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ADELES AND THE SPECTRUM OF COMPACT NILMANIFOLDS

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Let G be a nilpotent Lie group and Γ a discrete cocompact subgroup of G. A basic problem in harmonic analysis is to determine the structure of $L^2(G/\Gamma)$. We apply adelic techniques to determine the decomposition of $L^2(G/\Gamma)$. To do so, we first develop a "rational" Kirillov theory for the adele group $G_{\mathbb{A}}$. Once this is done, the decomposition and multiplicity formulas follow from elementary considerations.

Let G be a locally compact group and $\Gamma \subset G$ a closed subgroup such that G/Γ is compact. For $x \in G$, we can define $\lambda(x)$, a unitary operator on $L^2(G/\Gamma)$ by $(\lambda(x)f)(y) = f(x^{-1}y)$. The representation $x \to \lambda(x)$ is called the quasi-regular representation of G. A fundamental problem in representation theory is to decompose λ into irreducible representations. A theorem of Fell **[F]** says that λ will be discretely decomposable and each irreducible will occur with finite multiplicity. Thus we can write:

$$L^2(G/\Gamma) = \sum_{\pi \in \mathrm{Sp}(\Gamma)} H(\pi) \otimes V_\pi \tag{0.1}$$

where $0 < \dim(V_\pi) < \infty$ for each $\pi \in \mathrm{Sp}(\Gamma)$. The first task is to determine $\mathrm{Sp}(\Gamma)$ as a subset of $\hat{G}$ and then to determine $\dim(V_\pi)$, the multiplicity with which π occurs in $L^2(G/\Gamma)$. When G is a solvable Lie group, one may use the reduction procedures of Howe **[H2]**, Auslander-Brezin **[B]**, and Fox **[Fo]**. In principle, then, the multiplicities can be computed, but the answers seem unsatisfying, and much work needs to be done in this area. For nilpotent Lie groups, the problem was first addressed by Moore **[M]**, and later solved by Howe **[H1]** and Richardson **[R]**. The answer found by Howe and Richardson generalized the classical Frobenius reciprocity theorem for compact groups and provided a useful computational answer to the problem. Some time later, Corwin and Greenleaf **[C-G]** gave a beautiful solution, expressed in terms of canonical data attached to Γ and the representation π, when Γ satisfied some mild side conditions.

In spite of the above work, and perhaps because of the success of the above authors, an important question remained unanswered. The question was first formulated for unipotent groups by Moore [**M**], and its importance was later emphasized by Howe [**H1**]. To describe the problem we need to recall the notion of the commensurability class of a discrete group Γ. We say that Γ_1 is commensurable with Γ_2 if $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2. We will write $\{\Gamma\}$ for the class determined by Γ. The approach taken by the above authors is to pick a Γ out of the class $\{\Gamma\}$ and then determine the decomposition of $L^2(G/\Gamma)$. The approach we shall take is to formulate and solve the problem for the class $\{\Gamma\}$; from that solution the decomposition of a particular $L^2(G/\Gamma)$ will follow as a simple corollary.

For nilpotent Lie groups, Moore formulated the problem and proved the fundamental theorem necessary to complete the solution. The class $\{\Gamma\}$ uniquely determines and is determined by, the structure of an affine algebraic group defined over $\mathbb{Q}$ on G. Let $G_\mathbb{Q}$ be the $\mathbb{Q}$ rational points of G. If $\mathbb{Q}_\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$ and $G_\mathbb{A}$ the $\mathbb{Q}_\mathbb{A}$ points of G then we can embed $G_\mathbb{Q}$ diagonally into $G_\mathbb{A}$. The resulting quotient space, $G_\mathbb{A}/G_\mathbb{Q}$, is compact. Thus, $L^2(G_\mathbb{A}/G_\mathbb{Q})$ is discretely decomposable. Moore has determined $\mathrm{Sp}(G_\mathbb{Q})$ and showed that the multiplicity of each irreducible representation in $\mathrm{Sp}(G_\mathbb{Q})$ is one. Once this decomposition is available, the local information follows readily, in principle. The representations of $G_\mathbb{A}$ can be constructed as infinite tensor products, and it is as a consequence of this construction that the local information is obtainable. However, to use the adele machinery effectively, we need a description of the representations occurring in $L^2(G_\mathbb{A}/G_\mathbb{Q})$ that does not involve the infinite tensor product construction.

If G is a nilpotent Lie group, models for the representations of $G_\mathbb{A}$ that occur in $L^2(G_\mathbb{A}/G_\mathbb{Q})$ can be constructed by applying the adele functor systematically, thereby obtaining a "rational" Kirillov theory for these groups. Once this is done, the multiplicity formulas of Howe-Richardson and Corwin-Greenleaf follow from elementary considerations. Moore's multiplicity one theorem also follows from a straightforward computation once the "rational" Kirillov theory has been constructed.

In §1 we describe the rational Kirillov theory for $G_\mathbb{A}$ and give a simple proof of Moore's multiplicity one theorem. In §2, we use the multiplicity one result to obtain the Howe-Richardson multiplicity formula and outline how the Corwin-Greenleaf results fit into the adele picture. We then obtain a sharp upper bound on the rate of growth of multiplicities (in terms of Plancherel density) for the generic representations of G_∞ occurring in $L^2(G_\infty/\Gamma)$. It should be noted that by using Pukanszky's parameterization of all orbits occurring in $\mathfrak{g}^*$, it would be possible to produce a polynomial bound for all representations occurring in $L^2(G_\infty/\Gamma)$. (Consult **[C-G2]** for a very accessible description and application of this parameterization.)

What is most striking about the adelic approach to the multiplicity problem is the simplicity of the constructions and the ease with which the various multiplicity formulas follow from the adelic information. In particular, Moore's infinity tensor product construction is not needed.

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1. Rational Kirillov theory. Our basic references will be Weil's two books, **[W-1]** and **[W-2]**; from **[W-2]**, we only need the first two chapters, which discuss the ideas of adelic geometry. Also, the reader might consult Tamagawa's brief survey of adeles ([T], p. 113). We now recall some definitions and notation. Given a prime p of $\mathbb{Q}$, we can define a valuation $|\ |_p$ on $\mathbb{Q}$. If $x = p^n \frac{a}{b}$ with $b \in \mathbb{Z}$ and a, b relatively prime to p, then $|x|_p = p^{-n}$. If we complete $\mathbb{Q}$ with respect to $|\ |_p$, we obtain $\mathbb{Q}_p$, a locally compact field. If $|\ |$ is the usual absolute value on $\mathbb{Q}$, then we will call $|\ |$ the valuation at the infinite prime and write $|\ |_\infty$. In this case, $\mathbb{Q}_\infty = \mathbb{R}$, the real numbers. For p, a finite prime $\mathbb{Q}_p$ will denote the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$. Thus, $\mathbb{Z}_p$ is a compact open subgroup of $\mathbb{Q}_p$, which can also be described as $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$. Given $x \in \mathbb{Q}$, we have the product formula:

$$\prod_p |x|_p = 1. \tag{1.1}$$

(Here, the product is over all primes, including the infinite one.)

We now recall the construction of the "ring of adeles" of $\mathbb{Q}$ (see **[W-1]**, p. 59). Let P be a finite set of primes that includes the infinite prime, and let $\mathbb{Q}(P) = \prod_{p\in P} \mathbb{Q}_P \times \prod_{p\notin P} \mathbb{Z}_p$ with the product topology. If $P_1 \subseteq P_2$, then $\mathbb{Q}(P_1)$ is an open subring of $\mathbb{Q}(P_2)$. We set $\mathbb{Q}_A = \bigcup_P \mathbb{Q}(P)$ with the inductive limit topology and call $\mathbb{Q}_A$ the adele ring of $\mathbb{Q}$. Given $x \in \mathbb{Q}_A$ we will write $x = (x_\infty, x_{p_1}, \dots)$ if we need to exhibit x in terms of coordinates. Given $q \in \mathbb{Q}$, we can write $q = p_1^{n_1} \cdots p_e^{n_e}$.

Thus, if $p \neq p_1, \dots, p_e$ we have $q \in \mathbb{Z}_p$. It follows that we can consider $\mathbb{Q}$ to be diagonally embedded in $\mathbb{Q}_A$. With this embedding, $\mathbb{Q}$ is a discrete cocompact subgroup of $\mathbb{Q}_A$. We can also view $\mathbb{Q}_p$ as being embedded in $\mathbb{Q}_A$ by $\mathbb{Q}_p \simeq \{x \in \mathbb{Q}_A | x = (0, 0, \dots, x_p, 0, 0, \dots)\}$. The statement that strong approximation holds for $\mathbb{Q}$ is simply that $\mathbb{Q} + \mathbb{Q}_p$ is dense in $\mathbb{Q}_A$ for any p (**[W-1]**, p. 70). If we define $\mathbb{Q}_f = \{x \in \mathbb{Q}_A | x_\infty = 0\}$, then $\mathbb{Q}_A \simeq \mathbb{Q}_\infty \times \mathbb{Q}_f$ as topological spaces. If ψ is a character of $\mathbb{Q}_A$, then $\psi = \prod_p \psi_p$, with ψ_p a character of $\mathbb{Q}_p$; and for all but a finite number of p's (to be abbreviated a.e. p), $\psi_p|_{\mathbb{Z}_p} \equiv 1$. A character ψ of $\mathbb{Q}_A$ is called basic if ψ is non-trivial but $\psi|_{\mathbb{Q}} \equiv 1$. Note that strong approximation says that a basic character is determined by its restriction to $\mathbb{Q}_p$ (which is ψ_p) for any prime p. (See **[W-1]**, part 1, for more information on the above constructions.)

In **[W-2]**, Chapter 1, Weil constructs the adele functor from the category of algebraic varieties defined over $\mathbb{Q}$ to the category of topological spaces. The construction is very similar to the construction of $\mathbb{Q}_A$ (we refer the reader to **[W-2]** for details and also to **[T]**). For G, a unipotent algebraic group defined over $\mathbb{Q}$, it will be useful to phrase the construction of G_A in terms of restricted direct products. Moore (**[M]**, pp. 163–64) gives a detailed account of this construction, a review of which follows.

Let $G_{\mathbb{Q}}$ be the $\mathbb{Q}$ rational points of G and $\mathfrak{g}_{\mathbb{Q}}$ the Lie algebra of $G_{\mathbb{Q}}$. Let $X_1, \dots, X_n$ be a basis of $\mathfrak{g}_{\mathbb{Q}}$ over $\mathbb{Q}$, and let L be the $\mathbb{Z}$-span of $X_1, \dots, X_n$; so L is a lattice in $\mathfrak{g}_{\mathbb{Q}}$. By the Campbell-Baker-Hausdorff formula, there exists a polynomial $P\colon \mathfrak{g}_{\mathbb{Q}} \times \mathfrak{g}_{\mathbb{Q}} \to \mathfrak{g}_{\mathbb{Q}}$ such that $\exp(X) \cdot \exp(X) = \exp(P(X, Y))$. It follows that for $G_{\mathbb{Q}_p}$ multiplication is given by the same polynomial, but extended by continuity to $\mathfrak{g}_{\mathbb{Q}_p} \times \mathfrak{g}_{\mathbb{Q}_p}$. Let L_p be the closure of L in $\mathfrak{g}_{\mathbb{Q}_p}$. If the coefficients of P are in $\mathbb{Z}_p$ (which will happen for all but a finite number of primes), then $P\colon L_p \times L_p \to L_p$ and $\exp(L_p) = K_p$ is a compact open subgroup of $G_{\mathbb{Q}_p}$. If we change the basis, then only a finite number of the K_p's will change; thus, we say that the K_p's are defined for a.e. p. Now we can form the restricted direct product of the $G_{\mathbb{Q}_p}$ with respect to the K_p's. Let S be any finite set of primes containing those primes for which K_p is not defined. Let $G(S) = \prod_{p \in S} G_{\mathbb{Q}_p} \times \prod_{p \notin S} K_p$ with the product topology. Then $G_A = \bigcup_S G(S)$ with the inductive limit topology.

Let V be an algebraic variety defined over $\mathbb{Q}$; then V_A will be the corresponding adele space attached to V. Given two varieties, V_1 and V_2, with a morphism $F\colon V_1 \to V_2$ defined over $\mathbb{Q}$, there exists a canonical map $F_A\colon (V_1)_A \to (V_2)_A$ such that the following diagram commutes:

$$(1.2)\qquad \begin{array}{ccc} (V_1)_{\mathbf{Q}} & \hookrightarrow & (V_1)_A \\ F\downarrow & & \downarrow F_A \\ (V_2)_{\mathbf{Q}} & \hookrightarrow & (V_2)_A \end{array}.$$

As for $\mathbf{Q}_A$, one can define V_p, V_f, F_p, and F_f so that $V_A = V_\infty \times V_f$ and $F_A = F_\infty \times F_j$.

We now want to develop what might be called a rational Kirillov theory for unipotent algebraic groups defined over $\mathbf{Q}$. In what follows, we will use no subscript to denote the $\mathbf{Q}$ rational object and will employ the subscript p or A to denote the local or adele object. We fix the basic character ψ of $\mathbf{Q}_A$ such that $\psi_\infty(x) = \exp(2\pi i x)$.

Let G be a unipotent algebraic group defined over $\mathbf{Q}$ with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$ and choose $l \in \mathfrak{g}^*$. A subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is said to be a polarization for l if $l([\mathfrak{h}\mathfrak{h}]) = 0$ and if $\mathfrak{h}$ is a maximal subspace with respect to this property. We let $H = \exp(\mathfrak{h})$ be the subgroup of G with Lie algebra $\mathfrak{h}$ and H_A represent the corresponding subgroup of G_A. We can define a character of H_A (trivial on $H \subseteq H_A$) by the usual formula; for $h \in H_A$:

$$(1.3)\qquad \chi_l(h) = \psi(l(\log_A(h))).$$

DEFINITION 1.1. Let $\pi(l,H) = \operatorname{ind}_{H_A}^{G_A}(\chi_l)$.

LEMMA 1.1. *Let $l \in \mathfrak{g}^*$; then there exists a polarization $\mathfrak{h}$ for l such that $\pi(l,H)$ is irreducible. Furthermore, up to equivalence, $\pi(l,H)$ is independent of H and depends only on the* $\mathrm{Ad}^*(G)$ *orbit of l in $\mathfrak{g}^*$. If $\mathcal{O} = \mathrm{Ad}^*(G)\cdot l \subseteq \mathfrak{g}^*$, then we will write $\pi_{\mathcal{O}}$ for a representative of the class determined by $\pi(l,H)$.*

Proof. The proof is identical to the real case, so we will omit it. (However, one might consult **[K1]**, p. 71, and **[Wa]**, p. 326.) □

As in the real case, the representations $\pi_{\mathcal{O}}$, $\mathcal{O} \in \mathfrak{g}^*/\mathrm{Ad}^*(G)$, will all be CCR and they will possess an appropriate orbital integral formula for the character. Before describing this, we need to recall the notion of a standard function (cf. **[W-1]**, Ch. VII, §2). Let $C_c^\infty(G_p)$ be the C^∞-function with compact support if p is the infinite prime and the locally constant functions of compact support if p is a finite prime. Let θ_p be the characteristic function of K_p for those p's for which K_p is defined (so θ_p is defined for a.e. p).

DEFINITION 1.2. A function ϕ on G_A is called standard if

$$\phi((x_\infty, x_{p_1}, \dots)) = \prod_p \phi_p(x_p)$$

with $\phi_p \in C_c^\infty(G_p)$ and $\phi_p = \theta_p$ for a.e. p. We denote by $C_c^\infty(G_A)$ the $*$-algebra (under convolution) of standard functions.

LEMMA 1.2. *Let $\phi \in C_c^\infty(G_A)$; then ϕ is a finite sum of elements of the form $\phi_1 * \phi_2$ with $\phi_i \in C_c^\infty(G_A)$.*

Proof. Write $\phi = \phi_\infty \times \phi_f$ with $\phi_f = \prod_{p\neq\infty} \phi_p$. By the Dixmier-Malliavin factorization theorem **[D-M]**, $\phi_\infty = \sum_{i=1}^k \alpha_i * \beta_i$ with α_i, $\beta_i \in C_c^\infty(G_\infty)$. Now consider $\phi_f = \prod_{p\in\infty} \phi_p$. For each p, ϕ_p is locally constant and thus invariant under an open compact subgroup of G_p, say S_p. Now for a.e. p, $\phi_p = \theta_p$, so we can take $S_p = K_p$ for a.e. p. If we set $S_f = \prod_p S_p$, then ϕ_f is invariant under S_f and the characteristic function of S_f is in $C_c^\infty(G_f)$. Now we have;

$$(1.4)\qquad \begin{aligned} &\phi_f * (1/\operatorname{vol}(S_f)\chi_{S_f})(g) \\ &= \int_{G_f} \phi_f(gx)(1/\operatorname{vol}(S_f)\chi_{S_p}(x))\,dx = \phi_f(g). \end{aligned}$$

If we set $\gamma = 1/\operatorname{vol}(S_f)\cdot\chi_{S_f}$, then $\gamma \in C_c^\infty(G_f)$ and we have:

$$(1.5)\qquad \begin{aligned} \phi = \phi_\infty \times \phi_f &= \left(\sum \alpha_i * \beta_i\right) \times (\phi_f * \gamma) \\ &= \sum (\alpha_i \times \phi_f) * (\beta_i \times \gamma). \end{aligned} \qquad \square$$

Suppose π is an irreducible representation of G_A; then it is a consequence of Lemma 1.2 that $\pi(\phi)$ will be a trace class operator for all $\phi \in C_c^\infty(G_A)$ if $\pi(\phi)$ is a Hilbert Schmidt operator for all $\phi \in C_c^\infty(G_A)$. To show $\pi(\phi)$ is a Hilbert Schmidt operator, it suffices to show that $\pi(\phi * \phi^*)$ is trace class. The point here is that we are reduced to deciding if a positive operator is trace class, which simplifies the computations. If $\pi = \operatorname{ind}_{H_A}^{G_A}(\chi_l)$, then we can compute $\operatorname{tr}(\pi(\phi * \phi^*))$ using the standard computation for induced representations. Pukanszky's algorithm goes over almost word for word in the adele setting. After introducing some notation, we will describe how Pukanszky's algorithm can be adapted to this situation.

DEFINITION 1.3. Let V be a non-singular algebraic variety defined over $\mathbb{Q}$. An algebraic differential form ω on V is called a gauge form if $\deg(\omega) = \dim(V)$ and ω is everywhere holomorphic and non-zero (for a definition of holomorphic in the algebraic setting, see **[L]**, pg. 189).

If G is an affine algebraic group defined over $\mathbb{Q}$, then we have the usual definitions of right and left invariant differential forms on G. In particular, we say G is unimodular if every left invariant gauge form is also right invariant. If $E \subseteq G$ is a subgroup of G with E and G both unimodular, then there exists a gauge form on G/E which is G invariant ([**W-2**], p. 24). A standard induction on dimension of n shows that unipotent groups are unimodular.

Given a gauge form ω on a non-singular algebraic variety V, Weil ([**W-2**], p. 21) shows how to construct measures ω_p on the $\mathbb{Q}_p$ points of V for every p and how to match up these measures to form measure on V_A. The measure on V_A is essentially a product measure, which we denote by $(\omega)_A$. If ω is a left invariant gauge form on G, then $(\omega)_A$ is a left invariant measure on G_A. It follows that G_A is unimodular if G is unimodular. For unipotent groups, the basic case to consider is $G = \mathbb{Q}$. For each finite prime p, there is a unique Haar measure $(dX)_p$ on $\mathbb{Q}_p$ such that the volume of $\mathbb{Z}_p$ is one. We take on $\mathbb{R} = \mathbb{Q}_\infty$, the usual Lebesgue measure. For the gauge form $\omega = dX$ on $\mathbb{Q}$, the corresponding measure on $\mathbb{Q}_A$ is $(\omega)_A = \prod_p (dX)_p$.

If G is a unipotent algebraic group and $E \subseteq G$ a Zariski closed subgroup with Lie algebra $\mathfrak{g}$, then we can construct a global cross-section for G/E by means of a coexponential basis. Thus, we can find a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ such that $X_1, \ldots, X_s$ is a coexponential basis for $\mathfrak{e}$ in g and $X_{s+1}, \ldots, X_n$ span $\mathfrak{e}$ over $\mathbb{Q}$. Consequently, we have the isomorphism $F\colon \mathbb{Q}^s \times \mathfrak{e} \to \mathfrak{g}$ defined by:

$$F((t_1, \ldots, t_s), X) = \exp(l_1 X_1) \cdots \exp(t_s X_s) \cdot \exp(X). \tag{1.6}$$

Let G be a unipotent algebraic group with Lie algebra $\mathfrak{g}$. The exponential map $\exp\colon \mathfrak{g} \to G$ is an isomorphism of varieties. As in the real case (see, for instance, [**Wa**], p. 315), the pullback of a G invariant gauge form on G to $\mathfrak{g}$ via exp is a translation invariant gauge form on $\mathfrak{g}$. We fix a gauge form on $\mathfrak{g}$, say dX, and let $(dX)_A$ be the corresponding measure on $\mathfrak{g}_A$. If we put counting measure on the discrete cocompact subgroup $\mathfrak{g} \subseteq \mathfrak{g}_A$, then $(dX)_A$ is characterized by the fact that the volume of $\mathfrak{g}_A/\mathfrak{g}$ is one. The measure $(dX)_A$ is called the Tamagawa measure on $\mathfrak{g}_A$.

DEFINITION 1.4. If $\phi \in C_c^\infty(G_A)$, we define the Fourier transform of ϕ as a function on $\mathfrak{g}_A^*$ by:

$$\hat{\phi}(l) = \int_{\mathfrak{g}_A} \phi(X)\psi(l(X))(dX)_A. \tag{1.7}$$

(Recall that ψ is our fixed basic character of $\mathbb{Q}_A$.)

Now $\mathfrak{g}_A^*$ also has a Tamagawa measure $(dl)_A$. It is easy to see that, with respect to this measure, Fourier inversion holds ([W-1], p. 113):

$$\phi(0) = \int_{(\mathfrak{g}_A)^*} \hat{\phi}(l)(dl)_A. \tag{1.8}$$

Let $\pi = \operatorname{ind}_{H_A}^{G_A}(\chi_l)$. We now recall Pukanszky's algorithm for computing $\operatorname{tr}(\pi(\phi))$ as an orbital integral. Let $\mathfrak{h}$ be the Lie algebra of H and $X_1, \ldots, X_n$ basis of $\mathfrak{g}$ such that $X_1, \ldots, X_s$ is coexponential for $\mathfrak{h}$ in $\mathfrak{g}$ and $X_{s+1}, \ldots, X_n$ spans $\mathfrak{h}$. Set

$$\Gamma = \{\gamma = \exp(t_1 X_1) \cdots \exp(t_s X_s) | (t_1, \ldots, t_s) \in \mathbb{Q}^s\}$$

and let $\mathfrak{h}^\perp = \{\lambda \in \mathfrak{g}^* | \lambda(X) = 0 \ \forall X \in \mathfrak{h}\}$. If $\mathcal{O} = \operatorname{Ad}^*(G)(l) \subseteq \mathfrak{g}^*$, then $l + \mathfrak{h}^\perp \subseteq \mathcal{O}$. As in the real case, the map $L \colon \Gamma \times \mathfrak{h}^\perp \to \mathcal{O}$ defined by

$$L(\gamma, \lambda) = \operatorname{Ad}^*(\gamma)(l + \lambda) \tag{1.9}$$

is an isomorphism of varieties. If we identify Γ with $\mathbb{Q}^s$ by

$$(t_1, \ldots, t_s) \to \exp(t_1 X_1) \cdots \exp(t_s X_s),$$

then γ has the gauge from $d\gamma = dt_1 \wedge \cdots \wedge dt_s$. Let $X_1^*, \ldots, X_s^*$ be dual to $X_1, \ldots, X_s$; then if $\mathfrak{h}^\perp$, $\lambda = \sum_{i=1}^s u_i x_i^*$ with $u_i \in \mathbb{Q}$. This determines a gauge form $d\lambda = du_1 \wedge \cdots \wedge du_s$ on $l + \mathfrak{h}^\perp$.

Lemma 1.3. *Let $l \in \mathfrak{g}^*$, $\mathfrak{h}$ a polarization for l and $\pi = \operatorname{ind}_{H_A}^{G_A}(\chi_l)$. If $\phi \in C_c^\infty(G_A)$ is of the form $\phi = \alpha * \alpha^*$, then*

$$\operatorname{tr}(\pi(\phi)) = \int_{(\Gamma)_A} \int_{(\mathfrak{h}^\perp)_A} \hat{\phi}(\operatorname{Ad}^*(\gamma)(l + \lambda))(d\lambda)_A (d\gamma)_A. \tag{1.10}$$

Proof. See [Pu2], p. 267. □

Definition 1.5. Let $\omega_{\Gamma,H}$ be the gauge form on $\mathcal{O}$ such that $\omega_{\Gamma,H} = L^*(d\lambda \wedge d\gamma)$.

Equation (1.10) can be written

$$\operatorname{tr}(\pi(\phi)) = \int_{\mathcal{O}_A} \hat{\phi}(\omega_{\Gamma,H})_A, \tag{1.11}$$

where both sides are finite or both sides are infinite. To show that the right-hand side of (1.11) is finite, we need the following lemma:

Lemma 1.4. *Let $l \in \mathfrak{g}^*$ and $\mathcal{O}_A = \operatorname{Ad}^*(G_A)(l) = (\operatorname{Ad}^*(G)(l))_A$; then $\mathcal{O}_A$ is closed in $\mathfrak{g}_A^*$.*

Proof. Let $x_n = g_n \cdot l$ be a sequence in $\mathcal{O}_A$ such that x_n converges to $y \in \mathfrak{g}_A^*$. Let $\overline{g}_n$ be the image of g_n in G_A/G. Since G_A/G is compact

([**M-T**], p. 462), we can assume that $\overline{g}_n$ converges to $\overline{g}$ in G_A/G. Since the quotient map is open, we can find a sequence $z_n \in G_A$ such that z_n converges to z and $g_n = z_n \cdot w_n$ with $w_n \in G$. Now $\mathrm{Ad}^*(g_n) \cdot l$ converges and z_n is convergent, so $\mathrm{Ad}^*(z_n^{-1} g_n) \cdot l = \mathrm{Ad}^*(w_n) \cdot l$ is convergent in $\mathfrak{g}_A^*$. But $\mathrm{Ad}^*(w_n) l \in \mathfrak{g}_{\mathbb{Q}}^* \subseteq \mathfrak{g}_A^*$ and $\mathfrak{g}_{\mathbb{Q}}^*$ is discrete; thus, if $n > N$, we have that $\mathrm{Ad}^*(w_n) l$ is the constant sequence. Therefore, $\mathrm{Ad}^*(g_n)(l)$ converges to $\mathrm{Ad}^*(z) \cdot \mathrm{Ad}^*(w_N)(l) \in \mathcal{O}_A$. □

Let $\mathcal{O}_A = \mathcal{O}_\infty \times \mathcal{O}_f$; then $\mathcal{O}_\infty \subseteq \mathfrak{g}_\infty^*$ and $\mathcal{O}_f \subseteq \mathfrak{g}_f^*$ is also closed in $\mathfrak{g}_f^*$. Since $\hat{\phi} = \hat{\phi}_\infty \times \hat{\phi}_f$, we have:

$$(1.12)\quad \int_{\mathcal{O}_A} \hat{\phi} \cdot (\omega_{\Gamma,H})_A = \left(\int_{\mathcal{O}_\infty} \hat{\phi}_\infty \cdot (\omega_{\Gamma,H})_\infty \right) \times \left(\int_{\mathcal{O}_j} \hat{\phi}_f \cdot (\omega_{\Gamma,H})_f \right).$$

It follows from [**Pu**], p. 267, that $\int_{\mathcal{O}_A} \hat{\phi}_\infty \cdot (\omega_{\Gamma,H})_\infty$ is finite. As for the second factor, $\hat{\phi}_f$ has compact support in $\mathfrak{g}_f^*$. Since $\mathcal{O}_f$ is closed, $\mathrm{supp}(\hat{\phi}_f) \cap \mathcal{O}_f$ is compact in $\mathcal{O}_f$; from this, it follows that $\int_{\mathcal{O}_f} \hat{\phi}_f \cdot (\omega_{\Gamma,H})_f$ is finite. We can summarize this with:

COROLLARY 1.1. *If* $\pi = \mathrm{ind}_{H_A}^{G_A}(\chi_l)$ *and* $\phi \in C_c^\infty(G_A)$*, then* $\pi(\phi)$ *is a trace class operator.*

A priori, the gauge form $\omega_{\Gamma,H}$ depends on choices of a polarization $\mathfrak{h}$ for l and a coexponential basis for $\mathfrak{h}$. However, given two G-invariant gauge forms, ω_1 and ω_2, on $\mathcal{O}$, there exists a constant $c \in \mathbb{Q}$ such that $\omega_1 = c\omega_2$. It is now easy to check that $(\omega_1)_A = |c|_A \cdot (\omega_2)_A$ where $|c|_A =$ idéle norm of c ([**W-2**], p. 22). Since $c \in \mathbb{Q}$, $|c|_A = 1$ and $(\omega_1)_A = (\omega_2)_A$. We can now summarize our constructions with the following theorem.

THEOREM 1.1. *Let* G *be a unipotent algebraic group defined over* $\mathbb{Q}$ *with Lie algebra* $\mathfrak{g}$. *Given an* $\mathrm{Ad}^*(G)$ *orbit* $\mathcal{O} \subseteq \mathfrak{g}^*$, *we can associate with* $\mathcal{O}$ *an irreducible unitary representation* $(\pi = \pi_{\mathcal{O}})$ *of* G_A. *Given* $\phi \in C_c^\infty(G_A)$, $\pi(\phi)$ *is trace class and*

$$(1.13)\qquad \mathrm{tr}(\pi(\phi)) = \int_{\mathcal{O}_A} \hat{\phi} \cdot (\omega)_A,$$

where the canonical measure $(\omega)_A$ *is obtained from any rational* G-*invariant gauge form on* $\mathcal{O}$.

THEOREM 1.2 (*Moore*). *Let G be a unipotent algebraic group defined over $\mathbb{Q}$. Let $\rho = \operatorname{ind}_G^{G_A}(1)$; then we have:*

$$\rho = \bigoplus_{\mathcal{O}\in \mathfrak{g}^*/\operatorname{Ad}^*(G)} \pi_{\mathcal{O}}. \tag{1.14}$$

Proof. We use the Selberg trace formula in conjunction with Poisson summation formula, as in **[C-G]**. Let F be a fundamental domain of G in G_A, so $\operatorname{vol}(F) = 1$. Let $\phi \in C_c^\infty(G_A)$. Then we compute:

$$\begin{aligned} \operatorname{tr}(\rho(\phi)) &= \int_F \sum_{\gamma\in G} \phi(g^{-1}\gamma g)\, d\overline{g} \\ &= \int_F \sum_{X\in \mathfrak{g}} \phi(\operatorname{Ad}(g^{-1})(X))\, d\overline{g} \\ &= \int_F \sum_{l\in \mathfrak{g}^*} \hat{\phi}(\operatorname{Ad}^*(g)(l))\, d\overline{g}. \end{aligned} \tag{1.15}$$

In the last line, we use the Poisson summation formula for $\mathfrak{g} \subseteq \mathfrak{g}_A$ and $\mathfrak{g}^* \subseteq \mathfrak{g}_A^*$ (see **[W-1]**, §2).

Let S be a set of coset representatives for $\operatorname{Ad}^*(G)$ orbits in $\mathfrak{g}^*$. We then have:

$$\begin{aligned} &\int_F \sum_{l\in \mathfrak{g}^*} \hat{\phi}(\operatorname{Ad}^*(g)(l))\, d\overline{g} \\ &\quad = \sum_{l\in S} \left\{ \int_F \sum_{\lambda\in G/G(l)} \hat{\phi}(\operatorname{Ad}^*(g\lambda)(l))\, d\overline{g} \right\} \\ &\quad = \sum_{l\in S} \int_{G_A/G(l)} \hat{\phi}(\operatorname{Ad}^*(g)(l))\, d\overline{g}. \end{aligned} \tag{1.16}$$

The measure on $G_A/G(l)$ is determined by giving the discrete group $G(l)$ counting measure. Finally, (1.16) becomes:

$$\sum_{l\in S} \operatorname{vol}(G_A(l)/G(l)) \int_{G_A/G_A(l)} \hat{\phi}(\operatorname{Ad}^*(g)(l))\, d\overline{g}. \tag{1.17}$$

The measure on $G_A/G_A(l)$ is characterized by $\operatorname{vol}(G_A(l)/G(l))$. This measure comes from a G-invariant gauge form on $G/G(l)$ if and only if $\operatorname{vol}(G_A(l)/G(l)) = 1$. In this case, equations (1.13) and (1.17) yield:

$$\operatorname{tr}(\rho(\phi)) = \sum_{\mathcal{O}\in \mathfrak{g}^*/\operatorname{Ad}^*(G)} \operatorname{tr}(\pi_{\mathcal{O}}(\phi)). \tag{1.18}$$

2. Multiplicity formulas. In this section, we want to describe how the decomposition of $L^2(G_A/G)$ gives information about $L^\infty(G_\infty/\Gamma)$ when G is a unipotent algebraic group defined over $\mathbb{Q}$. For more information on this process, one can consult **[G-G-P]**, **[Ma]**, and **[S]**. For unipotent groups, the proofs in **[W-1]** carry over.

Recall that if E is a vector space over $\mathbb{Q}$, then $E_\infty + E$ is dense in E_A (**[W-1]**, p. 70). If G is an algebraic group defined over $\mathbb{Q}$, then strong approximation holds for G if $G_\infty \cdot G$ is dense in G_A.

LEMMA 2.1. *If G is a unipotent group defined over $\mathbb{Q}$, then strong approximation holds for G.*

Proof. Using that exp and log are isomorphisms, this follows from the corresponding fact for vector spaces. □

Let Γ be a discrete cocompact subgroup of G_∞. For each finite prime p, let Γ_p be the closure of Γ in G_p. Then Γ_p is a compact open subgroup of G_p, and for a.e. p. $\Gamma_p = K_p$ (see **[M]**, pp. 163–64). We can associate with Γ an open subgroup K_Γ of G_A by:

$$K_\Gamma = G_\infty \times \prod_p \Gamma_p. \tag{2.1}$$

Conversely, suppose we are given a family of compact open subgroups $\Gamma_p \subset G_p$ such that $\Gamma_p = K_p$ for a.e. p (equivalently, suppose we are given a compact open subgroup $\Gamma_f = \prod_{p\neq\infty} \Gamma_p$ of G_f); then we can form an open subgroup of G_A, $K = G_\infty \times \Gamma_f$. Let pr: $G_A \to G_\infty$ be the projection onto the first factor. If $\Gamma_K = \mathrm{pr}(K \cap G)$, then Γ_K is a discrete cocompact subgroup of G_∞. The two correspondences $\Gamma \to K_\Gamma$ and $K \to \Gamma_K$ are inverses of each other. The proofs of these facts are identical to Theorem 1 in **[W-1]**, p. 84.

Now fix a $\Gamma \subseteq G_\infty$; then we can define a natural map T from $L^2(G_\infty/\Gamma)$ to $L^2(G_A/G)$. Strong approximation says $G_A = K_\Gamma \cdot G$; thus, if $x \in G_A$, we can write $x = (g_\infty, g_f) \cdot \gamma$ with $(g_\infty, g_f) \in K_\Gamma$ and $\gamma \in G$. If $\phi \in L^2(G_\infty/\Gamma)$, define:

$$T(\phi)(x) = T(\phi)((g_\infty, g_f) \cdot \gamma) = \phi(g_\infty). \tag{2.2}$$

If $x = (g_\infty, g_f)\cdot\gamma = (\overline{g}_\infty, \overline{g}_f)\cdot\overline{\gamma}$, then $(g_\infty^{-1}\overline{g}_\infty, g_f^{-1}\overline{g}_f) = \overline{\gamma}\gamma^{-1} \in G \cap K$; so $g_\infty^{-1}\overline{g}_\infty \in \Gamma$ and $\phi(g_\infty) = \phi(\overline{g}_\infty)$. Thus, T is well-defined.

Next we want to show that $T(\phi)$ is in $L^2(G_{\mathbb{A}} G_Q)$. Since $G_{\mathbb{A}} = K_\Gamma \cdot G_Q$ we have:

$$\int_{G_{\mathbb{A}}/G_Q} |T(\phi)|^2 \le \int_{K_\Gamma/K_\Gamma \cap G_Q} |T(\phi)|^2. \tag{2.3}$$

Set $\overline{\Gamma} = K_\Gamma \cap G_Q = \{(\gamma, \gamma, \dots) | \gamma \in \Gamma\}$ and let $h \in C_c(K_\Gamma)$. We have by our normalizations of measures:

$$\int_{K_\Gamma/\overline{\Gamma}} \left(\sum_{\overline{\gamma}\in\overline{\Gamma}} h(g\overline{\gamma}) \right) = \int_{K_\Gamma} h(g). \tag{2.4}$$

Next suppose h is Γ_f invariant. Then $\exists f \in C_c(G_\infty)$ such that $f(q_\infty) = h(q)$ for all $q \in K_\Gamma$, thus we have:

$$\int_{K_\Gamma} h(q) = \operatorname{vol}(\Gamma_f) \int_{G_\infty} f(q_\infty) = \operatorname{vol}(\Gamma_f) \int_{G_\infty/\Gamma} \left(\sum_{\gamma\in\Gamma} f(q_\infty \cdot \gamma \right), \tag{2.5}$$

$$\int_{K_\Gamma} h(q) = \int_{K_\Gamma/\overline{\Gamma}} \left(\sum_{\overline{\gamma}\in\Gamma} h(q\overline{\gamma}) \right) = \int_{K_\Gamma/\overline{\Gamma}} T \left(\sum_{\gamma\in\Gamma} f(q_\infty \cdot \gamma) \right). \tag{2.6}$$

Thus (2.5) and (2.6) yield:

$$\operatorname{vol}(\Gamma_f) \int_{G_\infty/\Gamma} \phi = \int_{K_\Gamma/\overline{\Gamma}} T(\phi). \tag{2.7}$$

Since $T(|\phi|^2) = |T(\phi)|^2$ we get:

$$\int_{G_{\mathbb{A}}/G_Q} |T(\phi)|^2 \leq \int_{K_\Gamma/\overline{\Gamma}} |T(\phi)|^2 = \operatorname{vol}(\Gamma_f) \cdot \int_{G_\infty/\Gamma} |\phi|^2. \tag{2.8}$$

Thus T is a continuous mapping of $L^2(G_\infty/\Gamma)$ into $L^2(\Gamma_f/G_{\mathbb{A}}\backslash G_Q)$. To see that T is onto we produce an inverse. Let $\psi \in L^2(\Gamma_f/G_{\mathbb{A}}\backslash G_Q)$; then define $(S\psi)(q_\infty) = \psi((q_\infty, 1, 1, \dots))$. We note that $S\psi$ is left Γ invariant, for we have:

$$\begin{aligned} (S\psi)(q_\infty \cdot \gamma) &= \psi(q_\infty\gamma, 1, \dots)) \\ &= \psi((1, \gamma^{-1}, \gamma^{-1}, \dots)(q_\infty, 1, 1, \dots)(\gamma, \gamma, \gamma, \dots)) \\ &= \psi((q_\infty, 1, 1, \dots)) \end{aligned} \tag{2.9}$$

where we have used $(1, \gamma^{-1}, \gamma^{-1}, \dots) \in \Gamma_f$ for all $\gamma \in \Gamma$.

Similar computations as for T, show that S maps $L^2(\Gamma_f/G_A\backslash G_Q)$ continuously into $L^2(G_\infty/\Gamma)$.

Recall that ρ is left translation by G_A on $L^2(G_A/G)$ and λ is left translation by G_∞ on $L^2(G_\infty/\Gamma)$. If $g = (g_\infty, g_f) \in K_\Gamma$, then we have:

$$\rho(g)T(\phi) = T(\lambda(g_\infty)\phi). \tag{2.10}$$

LEMMA 2.2. *Let $\overline{\rho}$ denote the restriction of ρ to the subgroup $K_\Gamma = G_\infty \times \Gamma_f \subseteq G_A$ acting on the invariant subspace $L^2(\Gamma_f/G_A\backslash G)$. If $S_p(\Gamma)$ denotes the representations of G_∞ that occur with positive multiplicity in $L^2(G_\infty/\Gamma)$, then we have:*

$$\overline{\rho} = \sum_{\sigma \in S_p(\Gamma)} m(\sigma)(\sigma \otimes 1). \tag{2.11}$$

(Here $\sigma \otimes 1$ is the representation of K_Γ given by $(\sigma \otimes 1)(g_\infty, g_f) = \sigma(g_\infty)$.)

Proof. This is just a restatement of 2.10. □

Let π be an irreducible representation of G_A occurring in $L^2(G_A/G)$. Since $G_A = G_\infty \times G_f$ and G_∞ is a type 1 group, there exist irreducible representations π_∞, π_f of G_∞, G_f such that $\pi = \pi_\infty \times \pi_f$. Theorem 1.2 and formula (1.12), tell us that π is the only irreducible representation of G_A occurring in $L^2(G_A/G)$ with the ∞-factor being π_∞. If $\pi_f|_{\Gamma_f} = \sum_{\tau \in \Gamma_f} \mu(\pi_f, \tau)\tau$, then $\pi|_{K_\Gamma} = \sum_{\tau \in \widehat{\Gamma}_f} \mu(\pi_f, \tau)(\pi_\infty \otimes \tau)$. If we consider just the trivial representation of Γ_f, then we have that $\pi_\infty \otimes 1$ occurs in $L^2(\Gamma_f\backslash G_A/G)$ with multiplicity $\mu(\pi_f, 1)$. We can summarize the above discussion with the following lemma.

LEMMA 2.3. *Let G be a unipotent algebraic group defined over $\mathbb{Q}$. Let Γ be a discrete, cocompact subgroup of G_∞ and $\Gamma_f = \prod_{p_p \neq \infty} \Gamma_p$ the compact subgroup of G_A corresponding to Γ. Let $\pi = \pi_\infty \times \pi_f$ be an irreducible representation of G_A that occurs in $L^2(G_A/G)$. If $m(\pi_\infty)$ denotes the multiplicity of π_∞ in $L^2(G_\infty/\Gamma)$ and $\mu(\pi_f)$ denotes the multiplicity of the trivial representation of Γ_f in $\pi_f|_{\Gamma_f}$, then:*

$$m(\pi_\infty) = \mu(\pi_f). \tag{2.12}$$

In particular, for an irreducible representation π_∞ of G_∞ to occur in $L^2(G_\infty/\Gamma)$, it is necessary that $\pi = \pi_\infty \times \pi_f$ for some π of Theorem 1.2.

Since we have such a nice model for $\pi_{\mathcal{O}} = \mathrm{ind}_{H_A}^{G_A}(\chi_l)$, we can easily use Lemma 2.3 to compute the decomposition of $L^2(G_\infty/\Gamma)$. We see that $\mathrm{ind}_{H_A}^{G_A}(\chi_l) = \mathrm{ind}_{H_\infty}^{G_\infty}(\chi_l) \times \mathrm{ind}_{H_f}^{G_f}(\chi_l)$, so the representation π_f of Lemma 2.3 is an induced representation. Thus, we can apply the Mackey subgroup theorem to compute:

$$\pi_f|_{\Gamma_f} = \sum_{\pi \in H_f/G_f\backslash\Gamma_f} \mathrm{ind}_{xH_f x^{-1} \cap \Gamma_f}^{\Gamma_f}(\chi_l)^x. \tag{2.13}$$

We need to remark on two points in equation (2.13). First, since the subgroup Γ_f is open in G_f, we have that Γ_f and H_f are regularly related, $\Gamma_f \backslash G_f / H_f$ being discrete. Let $\overline{G}$ be the projection of $G \subseteq G_{\mathbb{A}}$ into $G_f \subseteq G_{\mathbb{A}}$, thus $\overline{G} = \{(1, x, x, \dots) | x \in G_Q\}$. It follows from **[W-1]**, pg. 83, Corollary 1, that $\overline{G}$ is dense in G_f, thus we have $G_f = \Gamma_f \cdot \overline{G}$. Therefore, (2.13) says:

$$(2.14) \qquad \mu(\pi_f) = \sum_{\substack{x \in H_f / G_f \backslash \Gamma_f \\ x \in \overline{G}}} \dim \operatorname{Hom}_{\Gamma_f}(1, \operatorname{ind}^{\Gamma_f}_{xH_f x^{-1} \cap \Gamma_f} (\chi_l)^x).$$

Now Γ_f is a compact group, so we can use the classical Frobenius reciprocity theorem for compact groups to conclude:

$$(2.15) \qquad \operatorname{Hom}_{\Gamma_f}(1, \operatorname{ind}^{\Gamma_f}_{xH_f x^{-1} \cap \Gamma_f} (\chi_l)^x) \simeq \operatorname{Hom}_{xH_f x^{-1} \cap \Gamma_f}(1, (\chi_l)^x).$$

In what follows, we will assume that x is the identity. Thus, we are in the situation where:

$$(2.16) \qquad \psi \circ l(\log_f(\cdot))|_{H_f \cap \Gamma_f} \equiv 1 \quad \text{and} \quad \psi \circ l(\log_A(\cdot))|_{H_{\mathbb{Q}}} \equiv 1.$$

If $\gamma \in H_{\mathbb{Q}}$, then we have $\psi_\infty \circ l(\log_\infty(\gamma)) = \psi_f \circ l(\log_f(\gamma))^{-1}$; and if $\gamma \in H_f \cap \Gamma_f$, then $\psi_f \circ l(\log_f(\gamma))^{-1} = 1$ by 2.16, so $\psi_\infty \circ l(\log_\infty(\gamma)) = 1$. Thus, we see that:

$$(2.17) \qquad (\chi_l)_f|_{H_f \cap \Gamma_f} \equiv 1 \quad \text{iff } (\chi_l)_\infty|_{H_\infty \cap \Gamma} \equiv 1.$$

Consequently, we have:

$$(2.18) \quad \dim \operatorname{Hom}_{xH_f x^{-1} \cap \Gamma_f}(1, (\chi_l)^x) = \begin{cases} 1 & \text{if } (\chi_l)_\infty|_{xHx^{-1} \cap \Gamma} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We can summarize the discussion with the following theorem.

THEOREM 2.1 (*Howe-Richardson*). *An irreducible representation π of G_∞ occurs in $L^2(G_\infty/\Gamma)$ if and only if there exist $l \in \mathfrak{g}^*_Q$ and a rational polarization H for $(\chi_l)_\infty$ such that $\pi = \operatorname{ind}^{G_\infty}_{H_\infty}((\chi_l)_\infty)$ and $(\chi_l)_\infty|_{H_\infty \cap \Gamma} \equiv 1$. The multiplicity of π_∞ in $L^2(G_\infty/\Gamma_\infty)$ is given by the number of rational double cosets in $H_\infty \backslash G_\infty / \Gamma$ such that $(\chi_l)^x_\infty|_{xH_\infty x^{-1} \cap \Gamma} \equiv 1$.*

We now indicate briefly how the results of Corwin-Greenleaf **[C-G]** fit into the present picture. We restrict ourselves to the case where $\Lambda = \log(\Lambda)$ is a lattice in $\mathfrak{g}$; such Γ's are called lattice subgroups. In this case, $\log(\Gamma_p) = \Lambda_p \subseteq \mathfrak{g}_p$ is a compact open $\mathbb{Z}_p$ module of $\mathfrak{g}_p$

([**M**], p. 161), so $\prod_p \Lambda_p = \Lambda_f$ is a compact open subgroup of $\mathfrak{g}_f$. Let $\phi \in C_c^\infty(G_A)$ be given by $\phi = \phi_1 \times \phi_2$, with ϕ_2 the characteristic function Γ_f (which we identify with the characteristic function of Λ_f via the exponential map). We have:

$$\text{(2.19)} \qquad \operatorname{tr}(\pi(\phi)) = \operatorname{tr}(\pi_\infty(\phi_1)) \cdot \operatorname{tr}(\pi_f(\phi_2)).$$

If P is the projection onto the space of Γ_f fixed vectors in $H(\pi_f)$, it is well known that $P = \operatorname{vol}(\Gamma_f)^{-1} \cdot \pi_f(\phi_2)$. Thus,

$$\operatorname{tr}(\pi_f(\phi_2)) = \operatorname{vol}(\Gamma_f) \cdot \operatorname{tr}(P) = \operatorname{vol}(\Gamma_f)\mu(\pi_f) = \operatorname{vol}(\Gamma_f) m(\pi_\infty).$$

To compute $m(\pi_\infty)$, we only need apply the character formula for $\operatorname{tr}(\pi(\phi))$. Since Γ is a lattice subgroup, this is particularly easy, since $\hat{\phi} = \hat{\phi}_1 \times \hat{\phi}_2$ and $\hat{\phi}_2 = \operatorname{vol}(\Lambda_f) \cdot$ (characteristic function of $\Lambda_f^\perp$ in $\mathfrak{g}^*$). (See [**W-1**], p. 107.) We then get:

$$\text{(2.20)} \qquad \operatorname{tr}(\pi(\phi)) = \left(\int_{\mathcal{O}_\infty} \hat{\phi}_\infty \, d\mu_\infty \right) \cdot \left(\int_{\mathcal{O}_f} \hat{\phi}_f \, d\mu_f \right).$$

If we break $\mathcal{O}_f \cap \Lambda_f^\perp$ into a sum of Γ_f orbits and evaluate the resulting integrals, we obtain the formula of Corwin and Greenleaf ([**C-G**], p. 12). We omit the details.

Next we will use the trace formula (1.12) to obtain a sharp upper bound for the rate of growth of the multiplicities of representations in the spectrum of Γ. When the representations are square-integrable mod the center, the estimate becomes exact and gives the Moore-Wolf multiplicity formula [**M-W**].

LEMMA 2.4. *Let $X_1, \dots, X_n$ be a Jordan-Hölder basis for $\mathfrak{g}$. Thus, if $\mathfrak{g}_i = \operatorname{span}_{\mathbf{Q}}\{X_1, \dots, X_i\}$, then $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$. Let $l_1, \dots, l_n$ be the corresponding dual basis of $\mathfrak{g}^*$. There exist complementary subsets S, T of $\{1, 2, \dots, n\}$ and an* $\operatorname{Ad}^*(G)$ *invariant Zariski-open dense subset $\mathscr{W}$ of $W = \operatorname{span}_{\mathbf{Q}}\{l_j | j \in T\}$ such that if $V = \operatorname{span}\{l_j | j \in S\}$, then:*

(1) *for almost every* $\operatorname{Ad}^*(G)$ *orbit $\mathcal{O} \subseteq \mu^*$, $\mathcal{O} \cap \mathscr{W}$ has only one element (if $\mathcal{O} \cap \mathscr{W} = \{l\}$, then we will write $\mathcal{O}_l$ for $\mathcal{O}$);*

(2) *for all $l \in \mathscr{W}$, there exists a polynomial map $P_l \colon V \to W$ with $\mathcal{O}_l = \operatorname{graph}(P_l)$; and*

(3) *the map $l \to P_l$ is rational in l.*

REMARK. Almost every orbit means a non-empty Zariski open subset of $\mathfrak{g}^*$.

Proof. See [**Pu1**], p. 55. □

The basis X_i, $i \in S$ for V determines a gauge form on V, which we denote by dX. Let $R_k : \mathcal{O}_l \to V$ be defined as follows:

$$R_l((x, P_l(x))) = x. \tag{2.21}$$

Then we can consider the gauge form on $\mathcal{O}_l$ defined by $R_l^*(dx) = \omega_l$.

LEMMA 2.5. *The gauge form $\omega_l = R_l^*(dx)$ is a G invariant form on $\mathcal{O}_l$.*

Proof. **[Pu1]**, p. 54. □

COROLLARY 2.1. *The canonical measure on $(\mathcal{O}_l)_A$ is given by $(\omega_l)_A$.*

Let π_l be the irreducible representation of G_A associated with $\mathcal{O}_l$. If $\phi \in C_c^\infty(G_A)$, then it follows from Corollary 2.1 that:

$$\operatorname{tr}(\pi_l(\phi)) = \int_{(V)_A} \hat{\phi}(x + P_l(x))(dx)_A. \tag{2.22}$$

If $\phi = \phi_\infty \times \phi_f$, then (2.22) becomes:

$$\operatorname{tr}(\pi_l(\phi)) = \int_{(V)_\infty} \hat{\phi}_\infty(x + P_l(x))(dx)_\infty \times \int_{V_f} \hat{\phi}_f(x + P_l(x))(dx)_f. \tag{2.23}$$

It is well known that there exists a polynomial function $l \to Pf(l)$ on $\mathcal{W}$ such that (**[C]**, Theorem 2):

$$\operatorname{tr}(\pi_\infty(\phi_\infty)) = |Pf(l)|_\infty^{-1} \int_{(V)_\infty} \hat{\phi}_\infty(x + P_l(x))(dx)_\infty. \tag{2.24}$$

It follows from (2.23) and (2.24) that:

$$\operatorname{tr}(\pi_f(\phi_f)) = |Pf(l)|_\infty \int_{(V)_f} \hat{\phi}_f(x + P_l(x))(dx)_f. \tag{2.25}$$

(Since $(Pf)(l)$ is rational if l is rational, we have $|Pf(l)|_A = 1$, where $|X|_A = \prod_p |X_p|_p$ is the idele norm, or $|Pf(l)|_\infty = |Pf(l)|_f^{-1}$.)

Now suppose $\Gamma \subseteq G_\infty$ is a lattice subgroup of G_∞, and set $\Lambda = \log(\Gamma)$. Let ϕ_f be the characteristic function of Λ_f. We saw before that $\operatorname{tr}((\pi_l)_f(\phi_f)) = \operatorname{vol}(\Lambda_f)m(\pi_\infty)$. Since ϕ_f is the characteristic function of Λ_f, $\hat{\phi}_f = \operatorname{vol}(\Lambda_f) \cdot K_{\Lambda_f^\perp}$, where $K_{\Lambda_f^\perp}$ is the characteristic function of $\Lambda_f^\perp = \{\lambda \in \mathfrak{g}_f^* | \psi_f(\lambda(\Lambda_f)) = 1\}$. Applying (2.25), we have:

$$m((\pi_l)_\infty) = 1/\operatorname{vol}(\Lambda_f) \cdot \operatorname{vol}(\Lambda_f)|Pf(l)|_\infty \int_{(V)_f} K_{\Lambda_f^\perp}(x + P_l(x))(dx)_f. \tag{2.26}$$

For $K_{\Lambda_f^\perp}(x + P_l(x))$ to be one, we need at least that $x \in V_f \cap \Lambda_f^\perp$; so we get:

$$\int_{V_f} K_{\Lambda_f^\perp}(x + P_l(x))(dx)_f \leq \text{vol}(\Lambda_f^\perp \cap V_f). \tag{2.27}$$

We can summarize the above with:

LEMMA 2.2. *Let $\Gamma \subseteq G_\infty$ be a lattice subgroup of G_∞. Let $\mathscr{W}$ be as above. Then for every $l \in \mathscr{W}$ we have:*

$$m((\pi_l)_\infty) \leq A|Pf(l)|_\infty, \quad \textit{where } A = \text{vol}(\Lambda_f^\perp \cap V_f). \tag{2.28}$$

REMARKS. (1) It is well known that $|Pf(l)|_\infty$ is Plancherel density with respect to the appropriate coordinates (see **[C]**, p. 6, or **[K2]**).

(2) Suppose G has square-integrable representations. Then, if $\mathfrak{z} =$ center of $\mathfrak{g}$, we can take $V \simeq \mathfrak{z}^\perp$, $W \approx \mathfrak{z}^*$, and $P_l(l) \equiv l$ **[M-W]**. Applying formula (2.26) yields:

$$m((\pi_l)_\infty) = \begin{cases} |Pf(l)| & \text{if } l \in (\mathfrak{z} \cap \Lambda)^\perp, \\ 0 & \text{otherwise.} \end{cases} \tag{2.29}$$

REFERENCES

[Bo] A. Borel, *Algebraic Groups*, Benjamin, New York, 1969.

[B] J. Brezin, *Harmonic Analysis on Compact Solvmanifolds*, LNM #602, Springer-Verlag, New York, 1977.

[C] L. Corwin, *A representation theoretic criterion for local solvability of left-invariant differential operators on nilpotent lie groups*, Trans. Amer. Math. Soc., **264** (1981), 113–120.

[C-G1] L. Corwin and F. Greenleaf, *Character formula and spectra of compact nilmanifolds*, J. Funct. Anal., **221** (1976), 123–154.

[C-G2] ——, *Direct integral decompositions and multiplicities for induced representations of nilpotent Lie groups*, (preprint).

[D-M] J Dixmier and P. Malliavin, *Factorization de fonctions et de vecteurs indefiniment differentiables*, Rulf. Sc. Math., 2^e Series, **102** (1978), 305–330.

[F] J. Fell, *The dual spaces of C^*-algebras*, Trans. Amer. Math. Soc., **94**(1960), 365–403.

[Fo] J. Fox, *Frobenius reciprocity and extensions of nilpotent Lie groups*, Trans. Amer. Math. Soc., **298**, no. 1 (Nov. 1986), 123–145.

[G-G-P] Gelfand, Graev, Pyateskii-Shapiro, *Representation Theory and Automorphic Functions*, W. B. Saunder, Philadelphia, 1969.

[H1] R. Howe, *On Frobenius reciprocity for unipotent algebraic groups over Q*, Amer. J. Math., **93** (1971), 163–172.

[H2] ——, *Harmonic analysis on solvable algebraic groups*, Pacific J. Math., **73** (1977), 383–437.

[K1] A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Russian Math. Surveys, **19** (1962), 53–104.

[K2] ——, *Elements of the Theory of Representations*, Springer-Verlag, New York, 1976.

[Ma] G. Mackey, *The Theory of Unitary Group Representations*, University of Chicago Press, Chicago, 1976.

[M] C. Moore, *Decomposition of unitary representations defined by discrete subgroups of nilpotent groups*, Ann. of Math., **82** (1965), 146–182.

[M-W] C. Moore and J. Wolf, *Square-integrable representations of nilpotent Lie groups*, Trans. Amer. Math. Soc., **185** (1973), 445–462.

[M-T] G. Mostow and T. Tamagawa, *The compactness of arithmetically defined homogeneous spaces*, Ann. of Math., **76** (1962), 446–463.

[Pu1] L. Pukanszky, *Leçons sur les répresentations des groupes*, Dunod, Paris, 1967.

[Pu2] ——, *On the characters and Plancherel formula of nilpotent Lie groups*, J. Funct. Anal., **1** (1967), 225–280.

[Ra] M. Raghunatham, *Discrete Subgroups of Lie Groups*, Springer-Verlag, New York, 1972.

[R] L. Richardson, *Decomposition of the L^2-space of a general compact nilmanifold*, Amer. J. Math., **93** (1971), 173–189.

[S] M. Silman, *Théorie de Mackey pour les groups adéliques*, Asterisque, **115** (1984).

[T] T. Tamagawa, *Adeles*, in *Algebraic Groups and Discontinuous Subgroups*, edited by Borel, Mostow, Proceedings of Symposia in Mathematics, **9**, Providence, American Math. Soc. (1966).

[Wa] N. Wallach, *Symplectic Geometry and Fourier Analysis*, Math. Science Press, Brookline, 1977.

[W-1] A. Weil, *Basic Number Theory*, Springer-Verlag, New York, 1974.

[W-2] ——, *Adeles and Algebraic Groups*, Progress in Math., **23**, Birkhauser, Boston, 1982.

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