then $H^*(F_0; \mathbf{Z})$ contains the polynomial ring $\mathbf{Z}[y_0]/(y_0^n)$ ([3], Corollary 0.1). We use the notation τ for tangent bundle, ν for normal bundle, c for Chern class, and P_* for total Pontrjagin class. For an abelian group H , let $H_f = H/\text{torsion}$ denote its free part, and if $x \in H$, its image in H_f is denoted by $f(x)$.

DEFINITION 5.1 (Dovermann, [3]). An action on X^{2n} with codimension 2 fixed point component is algebraically standard if

- (i) $P_*(X^{2n}) = (1 + y^2)^{n+1}$,
- (ii) $H_f^*(F_0; \mathbf{Z}) = \mathbf{Z}[f(y_0)]/(f(y_0)^n)$,
- (iii) $f(P_*(F_0)) = (1 + f(y_0)^2)^n$,
- (iv) $f(c(\nu(F_0, X^{2n}))) = 1 \pm f(y_0)$,
- (v) as a real representation, $\tau_{pt}(X^{2n}) = n\nu_x(F_0, X^{2n})$, where pt is the isolated fixed point and $x \in F_0$.

THEOREM 5.2. *If X^{2n} is a PL homotopy complex projective n -space with $n = 4$ or 5 , then every locally linear \mathbf{Z}_3 action on X^{2n} with a codimension 2 fixed submanifold is algebraically standard.*

Proof. If $n = 4$, F_0 is a 6-manifold and the normal block bundle, ν , is a real 2-plane bundle ([3], p. 492). If $j_i: H^k(F_0; \mathbf{Z}) \rightarrow H^{k+2}(X^{2n}; \mathbf{Z})$ is the Gysin map, then $c_1(\nu) = j^*j_i(1_{F_0})$ ([3], p. 503), and so there exists an integer, r , such that $f(c_1(\nu)) = rf(y_0)$. The block bundle equation $\tau(X^8)|_{F_0} = \tau(F_0) \oplus \nu$ together with formula (3.2) in the case $n = 4$, $i = 1$ (recall that $a_1(\sigma_2) = 24\sigma_2$) yield the fact that $P_1(F_0)$ is integral and $f(P_1(F_0)) + f(P_1(\nu)) = (5 + 24\sigma_2)f(y_0)^2$. Since ν is a real, orientable 2-plane bundle, $P_1(\nu) = c_1(\nu)^2$ ([7], p. 179), and so $f(P_1(\nu)) = r^2f(y_0)^2$. It follows that $f(P_1(F_0)) = sf(y_0)^2$, where

$$(5.3) \quad r^2 + s = 5 + 24\sigma_2.$$

Since the action is tame ([3], Theorem 1.1), we may apply the Atiyah-Singer G -Signature Theorem. If the above formulas for the characteristic classes and the formula $f(y_0)^3[F_0] = r$ ([3], Lemma 3.1) are

put into the signature formula for $G = \mathbf{Z}_3$, ([3], p. 504), the resulting equation is

$$(5.4) \quad r^2(2r^2 - s) = \pm 2.$$

Equations (5.3) and (5.4) imply that $r = \pm 1$, $s = 4$, and $\sigma_2 = 0$. Conditions (iii) and (iv) follow immediately and formula (3.2) implies (i) since $\sigma_2 = 0$. Since $r = \pm 1$, $f(c_1(\nu))^3 = f(y_0)^3$ generates $H^6(F_0; \mathbf{Z})$ and so $f(y_0)$ generates $H^*(F_0; \mathbf{Z})/\text{torsion}$ ([3], Lemma 3.1), and this is (ii). We remark that (v) holds automatically in the case $p = 3$ ([3], formula (1.4), p. 492).

The case $n = 5$ is similar. The block bundle equation $\tau(X^{10})|_{F_0} = \tau(F_0) \oplus \nu$ plus the equations $P_1(\nu) = c_1(\nu)^2$, $P_i(\nu) = 0$, $i > 2$ ([7], p. 174), together with formula (3.2) in the case $n = 5$, $i = 1, 2$ (see the remarks at the end of the proof of Theorem 3.1 for $a_1(\sigma_2)$ and $a_2(\sigma_2, \sigma_4)$) imply that there are integers r, s and t such that $f(c_1(\nu)) = rf(y_0)$, $f(P_1(F_0)) = sf(y_0)^2$, and $f(7P_2(F_0)) = tf(y_0)^4$. These integers satisfy the equations

$$(5.5) \quad r^2 + s = 6 + 24\sigma_2,$$

$$(5.6) \quad t + 7r^2s = 105 + 360\sigma_4 + 576\sigma_2^2 + 408\sigma_2.$$

If the above formulas for the characteristic classes and the formula $f(y_0)^4[F_0] = r$ ([3], Lemma 3.1), are put into the signature formula for $G = \mathbf{Z}_3$ ([3], p. 505), the resulting equation is

$$(5.7) \quad r(60r^4 - 20r^2s + t - s^2) = \pm 5.$$

Formulas (5.5), (5.6), and (5.7) plus some elementary number theory show that $r = \pm 5$ is impossible. Therefore, $r = \pm 1$ and hence (iv) holds. We conclude also that (ii) holds, that is, $H_f^*(F_0; \mathbf{Z}) = \mathbf{Z}[f(y_0)]/(f(y_0))^5$ ([3], Lemma 3.1). In particular, the index of the 8-manifold F_0 is 1. It follows from the Hirzebruch Index Theorem and formulas (5.5) and (5.6) that $(t - s^2)/45 = 1 + 8\sigma_4 = 1$, that is, $\sigma_4 = 0$. If the information that $r = \pm 1$ and $\sigma_4 = 0$ is put into (5.7), the only possible conclusion is $\sigma_2 = 0$. Condition (i) now follows easily from formula (3.2). Since $s = 5$ and $\sigma_2 = \sigma_4 = 0$, $t = 70$ and (iii) holds. As we remarked above, (v) holds automatically in the case $p = 3$.

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