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ON THE COHEN-MACAULAY PROPERTY IN COMMUTATIVE ALGEBRA AND SIMPLICIAL TOPOLOGY

DEAN E. SMITH

A ring R is called a "ring of sections" provided R is the section ring of a sheaf $(\mathscr{A}, X)$ of commutative rings defined over a base space X which is a finite partially ordered set given the order topology. Regard X as a finite abstract complex, where a chain in X corresponds to a simplex. In specific instances of $(\mathscr{A}, X)$, certain algebraic invariants of R are equivalent to certain topological invariants of X.

Introduction. The work of Reisner **[16]** shows a connection between the Cohen-Macaulay (CM) property in commutative algebra with a certain homological property of finite simplicial complexes. The purpose of this paper is to demonstrate a stronger connection. The main object of study in Reisner's Thesis is the face ring of a complex Σ with coefficients in a field F. In this paper the ring, hereby called the Stanley-Reisner ring and written $\mathrm{SR}(F,\Sigma)$, is also the main object of study.

The intent is to investigate the depth of factor rings of $\mathrm{SR}(F,\Sigma)$. The procedure is to regard $\mathrm{SR}(F,\Sigma)$ as the ring of sections of a sheaf of polynomial rings over a base space $X = X(\Sigma)$ where X is the partially ordered set of all simplices of Σ with order being reverse-inclusion. The method is to make statements about the depth of factor rings in the general section ring setting and then to particularize to the ring $\mathrm{SR}(F,\Sigma)$.

The homological property referred to in Reisner's Theorem **[16]** later proven to be a topological property **[13]** can be defined as follows. Let F be a field and Δ be a finite simplicial complex, or complex. Call Δ an *F-bouquet of spheres* if $\tilde{H}^i(\Delta, F) = 0$ for each $i < \dim\Delta$, the dimension of Δ, where $\tilde{H}^i(\Delta, F)$ denotes reduced singular cohomology with coefficients in F. A complex Σ is defined to be $\mathrm{CM}(F)$ provided the link subcomplex link (σ, Σ) is an *F-bouquet of spheres* for each $\sigma \in \Sigma$ (including $\phi \in \Sigma$).

Fix a field F. This paper shows $\mathrm{CM}(F)$ complexes are ubiquitous in the following sense. Let Σ be a complex with vertex set

$V = \{x_0, \ldots, x_n\}$. Let S be the polynomial ring $S = F[X_0, \ldots, X_n]$ and let S act on $\mathrm{SR}(F, \Sigma)$ by natural projection. Then $pd_s\mathrm{SR}(F, \Sigma)$ denotes the projective (or homological) dimension of the S-module $\mathrm{SR}(F, \Sigma)$. Let the invariant $\alpha(\Sigma) := n - pd_s\mathrm{SR}(F, \Sigma)$. $\alpha(\Sigma)$ is defined in the paper of Munkres **[13]**. It will be proven (Theorem 4.8) that $\alpha = \alpha(\Sigma)$ measures the dimension of the skeleton Σ^α maximal with respect to the property of being $\mathrm{CM}(F)$, i.e. Σ^α is $\mathrm{CM}(F)$ and if $j > \alpha$ then Σ^j is *not* $\mathrm{CM}(F)$. Theorem 4.8 was a consequence of looking carefully at the work of Munkres, knowing the result to be true in the special case where Σ is pure, i.e. where all maximal simplices have a fixed dimension.

Fix a field F and a complex Σ. Munkres proves the algebraic invariant $\alpha(\Sigma)$ is a topological invariant (Thm. 3.1, p. 116 **[13]**). It then follows from the last paragraph that the dimension $\alpha(\Sigma)$ of a maximal $\mathrm{CM}(F)$ skeleton is a topological invariant.

Fix a field F and a complex Σ. Stanley **[20]** shows that $(*)\alpha(\Sigma) = d - 1$ where $d := \mathrm{depth}_M\mathrm{SR}(F, \Sigma)$ with M the homogeneous maximal ideal of $\mathrm{SR}(F, \Sigma)$ and $\mathrm{depth}_M\mathrm{SR}(F, \Sigma)$ being the length of the longest regular sequence of $\mathrm{SR}(F, \Sigma)$ within M. In this paper $(*)$ is proven in case Σ is pure using sheaf theoretic methods (see Cor. 4.4)

Finally for a field F and complex Σ one can state:

If $d = \mathrm{depth}_M\mathrm{SR}(F, \Sigma)$, then $d - 1$ is a topological invariant of the complex equal to the dimension of a maximal $\mathrm{CM}(F)$ skeleton. This statement affords a generalization of Reisner's Theorem: simply set d equal to $\dim \Sigma + 1$.

1. The basic pair $(\mathscr{A}, X)$. In the following all partially ordered sets (posets) and all abstract simplicial complexes (complexes) will be finite. All rings will be associative and commutative with identity. All ring homomorphisms carry identity to identity. All modules over a ring are unitary.

Let X be a poset. The (order) topology on X is defined as the collection of all increasing subsets of X, i.e. $U \subseteq X$ is open if whenever $x \in U$ and $y \geq x$, then $y \in U$. For every $z \in X$ set $X_z = \{x \in X | x > z\}, \overline{X}_z = \{x \in X | x \geq z\}, X^z = \{x \in X | x < z\}, \overline{X}^z = \{x \in X | x \leq z\}$.

The sheaves considered in this paper will all be sheaves of rings with base space a poset X, with the above topology. The general definition of a sheaf on a topological space (see for example **[6]**) yields the following construction. A sheaf $\mathscr{A}$ of rings on X is a collection of rings $(\mathscr{A})_x = A_x$ for all $x \in X$ which are the stalks of $\mathscr{A}$, and ring homomorphisms $\rho_{yx}: A_x \to A_y$ for every $x, y \in X$ with $x \leq y$. The homomorphisms satisfy the following conditions: (1) $\rho_{xx}: A_x \to A_x$ is the identity map all $x \in X$ and (2) $\rho_{zy}\rho_{yx} = \rho_{zx}$ for every $x, y, z \in X$ with $x \leq y \leq z$. Note that the general notions of sheaf homomorphisms, restrictions to subsets, sub-sheaves and quotient sheaves could be found, for example in **[3]**, and easily specialized to this category of sheaves.

If $\mathscr{A}$ is a sheaf of rings on X, form the ring $S = \prod\{A_X | x \in X\}$ and if $s \in S$, denote by $s(x)$ its projection to A_x. Let $\Gamma(\mathscr{A}) = \{s \in S | \rho_{yx}s(x) = s(y), x, y \in X \text{ and } x \leq y\}$ and call the elements of $\Gamma(\mathscr{A})$ the sections of $\mathscr{A}$ on X. Clearly $\Gamma(\mathscr{A})$ is a subring of S, called the section ring of $\mathscr{A}$. For each subset $Y \subseteq X, Y \neq \emptyset$, denote by ρ_Y the restriction homomorphism $\Gamma(\mathscr{A}) \to \Gamma(\mathscr{A}|Y)$. The sheaf is called flasque if ρ_U is an epimorphism for all non-empty open subsets U of X. Note that $\Gamma(\mathscr{A}|\overline{X}_x) \cong A_x$ for each $x \in X$ where it is recalled that $\overline{X}_x = \{z \in X | z \geq X\}$ is open and $\Gamma(\mathscr{A}|\overline{X}_x)$ is the ring of sections defined on $\overline{X}_x$. If $\rho_x = \rho_U$, where $U = \overline{X}_X$ and $x \in X$, then $\rho_{yx}\rho_x = \rho_y$ for all $x \leq y$. In particular, if $\mathscr{A}$ is a flasque sheaf, all morphisms ρ_{yx} are epimorphisms. $\mathscr{A}$ is said to be a sharp sheaf provided $\operatorname{Ker}\rho_{yx} \neq 0$ all x and y with $x < y$. Set $P_x = \{r \in \Gamma(\mathscr{A}) | r(x) = 0\}$ for each $x \in X$. Clearly P_x is an ideal of $\Gamma(\mathscr{A})$ and $P_x \subseteq P_y$ for all $x \leq y$. The preceding definitions and notation will be used throughout this entire paper without further comment.

From now on the symbol "$(\mathscr{A}, X)$" will indicate a sheaf of rings $\mathscr{A}$ over a poset X. Furthermore, given $(\mathscr{A}, X)$, all stalk rings A_x will always be assumed to be integral domains. This means the ideals P_x are prime for each $x \in X$.

The following general lemmas will be utilized in the proof of Proposition 1.4.

LEMMA 1.1. *Let R be a unique factorization domain (UFD). Then every height 1 prime ideal is principal.*

Proof. Let P be a height 1 prime ideal of R, i.e. P is minimal over 0. By a basic result (p. 4 **[9]**), P contains a principal prime $(a) \neq 0$. By minimality of P over 0, $P = (a)$, and the argument is finished.

LEMMA 1.2. *Let R be an integral domain. For each $i, 1 \leq i \leq n$, let (a_i) be be a principal height 1 prime ideal of R with $(a_i) \neq (a_j)$ for $i \neq j$. Then $P = \bigcap\{(a_i) | 1 \leq i \leq n\}$, is a principal ideal with generator $\prod\{a_i | 1 \leq i \leq n\}$.*

Proof. The proof is by induction on n with trivial base step. So let $1 \leq k < n$ and for the inductive step assume $K = \bigcap\{a_i | 1 \leq i \leq k\}$, is principal with generator $\prod\{a_i | 1 \leq i \leq k\}$. Now consider $K \cap (a_{k+1})$. If $L = (\prod\{a_i | 1 \leq i \leq k+1\})$, obviously $L \subset K \cap (a_{k+1})$. It suffices to show the reverse containment. So let $b = \prod\{a_i | 1 \leq i \leq k\}$, and consider (1) $r = ab = ca_{k+1}, r \in K \cap (a_{k+1})$ and $a, c \in R$. Then $r \in (a_1)$. As (a_1) is prime and there are no inclusion relations amongst the $(a_i), c \in (a_1)$. So $r = c'a_1a_{k+1}$, some $c' \in R$. Arguing similarly, $r \in (a_2)$, so $c' \in (a_2)$ and so $r = c''a_2a_1a_{k+1}$, some $c'' \in R$. Continuing this argument inductively it is seen that $r \in L$. This finishes the inductive step and the proof.

Letting X be a poset, it is possible to provide X with a consistent enumeration which is a one-to-one monotone mapping $X \to \{1, \ldots, n\}$, n being the number of elements in X. Then every statement using $x \in X$ as a parameter can be proven by induction using this enumeration. This method of proof is referred to as "induction on x" or "induction with respect to X".

Let X be a poset. Define X as *lower-ranked* provided that for each $x \in X$ then every maximal chain ending at $x, y_1 < \cdots < y_n = x$, has the same length. Note that in a lower-ranked poset X it is possible to define a rank function rk: $X \to \mathcal{N}$ where $\mathrm{rk}(x) = n$ provided n is the length of any maximal chain of the above kind ending at x.

Let n be a positive integer and define a poset X to be *ranked of rank* n if the length of any two maximal chains is n. It is easy to see that a ranked poset is lower ranked so as a consequence, each ranked poset carries with it a rank function. Note also that a poset can be lower ranked but not ranked.

Let X be a poset and let $x, y \in X$. They y *covers* x provided $y > x$ and there does not exist $z \in X$ with $y > z > x$.

Let $(\mathscr{A}, X)$ be given, $\mathscr{A}$ a sheaf of integral domains, X a lower ranked poset. $(\mathscr{A}, X)$ is said to satisfy the *height* 1 *kernel condition* provided $\mathrm{ht}(\mathrm{Ker}\, \rho_{yx}) = 1$ whenever $x, y \in X$ with y covering x.

From now on, whenever $(\mathscr{A}, X)$ is a sheaf of rings over the poset X it will generally be assumed that all stalk rings A_x are Noetherian. (Recall that it is also understood that all stalk rings are integral domains.) It is a simple fact that as a consequence $R = \Gamma(\mathscr{A})$ is Noetherian (see Sec. 2.1, Exer. 9 **[9]**). In fact it can safely be assumed from now on that every ring is Noetherian unless otherwise specified.

LEMMA 1.3. *Let* $(\mathscr{A}, X)$ *be given,* $\mathscr{A}$ *a sharp flasque sheaf of Noetherian integral domains and* X *lower ranked. Consider the following statements*:

(a) $(\mathscr{A}, X)$ *satisfies the height one kernel condition*

(b) $\operatorname{ht} P_x = \operatorname{rk}(x) - 1$ *for each* $x \in X$.

Then (b) *implies* (a). *If* $\Gamma(\mathscr{A})$ *is catenary, then* (a) *implies* (b).

Proof. First prove (b) implies (a). Assume $\operatorname{ht} P_x = \operatorname{rk}(x) - 1$ each $x \in X$. Let $x, y \in X$ with y covering x. Now $\operatorname{ht} P_y/P_x + \operatorname{ht} P_x \leq \operatorname{ht} P_y$. But by (b), (1) $\operatorname{ht} P_y/P_x \leq 1$. As $\mathscr{A}$ is sharp, $P_y/P_x \neq 0$. By (1), (2) $\operatorname{ht} P_y/P_x = 1$. Given the sequence ($i$ the usual isomorphism and $R = \Gamma(\mathscr{A})$):

(3) $i \circ \rho_x \colon R \to \Gamma(\mathscr{A}|\overline{X}_x) \to A_x$ where $P_y \to \rho_x(P_y) \to \operatorname{Ker} \rho_{yx}$, one has the isomorphism of rings $R/P_x \cong A_x$ where P_y/P_x corresponds to $\operatorname{Ker} \rho_{yx}$. As this correspondence preserves height, (2) implies $\operatorname{ht} \operatorname{Ker} \rho_{yx} = 1$. This proves (a).

Now let R be catenary and assume (a). The proof is by induction on x. For the base step assume $\operatorname{rk}(x) = 1$, i.e. x is an atom of X. By a basic result (Prop. 1.4 **[21]**), $\operatorname{ht} P_x = 0 = \operatorname{rk}(x) - 1$.

For the inductive step assume $\operatorname{ht} P_z = \operatorname{rk}(z) - 1$ all z with $\operatorname{rk}(z) \leq \operatorname{rk}(x)$ and let y cover x. It suffices to show $\operatorname{ht} P_y = \operatorname{rk}(y) - 1 = \operatorname{rk}(x)$. As R is catenary, $\operatorname{ht} P_y/P_x + \operatorname{ht} P_x = \operatorname{ht} P_y$. By (a) and the inductive step derive $1 + \operatorname{rk}(x) - 1 = \operatorname{rk}(x) = \operatorname{ht} P_y$. The argument is complete.

Let $(\mathscr{A}, X)$ be a given sheaf pair with X a ranked poset of rank n. Let $\langle r_1, \ldots, r_{n-1}\rangle$ denote a sequence of length $n-1$ within the section ring $R = \Gamma(\mathscr{A})$. This sequence is said to be *ranked* if for each $i, 1 \leq i \leq n-1, r_i(x) \neq 0$ for all x with $\operatorname{rk}(x) \leq i$ and $r_i(x) = 0$ for all x with $\operatorname{rk}(x) > i$.

Let R be a ring. In the following, for I an ideal of $R, \operatorname{Rad}(I)$ is the usual nil-radical of I.

PROPOSITION 1.4. *Let* X *be a ranked poset of rank* $n, n > 1$, *and let* $\mathscr{A}$ *be a flasque sheaf of UFD's* (*possibly non-Noetherian*) *on* X *satisfying the height* 1 *kernel condition. Then*

(a) *there exists a ranked sequence* $\langle r_1, \ldots, r_{n-1}\rangle$ *in* $R = \Gamma(\mathscr{A})$ *and letting* $P(k) = \bigcap\{P_w | \operatorname{rk}(w) = k\}$, *for each positive integer* k, *then*

(b) $P(k) = \operatorname{Rad}(r_1, \ldots, r_{k-1})$.

Proof. First prove (a). Induct on n with base step $n = 2$. Pick a regular (i.e. non-zero divisor) element r_1 in the following manner. First fix x of rank 1. Let y cover x. By the assumptions, $\operatorname{Ker} \rho_{yx}$ is a height 1 prime ideal in A_x. By Lemma 1.1, $\operatorname{Ker} \rho_{yx}$ is principal and by Lemma 1.2, $0 \neq \bigcap\{\operatorname{Ker} \rho_{yx} | y \text{ covers } x\}$ is a principal ideal. Let $0 \neq r_1(x)$ be the generator of the above ideal. Define for each x of rank 1, $r_1(x)$ similarly.

Define $r_1 \in R$ by the projections $r_1(z) = 0, \operatorname{rk}(z) > 1$, and $r_1(z)$ as above if $\operatorname{rk}(z) = 1$. Note r_1 is a non-zero divisor of R and $\langle r_1 \rangle$ is a ranked sequence.

For the induction step assume $n > k > 1$ and that $\langle r_1, \ldots, r_{k-1} \rangle$ is a ranked sequence. The goal is to define $r_k \in R$. Fix z of rank k. Define $\bar{r}_k(z) \neq 0$ as in the base step as the generator of the ideal $\bigcap\{\operatorname{Ker} \rho_{wz} | w$ covers $z\}$. Now let $X(k-1) = X - \{x \in X | \operatorname{rk}(x) \leq k-1\}$. Define $r'_k \in \Gamma(\mathscr{A} | X(k-1))$ by $\bar{r}_k(z)$ for $\operatorname{rk}(z) = k$ and $r'(z) = 0$ for $\operatorname{rk}(z) > k$. Because $\mathscr{A}$ is flasque and $X(k-1)$ is an open subset of $X, r(k) \in R$ can be defined as any preimage of r'_k via the restriction epimorphism $R \to \Gamma(\mathscr{A} | X(k-1))$. This completes the proof of (a).

For the proof of (b) it is clear that $P(k) \supseteq \operatorname{Rad}(r_1, \ldots, r_{k-1})$ for each $k, 2 \leq k \leq n$. What remains to be proven is that $P(k) \subseteq \operatorname{Rad}(r_1, \ldots, r_{k-1})$ for each $k, 2 \leq k \leq n$. The proof is by induction.

For the base step let $k = 2$. Let $s \in P(2)$. For each x of rank 1, $s(x) = t(x) r_1(x), t(x) \in A_x$. Now $s^2(x) = t^2(x) r_1(x) r_1(x)$. The element $r \in \{\prod A_x | x \in X\}$, defined by $r(z) = t(z)t(z)r_1(z), \operatorname{rk}(z) = 1$, and $r(z) = 0, \operatorname{rk}(z) > 1$, is contained in R. So $s^2 \in (r_1)$. This shows $P(2) \subseteq \operatorname{Rad}(r_1)$.

For the induction step let $k > 2$ and assume the proposition that $P(j) \subseteq \operatorname{Rad}(r_1, \ldots, r_{j-1})$ all $j < k$. Take $s \in P(k)$. For all w of rank $k-1, s(w) = r_{k-1}(w) t(w)$, $t(w) \in A_w$. Now

$$s^2(w) = r_{k-1}(w)[r_{k-1}(w) t^2(w)].$$

Define the element $r' \in \Gamma(\mathscr{A} | X(k-2))$ by $r'(z) = r_{k-1}(z) t^2(z)$ for z of rank $k-1$ and $r'(z) = 0$ for $\operatorname{rk}(z) > k-1$. As before use the epimorphism $R \to \Gamma(\mathscr{A} | X(k-2))$ to produce a preimage r for $r', r \in R$.

Consider $s^2 - r_{k-1} r = q \in P(k-1)$. By the inductive assumption $q \in \operatorname{Rad}(r_1, \ldots, r_{k-2})$ so there exists an integer m such that $q^m \in (r_1, \ldots, r_{k-2})$. So $s^{2m} \in (r_1, \ldots, r_{k-1})$, and $s \in \operatorname{Rad}(r_1, \ldots, r_{k-1})$. This shows $P(k) \subseteq \operatorname{Rad}(r_1, \ldots, r_{k-1})$. The proof of (b) is done by induction.

Recall a definition from commutative algebra. (For instance see **[12]**.) Let R be a ring and $\langle r_1, \ldots, r_m \rangle$ be a sequence in R. This sequence is *regular* provided (i) for each integer $i, 1 \leq i \leq m, r_i$ is a non-zero divisor in the R-module $R/(r_1, \ldots, r_{i-1})$ and (ii) $(r_1, \ldots, r_m) \neq R$. This sequence is a *height sequence* provided $\operatorname{ht}(r_1, \ldots, r_i) = i$ for each

$i, 1 \leq i \leq k$. For R a Noetherian ring, every regular sequence is a height sequence (see the proof of Lemma 1.7).

THEOREM 1.5. *Let X be a ranked poset of rank $n, n > 1$, let $\mathscr{A}$ be a flasque sharp sheaf of Noetherian UFD's on X satisfying the condition* $\operatorname{ht} P_x = \operatorname{rk}(x) - 1$ *all $x \in X$. Then the ranked sequence $\langle r_1, \dots, r_{n-1}\rangle$ of Proposition* 1.4 *is a height sequence.*

Proof. By Lemma 1.3, $(\mathscr{A}, X)$ satisfies the height one kernel condition and by Proposition 1.4 the ranked sequence $\langle r_1, \dots, r_{n-1}\rangle$ exists. Fix k, $2 \leq k \leq n$, and let $B(k) = \{P \in \operatorname{Spec} R | P$ is a minimal over prime ideal of $(r_1, \dots, r_{k-1})\}$ where

Claim 1.

$$B(k) \subseteq \{P_z | \operatorname{rk}(z) = k\}.$$

Proof of Claim 1. Take P in the left hand side, i.e. let $P \supseteq (r_1, \dots, r_{k-1}), P$ a minimal over-prime of $(r_1, \dots, r_{k-1})$. By Proposition 1.4, $P \supseteq P(k) \supseteq (r_1, \dots, r_{k-1})$, so that P is a minimal over-prime of $P(k)$. By a fundamental result (see Prop. 1.4 **[21]**), $\{P_w + P(k) | \operatorname{rk}(w) = k\}$ is the complete set of minimal over-primes of 0 in $R/P(k) \cong \Gamma(\mathscr{A} | X(k-1))$. So $P = P_w$ some w of rank k and this completes the proof of the claim. (Note that Lemma 1.3 enabled Proposition 1.4 to be used above.)

Claim 2.

$$\{P_z | \operatorname{rk}(z) = k\} \subseteq B(k).$$

Proof of Claim 2. Take P_z with $\operatorname{rk}(z) = k$. Suppose by way of contradiction there exists $P \in \operatorname{Spec} R$ with $P_z \supset P \supseteq (r_1, \dots, r_{k-1})$. By Proposition 1.4, $P_z \supset P \supseteq P(k)$. But this contradicts P_z minimal over $P(k)$ as in the proof of Claim 1. The proof is complete. By use of Claims 1 and 2 above, $\{P_z | \operatorname{rk}(z) = k\} = B(k)$. All the P_z have height $k-1$ by hypothesis. So $\operatorname{ht}(r_1, \dots, r_{k-1}) = k-1$ by definition of height.

COROLLARY 1.6. *With hypotheses as in Theorem* 1.5, *if in addition R is Cohen-Macaulay* (*CM*) *then $\langle r_1, \dots, r_{n-1}\rangle$ is regular.*

Proof. By Theorem 1.5 $\langle r_1, \dots, r_{n-1}\rangle$ is a height sequence, so it suffices to prove the following general lemma, which is proven in the graded case by Smoke **[18]**.

LEMMA 1.7. *Let R be CM. Then the sequence $\langle r_1, \dots, r_{n-1}\rangle$ of R is a height sequence if and only if $\langle r_1, \dots, r_{n-1}\rangle$ is a regular sequence.*

Proof. Assume $\langle r_1, \dots, r_{n-1}\rangle$ is regular. We prove that $\langle r_1, \dots, r_k\rangle$ is a height sequence each $k, 1 \le k \le n-1$. The proof is by induction. For $k = 1$, $\langle r_1\rangle$ is regular so (r_1) is not contained in any height zero prime ideal of R. So $\mathrm{ht}(r_1) = 1$ by the Principal Ideal Theorem (see p. 104 **[9]**).

For the inductive step suppose $\mathrm{ht}(r_1, \dots, r_k) = k$, some $k, 1 \le k \le n-2$, and prove $\mathrm{ht}(r_1, \dots, r_{k+1}) = k+1$. By assumption r_{k+1} is not a zero divisor in $R/(r_1, \dots, r_k)$. So r_{k+1} is in no minimal over-prime of $(r_1, \dots, r_k)$. Using the Generalized Principal Ideal Theorem, $\mathrm{ht}(r_1, \dots, r_{k+1}) = k+1$. This finishes the induction. Note this part of the proof did not require R to be CM.

For the remainder, let $\langle r_1, \dots, r_{n-1}\rangle$ be a height sequence. Prove $\langle r_1, \dots, r_k\rangle$ is regular for each $k, 1 \le k \le n-1$. For the base step consider $\langle r_1\rangle$. As $\mathrm{ht}(r_1) = 1, r_1$ is not contained in any height zero prime. By **[12]** (Theorem 32) the associated primes of 0 are exactly the height zero primes. Thus r_1 is not in an associated prime of 0, and is regular. For the inductive step let $1 \le k \le n-2$ and assume $\langle r_1, \dots, r_k\rangle$ is regular. To prove: $\langle r_1, \dots, r_{k+1}\rangle$ is regular. r_{k+1} can be in no associated prime of $(r_1, \dots, r_k)$ since by the above result of **[12]**, each associated prime of $(r_1, \dots, r_k)$ has height k. So r_{k+1} is non-zero divisor of $R/(r_1, \dots, r_k)$ and $\langle r_1, \dots, r_{k+1}\rangle$ is regular. The induction is done.

DEFINITION 1.8. Let $(\mathscr{A}, X)$ be a pair with $\mathscr{A}$ a sheaf of integral domains on X a poset. Call $(\mathscr{A}, X)$ a *basic pair* if X is a ranked poset, $\mathscr{A}$ a sharp flasque sheaf of Noetherian UFD's on X such that $\mathrm{ht}\, P_x = \mathrm{rk}(x) - 1$ for each $x \in X$. Given $(\mathscr{A}, X)$ a basic pair call $\Gamma(\mathscr{A})$, the section ring arising from $(\mathscr{A}, X)$ the *section ring of a basic pair.*

2. Making depth statements for section rings $\Gamma(\mathscr{A})$ of a basic pair.

PROPOSITION 2.1. *Let $(\mathscr{A}, X)$ be a basic pair,* $\mathrm{rank}\, X = n$. *Suppose furthermore A_x is CM for each x of rank* 1. *Given the ranked height sequence $\langle r_1, \dots, r_{n-1}\rangle$ and R-module $P(2)$ as in* §1, *then $\langle r_1, \dots, r_{n-1}\rangle$ is a $P(2)$-regular sequence.*

Proof. First let $s \in P(2)$ and fix x of rank 1. For y of rank 2 with $y > x, \rho_{yx}s(x) = s(y) = 0$ so $s(x) \in \mathrm{Ker}\, \rho_{yx}$. Argue similarly for each $y > x$, and see that $s(x) \in \bigcap\{\mathrm{Ker}\, \rho_{yx} | y \text{ covers } x\}$. Arguing as in the proof of Proposition 1.4, $(r_1(x)) = \bigcap\{\operatorname{Ker}\rho_{yx} | y \text{ covers } x\}$ so write $s(x) = t(x)r_1(x)$ some $t(x) \in A_x$.

By varying the x above it is apparent that $P(2) \cong \bigoplus\{A_x r_1(x) | \operatorname{rk}(x) = 1\}$ where the direct sum is internal and the isomorphism is as left R-modules. As each of the A_x is an integral domain, the A_x-isomorphism $A_x \cong A_x r_1(x)$ induces the R-isomorphism $P(2) \cong \bigoplus\{A_x | \operatorname{rk}(x) = 1\}$ with R acting on the direct sum as follows: for $r \in R$ and $b \in \bigoplus\{A_x | \operatorname{rk}(x) = 1\}$, i.e. $b = (b(x), b(x'), \ldots,)$ write $rb = (r(x)b(x), r(x')b(x'), \ldots)$.

Let x have rank 1.

Claim. $(\mathscr{A}|\overline{X}_x, \overline{X}_x)$ is a basic pair.

Proof of Claim. First show that $\mathscr{B} = \mathscr{A}|\overline{X}_x$ is flasque. Suppose $U \subseteq \overline{X}_x$ is open. Then U is open in X. The restriction epimorphism $\rho_U : \Gamma(\mathscr{A}) \to \Gamma(\mathscr{A}|U)$ can be factored as

$$(1) \qquad \begin{tikzcd} \Gamma(\mathscr{A}) \arrow[rr] \arrow[dr, "\rho_x"'] & & \Gamma(\mathscr{A}|U) = \Gamma((\mathscr{A}|\overline{X}_x)|U) \\ & \Gamma(\mathscr{A}|\overline{X}_x) \arrow[ur, "\delta"'] & \end{tikzcd}$$

Conclude that the map δ is an epimorphism. This shows $\mathscr{B}$ is flasque. Now let $\rho_x : \Gamma(\mathscr{A}) \to \Gamma(\mathscr{B}), P_y \to \overline{P}_y$ be as in Section 1. It remains to show that $\operatorname{ht}\overline{P} = \operatorname{rk}(y) - 1$ each $y \in X_x$. Certainly $\operatorname{ht}\overline{P}_y \leq \operatorname{rk}(y) - 1$ each $y \in \overline{X}_x$ by reason that ρ_x can only lower height. But $\operatorname{ht}\overline{P}_y \geq \operatorname{rk}(y) - 1$ each $y \in \overline{X}_x$ since $\overline{X}_x$ contains a chain $x = x_{[1]} < \cdots < x_{[\operatorname{rk}(y)]} = y$. As $\mathscr{A}$ is sharp, $P_{x[1]} \subset P_{x[2]} \subset \cdots \subset P_y$ so that applying ρ_x it follows $\overline{P}_{x[1]} \subseteq \cdots \subseteq \overline{P}_y$ with all inclusions proper as ρ_x is the restriction map. Thus for each $y \in \overline{X}_x, \operatorname{ht}\overline{P}_y = \operatorname{rk}(y) - 1$. This finishes the proof of the Claim.

Now fix x of rank 1. The sequence $\langle\rho_x(r_1), \ldots, \rho_x(r_{n-1})\rangle$ is the ranked sequence in $\Gamma(\mathscr{A}|\overline{X}_x)$ as in Proposition 1.4 and since the Claim states that $\Gamma(\mathscr{A}|\overline{X}_x)$ is the section ring of a basic pair, $\langle\rho_x(r_1), \ldots, \rho_x(r_{n-1})\rangle$ is a height sequence by Theorem 1.5. By Lemma 1.7 and the assumption of $\Gamma(\mathscr{A}|\overline{X}_x) \cong A_x$ being CM, $\langle\rho_x(r_1), \ldots, \rho_x(r_{n-1})\rangle$ is $\Gamma(\mathscr{A}|\overline{X}_x)$-regular: but whereas $\rho_x(r_i) = r_i(x)$ under the isomorphism $\Gamma(\mathscr{A}|\overline{X}_x) \to A_x$, then conclude $(*)$ that $\langle r_1(x), \ldots, r_{n-1}(x)\rangle$ is regular for each x of rank 1. To finish the proof, as before regard $P(2) = \bigoplus\{A_x | \operatorname{rk}(x) = 1\}$ with the given action of R. It is clear by $(*)$ that $\langle r_1, \ldots, r_{n-1}\rangle$ is $P(2)$-regular.

Let R be a ring, N be an R-module and I be an ideal of R. Denote by $\operatorname{depth}_I N$ (see **[12]**) the length of the longest N-regular sequence of elements taken from I.

THEOREM 2.2. *Let $(\mathscr{A}, X)$ be a basic pair with* $\operatorname{rank} X = n$, *and suppose A_x is CM for each x of rank* 1. *Furthermore suppose X has a unique maximal element m with* $\operatorname{rk}(m) = n > 1$, *and let the Krull dimension of* $R(\dim R)$ *be* $n - 1$. *Letting* $M = P_m$,

$$\operatorname{depth}_M R/P(2) = \begin{cases} \text{(a) } \operatorname{depth}_M R - 1, & \textit{if } \operatorname{depth}_M R = n - 1, \\ \text{(b) } \operatorname{depth}_M R, & \textit{if } \operatorname{depth}_M R < n - 1. \end{cases}$$

Proof. Note by the definition of basic that $\operatorname{ht} M = n - 1 = \dim R$. It follows that M is a maximal ideal of R. (Note also that in Case (a), $\operatorname{depth}_M R$ is as big as it can be, i.e. in general for an ideal I of a ring R $\operatorname{depth}_I R \leq \operatorname{ht} I$. As a result the localization R_M is CM. See the proof of Theorem 3.6.)

Consider first Case (a). There is a short exact sequence of R-modules

$$(1) \qquad 0 \to P(2) \to R \to R/P(2) \to 0$$

where R acts on $R/P(2)$ by $r(s + P(2)) = rs + P(2)$ for all $r, s \in R$. (1) induces the long exact sequence in the usual derived functor $\operatorname{Ext}^i(R/M, \dots)$:

$$(2) \qquad \begin{aligned} \cdots \to \operatorname{Ext}^i(R/M, R) \to \operatorname{Ext}^i(R/M, R/P(2) \\ \to \operatorname{Ext}^{i+1}(R/M, P(2)) \to \cdots. \end{aligned}$$

By the last proposition and **[12]** (Theorem 28), in Case (a) $\operatorname{Ext}^i(R/M, R/P(2)) = 0$ all $i < n - 1$, and by assumption and **[12]** again, $\operatorname{Ext}^i(R/M, R) = 0$ all $i < n - 1$. Consider the following exact sequences extracted from (2) for $3 \leq j \leq n$:

$$(3) \qquad \begin{aligned} \operatorname{Ext}^{n-j}(R/M, R) \to \operatorname{Ext}^{n-j}(R/M, R/P(2)) \\ \to \operatorname{Ext}^{n-j+1}(R/M, P(2)). \end{aligned}$$

One must conclude

$$(4) \qquad \operatorname{Ext}^{n-j}(R/M, R/P(2)) = 0$$

for all j with $3 \leq j \leq n$. By **[12]** (Theorem 28), $\operatorname{depth}_M R/P(2) \geq n - 2$. To prove the result in Case (a) it suffices to prove the reverse inequality.

Looking at the ideal $P(2)$ of $R, \operatorname{ht} P(2) = 1$ whereas $P(2)$ could not be contained in a height 0 prime ideal as these are of the form $P_x, \operatorname{rk}(x) = 1$ ([**21**]). From the inequality $\operatorname{ht} P(2) + \dim R/P(2) \leq \dim R$ (see page 72 [**12**]) conclude

$$(5) \qquad \dim R/P(2) \leq n-2$$

Now assume by the way of contradiction that $\operatorname{depth}_M R/P(2) > n-2$. Then there is an $R/P(2)$-regular sequence $\langle s_1, \ldots, s_{n-1} \rangle$ inside of M. If $R \to R/P(2)$, where $s \to \bar{s}$, is given by the natural homomorphism, then $\langle \bar{s}_1, \ldots, \bar{s}_{n-1} \rangle$ is an $R/P(2)$-regular sequence in the maximal ideal $M/P(2)$ of the ring $R/P(2)$. By Lemma 1.7, $\operatorname{ht}(\bar{s}_1, \ldots, \bar{s}_{n-1}) = n-1$. Thus $\dim R/P(2) \geq n-1$. This contradicts (5) and concludes the proof in Case (a).

In case (b) let $\beta = \operatorname{depth}_M R$. By assumption $\beta < n-1$. Consider the exact sequence:

$$\begin{aligned} \cdots &\to \operatorname{Ext}^{\beta-1}(R/M, P(2)) \to \operatorname{Ext}^{\beta-1}(R/M, R) \\ &\to \operatorname{Ext}^{\beta-1}(R/M, R/P(2)) \to \operatorname{Ext}^{\beta}(R/M, P(2)) \\ &\to \operatorname{Ext}^{\beta}(R/M, R) \xrightarrow{f} \operatorname{Ext}^{\beta}(R/M, R/P(2)) \to \cdots . \end{aligned}$$

One sees that for all $j \leq \beta, \operatorname{Ext}^{j-1}(R/M, R) = 0 = \operatorname{Ext}^{j}(R/M, P(2))$, by the last proposition and assumption on R (see [**12**]). Thus $\operatorname{Ext}^{j-1}(R/M, R/P(2)) = 0$ for all $j \leq \beta$. But f must be a monomorphism and $\operatorname{Ext}^{\beta}(R/M, R) \neq 0$ so that $\operatorname{Ext}^{\beta}(R/M, R/P(2)) \neq 0$. By [**12**] (Theorem 28) $\operatorname{depth}_M R/P(2) = \beta = \operatorname{depth}_M R$. This completes the proof of the theorem.

3. Depth and the Stanley-Reisner ring. Recall the definition of the Stanley-Reisner ring of a complex Σ with coefficients in a field F, written $\operatorname{SR}(F, \Sigma)$ (see [**19**] for example). Let Σ be a complex (including $\varnothing$) with vertex set $V(\Sigma) = \{x_1, \ldots, x_m\}$ and F be a field. Denote by $I(\Sigma)$ the ideal of the polynomial ring $F[X_1, \ldots, X_m] = S$ generated by all square free monomials of the form $X_{i[1]}, \ldots, X_{i[k]}$ with the corresponding set $\{x_{i[1]}, \ldots, x_{i[k]}\} \notin \Sigma$. $\operatorname{SR}(F, \Sigma)$ is defined as $S/I(\Sigma)$.

What follows is a description showing that the Stanley-Reisner ring is the section ring of a sheaf of polynomial rings over a poset. Given a complex Σ define $X = X(\Sigma)$ to be the poset of all simplices of Σ with order relation the opposite of inclusion. Define a sheaf $\mathscr{A}$ of polynomial rings on X. (In fact $\mathscr{A}$ is a sheaf of F-algebras but this aspect will not be emphasized.) For each simplex $\sigma = \{x_{i[1]}, \ldots, x_{i[t]}\}$ put $A_\sigma = F[X_{i[1]}, \ldots, X_{i[t]}]$. In particular $F_\varnothing = F$. If $\sigma \subseteq \tau$, i.e. $\tau \leq \sigma$, define $\rho_{\sigma\tau}\colon A_\tau \to A_\sigma$ by $\rho_{\sigma\tau}(X_i) = X_i$ if $x_i \in \sigma$ and $\rho_{\sigma\tau}(X_i) = 0$ if $x_i \notin \sigma$. Clearly the collection of A_σ and $\rho_{\sigma\tau}$ form a sheaf $(\mathscr{A}, X)$ of rings on X. Note the stalk rings are Noetherian by the Hilbert Basis Theorem so by an earlier observation $\Gamma(\mathscr{A})$ is a Noetherian ring. The following proposition (see Prop. 7.6 **[21]**) is a basic for all of the results of this section.

PROPOSITION 3.1. *The sheaf $(\mathscr{A}, X)$ described above is flasque and* $\Gamma(A) \cong \mathrm{SR}(F, \Sigma)$.

Here are some definitions and easy observations which will allow the statement of the main results of this section.

Let X be any poset. For $x, y \in X, \{x, y\}$ is *bounded* provided there exists $w \in X$ with $x \leq w$ and $y \leq w$. X is a *prelattice* provided whenever $x, y \in X$ and $\{x, y\}$ is bounded, then $\{x, y\}$ has a least upper bound z, i.e. z is an upper bound for $\{x, y\}$ and if w is an upper bound for $\{x, y\}$, then $z \leq w$. It follows easily that for X a prelattice, $\{x, y\}$ has at most one least upper bound. Note that the poset $X(\Sigma)$ is a prelattice with $\sigma_v\tau = \sigma \cap \tau$, $\sigma_v\tau$ the least upper bound of $\{x, y\}$.

LEMMA 3.2. *In the poset $X = X(\Sigma), \Sigma$ a complex, whenever σ covers τ then* $\operatorname{ht}\operatorname{Ker}\rho_{\sigma\tau} = 1$.

Proof. For σ to cover τ means τ has one more vertex than σ. Say $\tau - \sigma = \{x_i\}$. Then $\operatorname{Ker}\rho_{\sigma\tau} = (X_i) \subseteq F[X_{i[1]}, \ldots, X_{i[t]}]$ where $\{x_{i[1]}, \ldots, x_{i[t]}\} = \tau$. The result follows easily (see Corollary p. 83 **[12]**).

Define a complex Σ to be pure of dimension N provided every maximal simplex has dimension N. (Alternatively every $\sigma \in \Sigma$ is a face of an N-dimensional simplex.) Note that if X is pure complex then the poset $X(\Sigma)$ is ranked.

Note. Let Σ be a pure complex with $\dim\Sigma = N$ and let $V(\Sigma)$ be the vertex set of Σ. For $\sigma \in \Sigma$ define the ideal I_σ of A_σ by $I_\sigma = (\{X_i \in S | x_i \in V(\Sigma),\ x_i \notin \sigma\})$. Then $A_\sigma \cong S|I_\sigma$. By Yuzvinsky (Prop. 1.10, p. 177 **[21]**) there is a natural ring homomorphism $\phi\colon \mathrm{SR}(F, \Sigma) \to \Gamma(\mathscr{A})$ defined by $\phi(a + I_\Sigma)(\sigma) = a + I_\sigma$ for $a \in S, \sigma \in \Sigma$. ϕ is the isomorphism referred to in Proposition 3.1. For some j, $1 \leq j \leq N+1$ let

$$a_j = \sum_{\{x_{i[1]}, \ldots, x_{i(N-j+2)}\} \in \Sigma} X_{i[1]} \cdots X_{i[N-j+2]} + I_\Sigma \in \mathrm{SR}(F, \Sigma).$$

Clearly a_j is a homogeneous element of degree $N - j + 2$. For every

$$\begin{aligned} \tau &= \{x_{i[1]}, \dots, x_{i[N-j+2]}\} \in \Sigma, \\ K_\tau &:= \bigcap \{\operatorname{Ker} \rho_{\sigma\tau} | \sigma \text{ covers } \tau\} \\ &= \bigcap \{(X_{i[k]}) | 1 \le k \le N - j + 2\} \\ &= (\prod \{X_{i[k]} | 1 \le k \le N - j + 2\}) \end{aligned}$$

by Lemmas 3.2 and 1.2. But for τ of dimension $N-j+1$ as above, $r_j(\tau)$ is defined as the generator of $K_\tau, c_\tau := \Pi\{X_{i[k]} | 1 \le K \le N-j+2\}$, and furthermore $\phi(a_j)(\tau) = c_\tau$. By the proof of Proposition 1.4, it follows that $\langle r_1, \dots, r_{N+1} \rangle$ may be chosen so that $\phi(a_j) = r_j$ for $1 \le j \le N+1$. It is clear that $\phi(a_j)(\sigma)$ is homogeneous in A_σ for each $\sigma \in \Sigma$. In summary, $\langle r_1, \dots, r_{N+1} \rangle$ can be chosen so that there is correspondence via ϕ to a homogeneous sequence of $\mathrm{SR}(F, \Sigma)$ with the property that for each j and for each $\sigma \in \Sigma, r_j(\sigma)$ is a homogeneous polynomial in A_σ.

Given any complex $\Sigma, X(\Sigma)$, and the sheaf of polynomial rings $\mathscr{A}$ on $X(\Sigma)$ described above, Lemma 3.2 states that $(\mathscr{A}, X)$ satisfies the height 1 kernel condition. Note also that $\Gamma(\mathscr{A}) \cong \mathrm{SR}(F, \cancel{E})$ is catenary (see Thm. 33, p. 111, **[12]**). If it is assumed Σ is pure, then as $X(\Sigma)$ is ranked, Lemma 1.3 yields the result that $\operatorname{ht} P_\sigma = \operatorname{rk}(\sigma) - 1$ for each $\sigma \in \Sigma$.

It is now possible to catalogue the above information.

PROPOSITION 3.3. *Let Σ be a pure complex with $(\mathscr{A}, X)$ as above. Then $X = X(\Sigma)$ is a ranked prelattice and $\mathscr{A}$ is a flasque sharp sheaf of Noetherian UFD's on X satisfying* $\operatorname{ht} P_\sigma = \operatorname{rk}(\sigma) - 1$ *for each $\sigma \in \Sigma$.*

In short, the proposition yields a basic pair $(\mathscr{A}, X)$. The theory developed in the last two sections can be applied in the context of Stanley-Reisner rings of pure complexes.

First here is a condition that insures purity in a complex. Define for a complex $\Sigma(\varnothing \in \Sigma)$ and $\sigma \in \Sigma, \operatorname{link}(\sigma, \Sigma) = \{\tau \in \Sigma | \tau \cup \sigma \in \Sigma$ and $\tau \cap \sigma = \varnothing\}$.

PROPOSITION 3.4. *Let Σ be a complex with the property that* $\operatorname{link}(\sigma, \Sigma)$ *is connected for each $\sigma \in \Sigma$ for which* $\dim \operatorname{link}(\sigma, \Sigma) > 0$. *Then Σ is pure.*

Proof. See "Proof, Step 1" (p. 117 **[13]**).

Let Σ be a complex and F be a field. Recall the definition of what it means for Σ to be Cohen-Macaulay (see **[16]**): For Δ a complex, $\tilde{H}^i(\Delta, F)$ denotes reduced singular cohomology with coefficients in F. Σ is said to be $\mathrm{CM}(F)$ provided for each $\sigma \in \Sigma, \tilde{H}^i(\operatorname{link}(\sigma, \Sigma), F) = 0$ for all $i < \dim \operatorname{link}(\sigma, \Sigma)$. Given Σ which is $\mathrm{CM}(F), \Sigma$ satisfies the hypothesis of the above proposition. Therefore any $\mathrm{CM}(F)$ complex is pure.

Here is a simple Lemma.

LEMMA 3.5. *Let X be a ranked poset with* $\operatorname{rank} X \geq 2$ *and $\mathscr{A}$ be a flasque sheaf of integral domains on X. Letting $X(1) = X - \{x \in X | \operatorname{rk}(x) = 1\}$ and $P(2) = \bigcap\{P_y | \operatorname{rk}(y) = 2\}$, then $R/P(2) \cong \Gamma(\mathscr{A}|X(1))$ as rings.*

Proof. The natural homomorphism $R = \Gamma(\mathscr{A}) \to \Gamma(\mathscr{A}|X(1))$ is an epimorphism of rings whereas $\mathscr{A}$ is flasque. It is clear that the following sequence is exact: $0 \to P(2) \to R \to \Gamma(\mathscr{A}|X(1)) \to 0$. The proof is complete by Noether's isomorphism theorem.

The following proposition follows from the work of Baclawski (Thm. 6.4, p. 247 **[1]**) and of Munkres (Cor. 6.6, p. 127 **[13]**). The following is a new proof using sheaf theory techniques.

PROPOSITION 3.6. *Let Σ be an N dimensional* $\mathrm{CM}(F)$ *complex. The $N-1$ skeleton* (Σ^{N-1}) *is* $\mathrm{CM}(F)$.

Proof. Given Σ and F with $\Sigma\,\mathrm{CM}(F)$. Let $\mathscr{M}$ denote the homogeneous maximal ideal of $\mathrm{SR}(F, \Sigma)$, i.e. if $\{x_1, \ldots, x_m\}$ is the vertex set for Σ then M is the image of $(X_1, \ldots, X_m)$ under the natural homomorphism $F[X_1, \ldots, X_m] \to \mathrm{SR}(F, \Sigma)$.

Fix the following notation, letting M and M' be the respective homogeneous maximal ideals of $\mathrm{SR}(F, \Sigma)$ and $\mathrm{SR}(F, \Sigma^{N-1})$ respectively. Let $\mathscr{M}$ and $\mathscr{M}'$ be ideals of $R = \Gamma(\mathscr{A})$ and $R' = \Gamma(\mathscr{A}|X(1))$ corresponding to M and M' respectively by means of the isomorphisms $R \cong \mathrm{SR}(F, \Sigma)$ and $R' \cong \mathrm{SR}(F, \Sigma^{N-1})$ of Proposition 3.1. (Recall $X(1) = X - \{x \in X | \operatorname{rk}(x) = 1\}$.) There exists a natural epimorphism $\eta\colon \mathrm{SR}(F, \Sigma) \to \mathrm{SR}(F, \Sigma^{N-1})$ such that the following diagram commutes:

$$\text{(1)} \qquad \begin{array}{ccc} R' & \longrightarrow & \mathrm{SR}(F, \Sigma^{N-1}) \\ {\scriptstyle \rho_{x[1]}}\uparrow & & \uparrow{\scriptstyle \eta} \\ R & \longrightarrow & \mathrm{SR}(F, \Sigma) \end{array} \qquad \text{where} \qquad \begin{array}{ccc} \mathscr{M}' & \longrightarrow & M' \\ \uparrow & & \uparrow \\ \mathscr{M} & \longrightarrow & M \end{array}$$

and the horizontal maps are the above isomorphisms.

Consider the basic pair $(\mathscr{A}, X)$ of Propositions 3.1 and 3.3. Note the hypotheses of Theorem 2.2, are satisfied. In this case $\varnothing$ is the unique maximal element of $X = X(\Sigma)$ and $N+1 = \dim \mathrm{SR}(F,\Sigma) = \dim \Gamma(\mathscr{A})$ (see p. 63 **[19]**), with $\mathscr{M} = P_\varnothing$. By Theorem 2.2 and Lemma 3.5, $N = \operatorname{depth}_{\mathscr{M}} R/P(2) = \operatorname{depth}_{\mathscr{M}} R'$ where R acts on R' via restriction; $ra = \rho_{X[1]}(r)a$ for $r \in R, a \in R'$. Consider operations in the ring R' and conclude (2) $\operatorname{depth}_{\mathscr{M}'} R' = N$.

Let $\langle r_1, \dots, r_{N+1}\rangle$ denote the usual height sequence in $\Gamma(\mathscr{A})$. Note $\langle r'_2, \dots, r'_{N+1}\rangle$ is the ranked height sequence of R' where for each $i, r'_i = \rho_{X[1]}(r_i)$. Noting the pair $(\mathscr{A}|X(1), X(1))$ is basic, Theorem 1.5, implies $\langle r'_2, \dots, r'_{N+1}\rangle$ is a height sequence in the ring R'. But $C' = (r'_2, \dots, r'_{N+1}) \subseteq \mathscr{M}'$. Thus $N = \operatorname{ht} C' \le \operatorname{ht} \mathscr{M}' \le \dim \Gamma(\mathscr{A}|X(1)) = N$. Conclude (3) $\operatorname{ht}\mathscr{M}' = N$.

Consider the localization ring $R'_{\mathscr{M}'}$. By (2) and (3) above: $N = \operatorname{depth}_{\mathscr{M}'} R' \le \operatorname{depth} R'_{\mathscr{M}'} \le N$. Conclude $\operatorname{depth} R'_{\mathscr{M}'} = \dim R'_{\mathscr{M}'}$ and $R'_{\mathscr{M}'}$ is CM.

Appealing to (1), $\mathrm{SR}(F, \Sigma^{N-1})_{M'}$ is CM. Now M' is the homogeneous maximal ideal of a graded ring so by a well known result (see p. 125 **[11]**), $\mathrm{SR}(F, \Sigma^{N-1})$ is CM. The argument is finished by an application of Reisner's Theorem (see **[16]**).

THEOREM 3.7. *Let Σ be a pure complex of dimension N, F be a field, and M be the homogeneous maximal ideal of* $\mathrm{SR}(F,\Sigma)$. *The following are equivalent.*

(a) $\operatorname{depth}_M \mathrm{SR}(F,\Sigma) = d$.

(b) Σ *contains the skeleton Σ^{d-1} maximal in the property of being* CM(F), *i.e. if $j > d-1$ then Σ^j is not* CM(F).

Proof. To simplify notation let $\mathrm{SR}(F,\Sigma^j) = \mathrm{SR}(\Sigma^j)$ for each $j, 0 \le j \le N$. Prove first that (a) implies (b). Given $\operatorname{depth}_M \mathrm{SR}(\Sigma) = d$. Write $d = N-k, k \in \{-1, 0, 1, \dots, N-1\}$.

Now prove (a) implies (b) in case $k = -1$, i.e. $d = N+1$. In this situation $\operatorname{depth}_M \mathrm{SR}(\Sigma) = \dim \mathrm{SR}(\Sigma) = N+1$. In the localization $\mathrm{SR}(\Sigma)_M$, $\operatorname{depth} \mathrm{SR}(\Sigma)_M = \dim \mathrm{SR}(\Sigma)_M = N+1$ so $\mathrm{SR}(\Sigma)_M$ is CM. But as M is the homogeneous maximal ideal (as in the proof of the last proposition), $\mathrm{SR}(\Sigma)$ is CM. By Reisner's Theorem $\Sigma = \Sigma^N$ is CM(F). (b) is proven.

Now assume $d = N-k, k \in \{0, 1, \dots, N-1\}$. Consider the following figure where the $M(i)$ are the homogeneous maximal ideals in the respective rings $\mathrm{SR}(\Sigma^{N-i})$ for each $i, 0 \le i \le N$.

	$\operatorname{depth}_{M[i]}(-)$	$\dim(-)$
$\mathrm{SR}(\Sigma)$	$N-k$	$N+1$
$\mathrm{SR}(\Sigma^{N-1})$	$N-k$	N
$\vdots$	$\vdots$	$\vdots$
$\mathrm{SR}(\Sigma^{N-k-1})$	$N-k$	$N-k$

The Krull dimension numbers are verified as in the proof of the last proposition.

The top entry in the middle column states: $N-k = \operatorname{depth}_{M[0]}\mathrm{SR}(\Sigma)$, which is (a).

Note $N+1 > N-k$, by choice of k. So applying the isomorphism $\Gamma(\mathscr{A}) \cong \mathrm{SR}(\Sigma)$ (Proposition 3.1) and recalling Lemma 3.5, $\Gamma(\mathscr{A})/P(2) \cong \Gamma(\mathscr{A}|X(1)) \cong \mathrm{SR}(\Sigma^{N-1})$; and through a use of Theorem 2.2, it follows that $N-k$ belongs in the second row, i.e. $\operatorname{depth}_{M[1]}\mathrm{SR}(\Sigma) = N-k$. Argue in exactly the same fashion and see that $N-k$ belongs in rows 2 through $k+1$ also.

Note $\operatorname{depth}_{M[k+1]}\mathrm{SR}(\Sigma^{N-k-1}) = \dim \mathrm{SR}(\Sigma^{N-k-1}) = N-k$. By precisely the same argument as for the case $k=-1$, $\Sigma^{N-k-1} = \Sigma^{d-1}$ is $\mathrm{CM}(F)$. To finish the argument it suffices to show none of $\Sigma, \cdots, \Sigma^{N-k}$ are $\mathrm{CM}(F)$. So choose $i, 0 \leq i \leq k$ and consider Σ^{N-i}. By Reisner's Theorem it suffices to show $\mathrm{SR}(\Sigma^{N-i})$ is not CM. Suppose by way of contradiction $\mathrm{SR}(\Sigma^{N-i})$ is CM. Then $\Gamma(\mathscr{B}) \cong \mathrm{SR}(\Sigma^{N-i})$ is CM where $\mathscr{B}$ is the sheaf of polynomial rings over the poset of simplices of Σ^{N-i}. Then the ranked height sequence of $\Gamma(\mathscr{B})$ is regular by Lemma 1.7. So $\operatorname{depth}_{M[1]}\mathrm{SR}(\Sigma^{N-i}) = N-i+1$ by this train of thought and contradicts the table's assumption of $N-k = \operatorname{depth}_{M[i]}\mathrm{SR}(\Sigma^{N-i}) < N-i+1$. As i was arbitrary the proof of (a) implies (b) is complete.

Now suppose Σ^{d-1} is a maximal $\mathrm{CM}(F)$ skeleton in the sense of (b). Look at the basic pair $(\mathscr{B}, Y)$ where $\mathscr{B}$ is the usual sheaf of polynomial rings over the poset $Y = Y(\Sigma^d)$ of all simplices of Σ^d. By (b) and Reisner's Theorem, $\Gamma(\mathscr{B}) \cong \mathrm{SR}(\Sigma^d)$ is not CM and $\Gamma(\mathscr{B}|Y(1) \cong \mathrm{SR}(\Sigma^{d-1})$ is CM where $Y(1) = Y - \{\sigma \in \Sigma | \dim \sigma = d\}$. Let $M(d-i)$ be the homogeneous maximal ideal of $\mathrm{SR}(\Sigma^{d-i})$, for $i=0$ and 1. As before, as $\mathrm{SR}(\Sigma^{d-1})$ is CM, $\operatorname{depth}_{M[d-1]}\mathrm{SR}(\Sigma^{d-1}) = d$. (Look at the ranked height sequence of length d in $\Gamma(\mathscr{B}|Y(1))$. It is regular by Lemma 1.7, and by Theorem 2.2, $\operatorname{depth}_{M[d]}\mathrm{SR}(\Sigma^d) = d$.

The last paragraph of the proof is iterated to skeleta $\Sigma^{d+1}, \ldots, \Sigma^N = \Sigma$ and one obtains $\operatorname{depth}_{M[N]}\mathrm{SR}(\Sigma) = d, M(N)$ being the maximal homogeneous ideal of $\mathrm{SR}(\Sigma)$. This is (a).

COROLLARY 3.8. *Let Σ be a pure complex of dimension N. Let M be the homogeneous maximal ideal of* $\mathrm{SR}(F,\Sigma)$. *Then* $\operatorname{depth}_M \mathrm{SR}(F,\Sigma) = \operatorname{depth} \mathrm{SR}(F,\Sigma)_M$.

Proof. Consider the isomorphism $\eta\colon \mathrm{SR}(F,\Sigma) \to \Gamma(\mathscr{A})$ and let $\mathscr{M} = \eta(M)$. By general facts about localization (see p. 179 [**21**]), $\Gamma(\mathscr{A})_{\mathscr{M}} = \Gamma(\mathscr{B})$ where $\mathscr{B}$ is a sheaf of rings over the poset $X = X(\Sigma)$ with $(\mathscr{B})_\sigma = (A_\sigma)_{\mathscr{M}/P\sigma}$ for all $\sigma \in X$. Note $(\mathscr{B}, X)$ is a basic pair: The stalks are regular local rings which are UFD's by an Auslander-Buchsbaum Theorem (p. 142, [**12**]). For the flasque property see [**21**] (Theorem 2.1). Also the height 1 kernel condition holds for $\mathscr{B}$ and $\Gamma(\mathscr{B})$ is a catenary, so by Lemma 1.3, $\operatorname{ht} P'_\sigma = \operatorname{rk}(\sigma) - 1$ for all $\sigma \in \Sigma$ where $P'_\sigma = \{r \in \Gamma(\mathscr{B}) | r(\sigma) = 0\}$. By the same argument as in the proof of Theorem 3.7, if $d = \operatorname{depth} \mathrm{SR}(F,\Sigma)_M$, then $\mathrm{SR}(F,\Sigma^{d-1})_{M'}$ is CM where M' is the image of M under the natural epimorphism $\mathrm{SR}(F,\Sigma) \to \mathrm{SR}(F,\Sigma^{d-1})$. But M' is the homogeneous maximal ideal of $\mathrm{SR}(F,\Sigma^{d-1})$ and reasoning as in the proof of Theorem 3.7, $\mathrm{SR}(F,\Sigma^{d-1})$ is CM. Then Σ^{d-1} is CM(F) by Reisner's Theorem. By Theorem 3.7, $d \leq \operatorname{depth}_M \mathrm{SR}(F,\Sigma)$. Whereas the reverse inequality is always true, the argument is complete.

4. A topological invariant for finite complexes. Given a complex Σ $|\Sigma|$ will denote the realization of Σ (for details see [**14**]). Given a property P of abstract simplicial complexes (e.g. dimension), P is a topological invariant means if Σ and Σ' are abstract simplicial complexes and $|\Sigma|$ is homeomorphic with $|\Sigma'|$ and furthermore Σ has P, then Σ' has P.

The following property of finite complexes is the main subject of this section. Fix F a field, Σ a complex of dimension N, and let d be an integer $0 < d \leq N$. $P(F,d,\Sigma)$ denotes the property that Σ^{d-1} is CM(F), i.e. Σ^{d-1} is a maximal CM(F) skeleton. In this section it will algebraically be proven that $P(F,d,\Sigma)$ is a topological invariant for pure complexes by showing that if $d = \operatorname{depth}_M \mathrm{SR}(F,\Sigma)$ then $d - 1 = \alpha(\Sigma)$, where M is the homogeneous maximal ideal of $\mathrm{SR}(F,\Sigma)$ and $\alpha(\Sigma)$ is the topological invariant found in the work of Munkres [**13**]. (See Theorem 4.3). Then it will be proven by doing a variation on [**13**] that $P(F,d,\Sigma)$ is a topological invariant for *all* complexes.

The following notation is fixed for this entire section. Let Σ be a complex with vertex set $\{x_0,\dots,x_n\}$. Let F be a field and $F[X_0,\dots,X_n] = S$ be the polynomial ring in indeterminants X_i. Regard $\mathrm{SR}(F,\Sigma)$ as a cyclic S-module. The action of S on $\mathrm{SR}(F,\Sigma)$ is: $sr = (s + I_\Sigma)r$ for

$s \in S$ and $r \in \mathrm{SR}(F,\Sigma)$. Then $\mathrm{pd}_s\mathrm{SR}(F,\Sigma)$ will denote the projective (or homological) dimension of the S-module $\mathrm{SR}(F,\Sigma)$. For a maximal ideal $\mathbf{n}$ of S with $S' = S_{\mathbf{n}}$ let $\mathrm{pd}_{S'}\mathrm{SR}(F,\Sigma)$ denote the projective dimension of the (localization) S'-module $S_n \otimes_S \mathrm{SR}(F,\Sigma) =_{S'} \mathrm{SR}(F,\Sigma)$. Let $\mathbf{m}$ be the ideal $(X_0,\ldots,X_n) \subseteq S$.

LEMMA 4.1. *Let Σ be a pure complex of dimension N. Then for any $P \in \mathrm{Spec}(\mathrm{SR}(F,\Sigma))$,*

$$N+1 = \dim \mathrm{SR}(F,\Sigma) = \mathrm{ht}\, P + \dim \mathrm{SR}(F,\Sigma)/P.$$

In particular for M' any maximal ideal of $\mathrm{SR}(F,\Sigma), \mathrm{ht}\, M' = N+1$.

Proof. Let P be any minimal prime ideal of $\Gamma(\mathscr{A}) \cong \mathrm{SR}(F,\Sigma)$ where $\mathscr{A}$ is the sheaf of polynomial rings over the poset of simplices X of Σ as in Proposition 3.1. By a basic result (see Prop. 1.4 **[21]**) $P = P_\sigma$ for some minimal σ, i.e. σ is maximal in Σ. There exists a chain $\sigma = \sigma_1 < \sigma_2 < \cdots < \sigma_{N+2} = \varnothing$ in X, so the strictly ascending chain of prime ideals $P = P_\sigma \subset P_{\sigma_2} \subset \cdots \subset P_{\sigma_{N+1}} \subset P_\varnothing$ is of length $N+2$. Thus $\dim R/P \geq N+1 = \dim R$. As the reverse inequality is automatic it is established that $\dim R/P = N+1$ for each minimal $P \in \mathrm{Spec}(\mathrm{SR}(F,\Sigma))$.

As a consequence of the Noether Normalization Theorem (Cor. 3.6, p. 53 **[10]**), $\dim \mathrm{SR}(F,\Sigma) = \mathrm{ht}\, P + \dim \mathrm{SR}(F,\Sigma)/P$ for each $P \in \mathrm{Spec}(\mathrm{SR}(F,\Sigma))$.

PROPOSITION 4.2. *With notation as above, Σ a pure complex, and $S'' = S_{\mathbf{m}}, \mathrm{pd}_{S'}\mathrm{SR}(F,\Sigma) \leq \mathrm{pd}_{S''}\mathrm{SR}(F,\Sigma)$, for each maximal ideal $\mathbf{n}$ of S. Consequently* $\mathrm{pd}_{S''}\mathrm{SR}(F,\Sigma) = \mathrm{pd}_S\mathrm{SR}(F,\Sigma)$.

Proof. First see the second statement follows from the first: It is known that $\mathrm{pd}_S\mathrm{SR}(F,\Sigma) = \sup\{\mathrm{pd}_{S'}\mathrm{SR}(F,\Sigma) | S' = S_{\mathbf{n}}, \mathbf{n} \subseteq S \text{ maximal}\}$ (Lemma 5, p. 129 **[12]**).

Now set some notation. Let $\gamma\colon S \to \mathrm{SR}(F,\Sigma), s \to s + I_\Sigma$ be the projection epimorphism. Let $\eta\colon \mathrm{SR}(F,\Sigma) \to \Gamma(\mathscr{A})$ be the usual isomorphism. Let $\lambda = \eta \circ \gamma$. S acts on $\Gamma(\mathscr{A})$ by $sa = \lambda(s)a$ for $s \in S$ and $a \in \Gamma(\mathscr{A})$.

Now let $\mathbf{n}$ be a fixed maximal ideal in S and $S' = S_{\mathbf{n}}$. Because localization gives an exact functor and $0 \to P(2) \to \Gamma(\mathscr{A}) \to \Gamma(\mathscr{A}|X(1)) \to 0$ is an exact sequence of S-modules, $0 \to_{S'} P(2) \to_{S'} \Gamma(\mathscr{A}) \to_{S'}$

$\Gamma(\mathscr{A}|X(1)) \to 0$ is exact. (The action of S on $\Gamma(\mathscr{A})$ is gotten by using λ as above and S acts on $\Gamma(\mathscr{A}|X(1))$ via projection.) By a standard result (Exercise 9.12, p. 243 **[17]**),

$$(0) \qquad \mathrm{pd}_{S'}\Gamma(\mathscr{A}) \leq \max\{\mathrm{pd}_{S'}P(2), \mathrm{pd}_{S'}\Gamma(\mathscr{A}|X(1))\}.$$

Claim. $\mathrm{pd}_{S'}P(2) \leq \mathrm{pd}_{S'}\Gamma(\mathscr{A}|X(1)$.

Note by (0) that the following is true once the Claim is established: (+) $\mathrm{pd}_{S'}\Gamma(\mathscr{A}) \leq \mathrm{pd}_{S'}\Gamma(\mathscr{A}|X(1))$ will hold.

Proof of Claim. ${}_{S'}P(2) \cong \bigoplus_{S'} A_x, \mathrm{rk}(x) = 1$, by a standard isomorphism. By additivity of the $\mathrm{Ext}^i_{S'}(\_, B)$ functor for B an S'-module, to prove the Claim it suffices to prove

(1) $\mathrm{pd}_{S'}A_x \leq \mathrm{pd}_{S'}\Gamma(\mathscr{A}|X(1))$ for x of rank 1. Consider A_x with x of rank 1. By the Auslander Buchsbaum (A-B) Theorem (p. 263 **[15]**), $\mathrm{pd}_{S'}A_x = \operatorname{depth} S' - \operatorname{depth}_{S'}A_X$. By a basic result (see Cor. 3 p. 92 **[12]**),

(2) $\mathrm{pd}_{S'}A_x = (n+1) - \operatorname{depth}_{S'}A_x$. Now $\operatorname{depth}_{S'}A_x = \operatorname{depth}(A_x)_{\tau[\mathbf{n}]}$ where δ: $\Gamma(\mathscr{A}|\overline{X}_x) \to A_x$ is the usual isomorphism and $\tau = \delta \circ \rho_x \circ \lambda$: $S \to \Gamma(\mathscr{A}) \to \Gamma(\mathscr{A}|\overline{X}_x) \to A_x$. But $A_x = F[Z_1, \ldots, Z_{N+1}]$ where $\{Z_1, \ldots, Z_{N+1}\} \subseteq \{X_0, \ldots, X_n\}$. But by the same basic result quoted above, conclude from (2),

(3) $\mathrm{pd}_{S'}A_x = n+1-(N+1) = n-N$ for all x of rank 1. By the A-B Theorem, where $\rho' = \rho_{x[1]}, \mathrm{pd}_{S'}\Gamma(\mathscr{A}|X(1)) = n+1-\operatorname{depth}_{S'}\Gamma(\mathscr{A}|X(1))$ $= n+1-\operatorname{depth}\Gamma(\mathscr{A}|X(1))_{\rho'\circ\lambda[\mathbf{n}]} \geq n+1-\dim\Gamma(\mathscr{A}|X(1))_{\rho'\circ\lambda[\mathbf{n}]} =$ $n+1-N$ with the last equality following from Lemma 4.1. Thus the following is established:

(4) $\mathrm{pd}_{S'}\Gamma(\mathscr{A}|X(1)) \geq n+1-N$. (3) and (4) yield (1) immediately. This finishes the proof of the Claim.

Now use (+) repeatedly where $M = \gamma(\mathbf{m})$ and

$$(*) \qquad d = \operatorname{depth}_{S''}\mathrm{SR}(F, \Sigma) = \operatorname{depth}\mathrm{SR}(F, \Sigma)$$

by Corollary 3.8:

$$\begin{aligned}
\mathrm{pd}_{S'}\mathrm{SR}(F,\Sigma) &\leq \mathrm{pd}_{S'}\mathrm{SR}(F,\Sigma^{n-1}) \leq \cdots \leq \mathrm{pd}_{S'}\mathrm{SR}(F,\Sigma^{d-1}) \\
&= n+1-\operatorname{depth}_{S'}\mathrm{SR}(F,\Sigma^{d-1}), \quad \text{by the A-B Theorem,} \\
&= n+1-\operatorname{depth}\mathrm{SR}(F,\Sigma^{d-1})_{M'}, \\
&\quad \text{where } M' \text{ is the image of } \mathbf{n} \text{ under } S \to \mathrm{SR}(F,\Sigma^{d-1}), \\
&= n+1-d,
\end{aligned}$$

whereas by $(*)$, Theorem 3.7, and Reisner's Theorem $\mathrm{SR}(F,\Sigma^{d-1})$ is CM and by Lemma 4.1,

$$\begin{aligned} d &= \dim \mathrm{SR}(F,\Sigma^{d-1})_{M'} = \operatorname{depth} \mathrm{SR}(F,\Sigma^{d-1})_{M'} \\ &= n+1-\operatorname{depth}_{S''}\mathrm{SR}(F,\Sigma), \text{ by } (*), \\ &= \mathrm{pd}_{S''}\mathrm{SR}(F,\Sigma) \qquad \text{by the A-B Theorem.} \end{aligned}$$

Putting together the two ends of the string of inequalities, the proof is complete.

THEOREM 4.3. *Let Σ be a pure complex and F be a field. Then $P(F,d,\Sigma)$ is a topological invariant.*

Proof. Let the notation be as in the proposition preceding. Munkres (Thm. 3.1, p. 116 **[13]**) has proven that $\alpha(\Sigma) = n - \mathrm{pd}_S\mathrm{SR}(F,\Sigma)$ is a topological invariant. By the proposition above and the Auslander-Buchsbaum Theorem,

$$\begin{aligned} \alpha(\Sigma) &= n - \mathrm{pd}_{S''}\mathrm{SR}(F,\Sigma) = n - [n+1-\operatorname{depth}_{S''}\mathrm{SR}(F,\Sigma)] \\ &= -1 + \operatorname{depth}_{S''}\mathrm{SR}(F,\Sigma). \end{aligned}$$

Since $\alpha(\Sigma)$ is a topological invariant, then so is $\operatorname{depth}_{S''}\mathrm{SR}(F,\Sigma) = \operatorname{depth}\mathrm{SR}(F,\Sigma)_M = \operatorname{depth}_M\mathrm{SR}(F,\Sigma)$ by Corollary 3.8. Theorem 3.7 implies the dimension $\alpha(\Sigma)$ of the maximal CM(F) skeleton is a topological invariant. This is what was required to be shown.

The following result is contained within the body of the proof above.

COROLLARY 4.4. *Let Σ be pure and $d = \operatorname{depth}_M\mathrm{SR}(F,\Sigma)$, M the homogeneous maximal ideal of* $\mathrm{SR}(F,\Sigma)$. *Then $d-1=\alpha(\Sigma)$.*

Corollary 4.4 is a special case of the following.

THEOREM 4.5 (*Stanley* **[19]**). *Let Σ be any complex and $d =$* $\operatorname{depth}_M\mathrm{SR}(F,\Sigma)$, *$M$ the homogeneous maximal ideal of* $\mathrm{SR}(F,\Sigma)$. *Then $d-1=\alpha(\Sigma)$*

Theorem 4.5 may be used in the proof of the following.

PROPOSITION 4.6. *With notation as above and Σ any complex the following are equivalent.*

(a) $\operatorname{depth}_M\mathrm{SR}(F,\Sigma) = \operatorname{depth}\mathrm{SR}(F,\Sigma)_M$.

(b) $\mathrm{pd}_S \mathrm{SR}(F,\Sigma) = \mathrm{pd}_{S'}(S' \otimes_S \mathrm{SR}(F,\Sigma))$, $S' = S_{\mathbf{m}}$.

Proof.

$$\begin{aligned}
\mathrm{depth}_M \mathrm{SR}(F, \Sigma &= \alpha(\Sigma) + 1, \quad \text{by Theorem 4.5,}\\
&= n + 1 - \mathrm{pd}_S \mathrm{SR}(F,\Sigma), \quad \text{by Hochster's formula (see p. 114 [13]),}\\
&\leq \mathrm{depth}\, S_{\mathbf{m}} - \mathrm{pd}_{S'}(S' \otimes \mathrm{SR}(F,\Sigma)), \quad \text{by [12] (Lemma 5, p. 129),}\\
&= \mathrm{depth}_{S'_{\mathbf{m}}}(S' \otimes \mathrm{SR}(F,\Sigma)), \text{ by the Auslander-Buchsbaum Theorem,}\\
&= \mathrm{depth}\, \mathrm{SR}(F,\Sigma)_M, \quad \text{by definitions of localization.}
\end{aligned}$$

The result follows.

Note 4.7. In case Σ is pure, Corollary 3.7 implies that

$$\mathrm{depth}_M \mathrm{SR}(F,\Sigma) = \mathrm{depth}\, \mathrm{SR}(F,\Sigma)_M.$$

Now Proposition 4.2 is a consequence of Proposition 4.6.

Techniques developed by Munkres allow a generalization of Theorem 4.3. The assumption "Σ pure" can be eliminated from the hypothesis.

THEOREM 4.8. *Let Σ be a complex and F be a field. Let $\alpha = \alpha(\Sigma)$ be as above. Then*

(a) *Σ^α is a* CM(F) *subcomplex.*

(b) *Σ^j is not* CM(F) *for each j with $N \geq j > \alpha$, i.e. Σ^α is a maximal* CM(F) *skeleton.*

First prove an easy result.

LEMMA 4.9. *Let Σ be a complex, $0 \leq i \leq \dim \Sigma$ and $\sigma \in \Sigma^i$. Then* $\mathrm{link}(\sigma, \Sigma^i) = \mathrm{link}(\sigma, \Sigma)^{i - \dim \sigma - 1}$.

Proof. Take $\tau \in \mathrm{link}(\sigma, \Sigma^i)$. Then $\tau \cup \sigma \in \Sigma^i$ and $\tau \cap \sigma = \varnothing$. Thus $\dim \tau + \dim \sigma = \dim(\tau \cup \sigma) - 1 \leq i - 1$. So $\dim \tau \leq i - \dim \sigma - 1$, and $\tau \in \mathrm{link}(\sigma, \Sigma)^{i - \dim \sigma - 1}$.

Take $\tau \in \mathrm{link}(\sigma, \Sigma)^{i - \dim \sigma - 1}$. Then $\dim \tau \leq i - \dim \sigma - 1, \sigma \cup \tau \in \Sigma$, and $\sigma \cap \tau = \varnothing$. So $\dim(\tau \cup \sigma) - 1 = \dim \tau + \dim \sigma \leq i - 1$ and $\tau \cup \sigma \in \Sigma^i$. Thus $\tau \in \mathrm{link}(\sigma, \Sigma^i)$.

Proof of Theorem 4.8. Let Σ be a complex, $X = |\Sigma|$ and singular cohomology groups $H^i(X)$ be defined with coefficients in F. By **[13]**

(Theorem 3.1) $\sigma(\Sigma) = \alpha$ is the smallest integer j such that at least one of $\tilde{H}^j(X)$ or $\{H^j(X, X - p) | p \in X\}$ is non-trivial.

Record the fact (1) for $0 < i \leq \dim \Sigma, H^j(\Sigma) \cong H^j(\Sigma^i)$ whenever $j < i$ (see Prop. 3.1, p. 166 **[2]**).

For the proof of (a) let $X^\alpha = |\Sigma^\alpha|$. By **[13]** (Corollary 3.4) it suffices to prove $\tilde{H}^j(X^\alpha) = H^j(X^\alpha, X^\alpha - p) = 0$ for all $j < \alpha$ and all $p \in X^\alpha$. But (2) $\tilde{H}^j(X^\alpha) = \tilde{H}^j(X) = 0$ for $j < \alpha$ by (1) and the definition of α.

Let $p \in X^\alpha$ and choose $\sigma \in \Sigma^\alpha$ with $p \in \operatorname{Int}|\sigma|$, the interior of $|\sigma|$. Then

$$\begin{aligned} &H^j(X^\alpha, X^\alpha - p) \\ &\quad \cong \tilde{H}^{j-\dim - \sigma - 1}(\operatorname{link}(\sigma, \Sigma^\alpha)), \quad \text{by } \textbf{[13]} \text{ (Lemma 3.3)}, \\ &\quad = \tilde{H}^{j-\dim \sigma - 1}(\operatorname{link}(\sigma, \Sigma)^{\alpha - \dim \sigma - 1}), \quad \text{by Lemma 4.9}, \\ &\quad \cong \tilde{H}^{j-\dim \sigma - 1}(\operatorname{link}(\sigma, \Sigma)), \quad \text{by (1) for } j < \alpha, \\ &\quad \cong H^j(X, X - p), \quad \text{by } \textbf{[13]} \text{ (Lemma 3.3)}, \\ &\quad \cong 0 \text{ by definition of } \alpha \text{ for all } j < \alpha. \end{aligned}$$

So (3) for all $p \in X^\alpha$ and for all $j < \alpha, H^j(X^\alpha, X^\alpha - p) = 0$. (2), (3) and **[13]** (Corollary 3.4) finish the proof of (a).

To prove (b), note by Proposition 3.6 if any Σ^i is CM(F), then Σ^j is CM(F) for all $j \leq i$. Thus it suffices to prove (4) $\Sigma^{\alpha+1}$ is not CM(F).

Assume by way of contradiction $\Sigma^{\alpha+1}$ is CM(F). By **[13]** (Corollary 3.4) $\tilde{H}^j(\Sigma^{\alpha+1}) = 0$ for all $j < \alpha + 1$ and $H^j(\Sigma^{\alpha+1}, \Sigma^{\alpha+1} - p) = 0$ for all $p \in X^{\alpha+1}$ and for all $j < \alpha + 1$.

As before, (5) $0 = \tilde{H}^j(\Sigma^{\alpha+1}) \cong \tilde{H}^j(\Sigma)$ for all $j < \alpha + 1$. For $p \in X^{\alpha+1}, \sigma \in \Sigma^{\alpha+1}, p \in \operatorname{Int}|\sigma|$, and for all $j < \alpha + 1$,

$$\begin{aligned} 0 = H^j(X^{\alpha+1}, X^{\alpha+1} - p) &\cong \tilde{H}^{j-\dim \sigma - 1}(\operatorname{link}(\sigma, \Sigma^{\alpha+1})) \\ = \tilde{H}^{j-\dim \sigma - 1}&(\operatorname{link}(\sigma, \Sigma)^{\alpha+1-\dim \sigma - 1}) \\ \cong \tilde{H}^{j-\dim \sigma - 1}&(\operatorname{link}(\sigma, \Sigma)) \cong H^j(X, X - p). \end{aligned}$$

So (6) for all $p \in X^{\alpha+1}$, for all $j < \alpha + 1, H^j(X, X - p) = 0$. Furthermore (7) for all $p \in X - X^{\alpha+1}$, with $\sigma \in \Sigma - \Sigma^{\alpha+1}$ and $p \in \operatorname{Int}|\sigma|, H^j(X, X - p) \cong \tilde{H}^{j-\dim \sigma - 1}(\operatorname{link}(\sigma, \Sigma)) = 0$ for all $j < \alpha + 1$. (5), (6) and (7) contradict the definition of α. This finishes the proof.

THEOREM 4.10. *$P(F, \beta, \Sigma)$ is a topological invariant for finite simplicial complexes.*

Proof. Assume Σ satisfies $P(F, \beta, \Sigma)$. Then $\Sigma^{\beta-1}$ is CM(F) and for all j with $j > \beta - 1, \Sigma^j$ is not CM(F). But replacing β by $\alpha = \alpha(\Sigma)$ in the last two sentences, conclude by means of Theorem 4.8 that $\beta = \alpha$. The proof is done by **[13]** (Theorem 3.1).

5. The notion of regularity has topological consequences–a beginning. Let $(\mathscr{A}, X)$ be as in §2 a basic pair. Let $\langle r_1, \ldots, r_{n-1}\rangle$ be the usual ranked height sequence of $R = \Gamma(\mathscr{A})$ and consider the subsequence $\langle r_1, r_2\rangle$. In case $\langle r_1, r_2\rangle$ is not regular one can ask whether one can find an obstruction in $X = X(\Sigma)$ preventing regularity. Theorem 5.2 below will answer this question. Now ask in case $n > 3$ whether, given $\langle r_1, r_2\rangle$ is regular, an obstruction in X preventing $\langle r_1, r_2, r_3\rangle$ from being regular can be found. This question seems difficult to answer.

First a general lemma:

LEMMA 5.1. *Let X be a ranked poset with rank $X \geq 2, \mathscr{A}$ be a sheaf of rings on X. Suppose*

(1) *For all $x \in X$ with* $\mathrm{rk}(x) > 2, X^x = \{z \in X | z < x\}$ *is connected and*

(2) *There exists $b' \in \Pi\{A_x | \mathrm{rk}(x) = 1\}$ (the Cartesian product), such that whenever $\mathrm{rk}(x) = \mathrm{rk}(x') = 1$ and y covers x and x' then $\rho_{yx}b'(x) = \rho_{yx'}b'(x')$. Then there is an element $b \in R = \Gamma(\mathscr{A})$ with $b(x) = b'(x)$ for all x of rank* 1.

Proof. b is constructed by induction on x. Let $\mathscr{P}(k)$ be the proposition defined for all positive integers k by "$b(w)$ is defined for all $w \in X$ such that $\mathrm{rk}(w) \leq k$ and for $u, u' \leq w$ such that $\mathrm{rk}(w) \leq k, \rho_{wu}b(u) = \rho_{wu'}b(u')$." In other words $\mathscr{P}(k)$ says that b is defined up to the kth rank.

For the base step let $k = 2$ and prove $\mathscr{P}(2)$. Let $b(x) = b'(x)$ for ank $x = 1$, and for y of rank 2 with $y > x$, let $b(y) = \rho_{yx}b(x)$. By (2), $b(y)$ is well defined for y of rank 2. $\mathscr{P}(2)$ is proved.

For the induction step assume $\mathscr{P}(k-1)$ is true and prove $\mathscr{P}(k)$ as follows: Let $\mathrm{rk}(w) = k$ and consider $Y = X^w$ which is connected by (1). First consider x, x' atoms of Y. There is a path in Y from x to x':

(3)
$$\begin{array}{ccccccccc} & y_1 & & y_2 & & \cdots & y_{n-1} & & \\ x = x_1 \nearrow & & \searrow x_2 \nearrow & & \searrow x_3 & \cdots & & \searrow & x' = x_n \end{array}$$

By running a chain up to w from each y_i, it may be assumed each y_i has rank $k-1$. In order to show $\rho_{wx}b(x) = \rho_{wx'}b(x')$ it suffices to show by (3) that $\rho_{wx}b(x) = \rho_{wx_2}b(x_2)$. Now $\rho_{y_1x}b(x) = \rho_{y_1x_2}b(x_2)$

by $\mathscr{P}(k-1)$. So (4) $\rho_{wx}b(x) = \rho_{wy_1}\rho_{y_1x}b(x) = \rho_{wy_1}\rho_{y_1x_2}b(x_2) = \rho_{wx_2}b(x_2)$.

Now consider any $x, x' \in Y$ and prove $\rho_{wx}b(x) = \rho_{wx'}b(x')$. Let $u \leq x$ and $u' \leq x'$ where u and u' are atoms of Y. Using (4) and $\mathscr{P}(k-1)$,

$$\begin{aligned}\rho_{wx}b(x) &= \rho_{wx}\rho_{xu}b(u) = \rho_{wu}b(u) = \rho_{wu'}b(u') \\ &= \rho_{wx'}\rho_{x'u'}b(u') = \rho_{wx'}b(x').\end{aligned}$$

So $b(w)$ is well defined and the induction step is finished.

THEOREM 5.2. *Let $(\mathscr{A}, X)$ be a basic pair with X a prelattice of rank greater than 2 and $R = \Gamma(\mathscr{A})$. The following are equivalent.*

(a) *X^x is connected for all x of rank bigger than 2.*

(b) *$\langle r_1, r_2 \rangle$ is regular.*

Proof. First prove (b) implies (a) by contraposition. Suppose X^z is not connected for some $z \in X, \mathrm{rk}(z) > 2$. Consider the localization R_{P_z}, and let $\varphi\colon R \to R_{P_z}, r \to [r/1]$ be the standard homomorphism. As $r_1, r_2 \in P_z$, assuming $\langle r_1, r_2 \rangle$ is regular then implies $\langle [r_1/1], [r_2/1] \rangle$ is regular in R_{P_z}. Using the fact that $R_{P_z} \cong \Gamma(\mathscr{B})$ where $\mathscr{B}$ is a sheaf of rings over the disconnected poset X^z this contradicts **[21]** (Proposition 6.1). One is forced to conclude $\langle r_1, r_2 \rangle$ is not regular.

Now assume (a). Let $\overline{R}$ denote the R-module $\overline{R} = R/(r_1)$.

Claim. $\mathrm{Ass}_R\overline{R} = \{P_y | \mathrm{rk}(y) = 2\}$.

Proof of Claim. First suppose $\mathrm{rk}(y) = 2$. Then P_y is a minimal over-prime of $P(2)$ as in the proof of Theorem 1.5. But by Proposition 1.4, $P(2) = \mathrm{Rad}(r_1)$. It follows that P_y is a minimal over-prime of (r_1). But such ideals are in $\mathrm{Ass}_R\overline{R}$. Thus the left hand side contains the right hand side.

Next argue by contradiction and suppose $P \in \mathrm{Ass}_R\overline{R}$ but $P \neq P_y$ for each y of rank 2. By minimality of P_y in $\mathrm{Ass}_R\overline{R}$, for all y of rank 2, $P \not\subseteq P_y$. Supposing $P \subseteq \bigcup\{P_y | \mathrm{rk}(y) = 2\}$, then by a basic fact, $P \subseteq P_y$ for some y of rank 2. This contradiction establishes the existence of an $s \in P - \bigcup\{P_y | \mathrm{rk}(y) = 2\}$. Say $P = \mathrm{Ann}(r + (r_1))$, for some $r \in R$. Then $sr \in (r_1)$ so (1)$sr = cr_1$, some $c \in R$. By (1) for each y of rank 2, $s(y)r(y) = 0$. Thus $r(y) = 0$ all y of rank 2 (by choice of s). So $r \in P(2)$ and thus for each x of rank 1, $r(x) = b'(x)r_1(x)$ where

$b'(x) \in A_x$. From (1) it follows that $s(x)b'(x) = c(x)$ for each x of rank 1. Now let y be of rank 2 and cover x and x'. Then

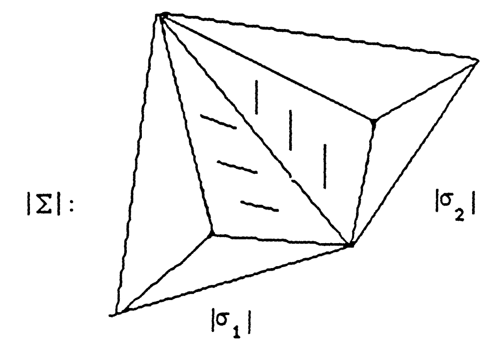
$$\begin{aligned} s(y)\rho_{yx}(b'(x)) &= \rho_{yx}s(x)\rho_{yx}b'(x) = \rho_{yx}(s(x)b'(x)) \\ &= \rho_{yx}c(x) = \rho_{yx'}c(x') = \rho_{yx}(s(x')b'(x')) \\ &= \rho_{yx'}s(x')\rho_{yx'}b'(x') = s(y)\rho_{yx'}b'(x'). \end{aligned}$$

Using the fact $s(y) \neq 0, \rho_{yx}b'(x) = \rho_{yx'}b'(x')$. This shows condition (2) of Lemma 5.1 is satisfied. As condition (1) is hypothesis, conclude by Lemma 5.1 that b' defines an element $b \in R$ such that $b(x) = b'(x)$ for all x of rank 1. But then from the definition of $b, r = br_1$ and then $r \in (r_1)$. This contradicts the choice of P and the Claim is proved.

By a basic result, $\mathrm{Ass}_R\overline{R}$ is a union of the zero divisors of $\overline{R}$. Whereas $r_2 \notin P_y$ for each y of rank 2, by the Claim $r(2)$ is not a zero divisor of the R-module $\overline{R}$. So $r_2 + (r_1)$ is not a zero divisor of the ring $\overline{R}$. So $\langle r_1, r_2\rangle$ is regular. The proof is complete.

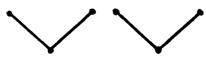
Note 5.3. Given $(\mathscr{A}, X)$ is basic with X a prelattice of rank bigger than two and $R = \Gamma(\mathscr{A})$. Assuming R is local with depth $R > 1$, does $\langle r_1, r_2\rangle$ have to be regular? The example here constructed shows the answer to the question is no.

Let Σ consist of two tetrahedra σ_1 and σ_2 joined on a common 1-simplex.



If F is a field and M is the homogeneous maximal ideal of $\mathrm{SR}(F, \Sigma)$, then depth $\mathrm{SR}(F, \Sigma)_M = 3$. The reason for this is that $\Sigma = \Sigma^3$ is not $\mathrm{CM}(F)$ for $\Delta = \mathrm{link}(\sigma_1 \cap \sigma_2, \Sigma)$ is of dimension 1 but not connected so $\tilde{H}^0(\Delta, F) \neq 0$. To show Σ^2 is $\mathrm{CM}(F)$ note if $\tau \in \Sigma^2$, then $\mathrm{link}(\tau, \Sigma^2)$ is empty or of dimension 0 or a complex of dimension 1 and in all cases an F-bouquet of spheres. So conclude that Σ^2 is a maximal $\mathrm{CM}(F)$ skeleton and by Theorem 3.7 and Corollary 3.8 depth $\mathrm{SR}(F, \Sigma)_M = 3$.

Let $\sigma = \sigma_1 \cap \sigma_2$. The rank of σ in the poset X is 3. X^σ is:

X^σ 

Obviously X^σ is not connected. This shows (a) and thus (b) of Theorem 5.2 is not satisfied.

Next a general Lemma of some usefulness.

LEMMA 5.4. *Let X be a prelattice. If $x \in X$ is not a join of atoms, then X^x is contractible.*

Proof. Suppose $x \in X$ is not a join of atoms. Let $A = \{z \in X | z$ is an atom and $z < x\}$. Then $z_0 = \bigvee\{z | z \in A\}$, exists and $z_0 < x$ by assumption so $z_0 \in X^x$.

Now let $y \in X^x$. If z is an atom of X and $z < y$ then $z \in A$ by transitivity of $\leq$. So $u(y) = \bigvee\{z | z$ an atom and $z \leq y\}$ exists in $X^x, u(y) \leq z_0$ and $u(y) \leq y$. Now define a function $f\colon X^x \to X^x$ by $y \to u(y)$. It is clear f is a poset map. Now for all $y \in X^x$, $y \geq f(y) \leq z_0$. Thus the identity map $|\mathrm{Id}|\colon |X^x| \to |X^x|$ is homotopic to the constant map $|z_0|\colon |X^x| \to |X^x|$ (see p. 103 **[15]**). Thus X^x is contractible.

In the following proposition $R = \Gamma(\mathscr{A})$ is the section ring of the sheaf of polynomial rings over the poset as in §3. Elements r_1 and r_2 are from the usual ranked height sequence of $\Gamma(\mathscr{A})$.

PROPOSITION 5.5. *Let Σ be a simplicial complex of dimension $N > 0$ and F be a field. The following are equivalent.*

(a) *Σ is pure and $\langle r_1 r_2\rangle$ is regular.*

(b) $\operatorname{link}(\sigma, \Sigma)$ *is connected for all $\sigma \in \Sigma$ such that* $\operatorname{link}(\sigma, \Sigma) > 0$.

(c) $\operatorname{link}(\sigma, \Sigma)$ *is connected for all $\sigma \in \Sigma$ such that σ is the intersection of a set of maximal simplices and* $\dim \operatorname{link}(\sigma, \Sigma) > 0$.

Proof. (c) implies (b) for if $\sigma \in \Sigma$, by Lemma 5.4 it suffices to consider σ a join of atoms, i.e. an intersection of maximal simplices. (b) trivially implies (c).

Assume (a). Choose $\sigma \in \Sigma$ with $\dim \operatorname{link}(\sigma, \Sigma) > 0$. Then $\dim \sigma \leq N - 2$ by the formula $\dim \sigma + \dim \operatorname{link}(\sigma, \Sigma) = N - 1$. So $\operatorname{rk}(\sigma) > 2$ in X and X^σ is connected by Theorem 5.2. But $X^\sigma \cong X(\operatorname{link}(\sigma, \Sigma)) - \varnothing$ (poset isomorphism). Thus $\operatorname{link}(\sigma, \Sigma)$ is connected. This proves (b).

Assume (b). Σ is pure by Proposition 3.4. Consider $\sigma \in \Sigma$ with $\operatorname{rk}(\sigma) > 2$, i.e. $\dim \sigma \leq N - 2$. By the dimension formula above,

$\dim \operatorname{link}(\sigma, \Sigma) > 0$. $\operatorname{Link}(\sigma, \Sigma)$ is connected by hypothesis so $X^\sigma \cong X(\operatorname{link}(\sigma, \Sigma)) - \emptyset$ is connected. σ is arbitrary so Theorem 5.2 finishes the argument. This proves (a).

6. Explicit regular sequences in Stanley-Reisner rings. The following lemmas are known and are recorded here for sake of completeness (see p. 103, Exercise 14 **[9]**).

LEMMA 6.1. *Let R be a ring and $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. Suppose a given sequence $\langle r_1, \ldots, r_n \rangle$ in R is a C-regular sequence. Then $0 \to A/(r, \ldots, r_n)A \to B/(r_1, \ldots, r_n)B \to C/(r_1, \ldots, r_n)C \to 0$ is an exact sequence of R-modules.*

LEMMA 6.2. *With the same assumptions as in Lemma 6.1 and in addition $\langle r_1, \ldots, r_n \rangle$ is B-regular, then $\langle r_1, \ldots, r_n \rangle$ is A-regular.*

LEMMA 6.3. *Given the exact sequence of R-modules $0 \to A \to B \to C \to 0$ and given the sequence $\langle r_1, \ldots, r_n \rangle$ of R which is A-regular and C-regular, then $\langle r_1, \ldots, r_n \rangle$ is B-regular.*

Let R and $\overline{R}$ be rings with a ring homomorphism $\varphi: R \to \overline{R}$. The following theorem uses Lemma 6.3 to show how to pull back a regular sequence in $\overline{R}$ to R in the special case where $\overline{R}$ and R are section rings with φ being the restriction map.

THEOREM 6.4. *Let $(\mathscr{A}, X)$ be basic with* $\operatorname{rank} X = n$ *and $\langle r_1, \ldots, r_{n-1} \rangle$ the usual ranked height sequence for $R = \Gamma(\mathscr{A})$. Let m be an integer, $2 \leq m < n$ $(= \operatorname{rank} X)$ and assume A_x is CM for all $x \in X$ with* $\operatorname{rk}(x) \leq m - 1$. *Assume either*

(a): *A_x is local for all x with* $\operatorname{rk}(x) \leq m - 1$ *or*

(b): *A_x is a non-negatively $\mathscr{Z}$-graded ring and each $\rho_x(r_i)$ is homogeneous of positive degree for each i and all x with* $\operatorname{rk}(x) \leq m - 1$. *(Recall $\rho_x: \Gamma(\mathscr{A}) \to \Gamma(\mathscr{A} | \overline{X}_x) \cong A_x$.)*

Finally let $X(m-1) = X - \{x \in X | \operatorname{rk}(x) \leq m-1\}$, and let $r \to \overline{r}$ denote the restriction map $\Gamma(\mathscr{A}) = R \to \Gamma(\mathscr{A} | X(m-1)) = \overline{R}$. If $(*)$ $U = \langle \overline{r}_m, \ldots, \overline{r}_{n-1} \rangle$ is $\overline{R}$-regular, U the ranked height sequence of $\overline{R}$, then $\langle r_m, \ldots, r_{n-1} \rangle$ is R-regular.

Proof. First prove the statement in the special case $m = 2$. By Lemma 3.5 $\Gamma(\mathscr{A} | X(1)) = \overline{R} \cong R/P(2)$ and regarding $\overline{R}$ as an R-module via projection $R \to \overline{R}$ note that (1): $\langle r_2, \ldots, r_{n-1} \rangle$ is $R/P(2)$-regular.

Consider the exact sequence of R-modules $0 \to P(2) \to R \to R/P(2) \to 0$. Supposing $\langle r_2, \ldots, r_{n-1} \rangle$ is $P(2)$-regular then using (1), Lemma 6.3 would finish off the argument.

Claim: $\langle r_2, \ldots, r_{n-1} \rangle$ is $P(2)$-regular.

Proof of Claim. $P(2) \cong \bigoplus\{A_x | \mathrm{rk}(x) = 1\}$, as R-modules. So it suffices to prove for each x of rank 1 that $\langle r_2, \ldots, r_{n-1} \rangle$ is A_x-regular where the R-action on A_x is given by restriction. So fix x of rank 1. Then A_x is CM by assumption and $\langle \rho_x r_1, \ldots, \rho_x r_{n-1} \rangle$ is the ranked height sequence of $\Gamma(\mathscr{A} | \overline{X}_x) \cong A_X$ so regular by Lemma 1.7. In case (a) or in case (b), considering the local ring case or the graded ring case, $\langle \rho_x r_1, \ldots, \rho_x r_{n-1} \rangle$ is quasi-regular (see p. 98, **[12]** with thanks to Marie Vitulli), so that $\langle \rho_x r_2, \ldots, \rho_x r_{n-1} \rangle$ is $\Gamma(\mathscr{A} | \overline{X}_x)$-regular. Taking into account the R-action, $\langle r_2, \ldots, r_{n-1} \rangle$ is A_x-regular. This is what was needed for the Claim and completes the proof of the special case.

Now consider m, $1 \leq m \leq n$. Let $\langle r_{m-1}, \ldots, r_{n-1} \rangle$ be the ranked height sequence of $\Gamma(\mathscr{A} | X(m-2))$ where $X(m-1) = X - \{x \in X | \mathrm{rk}(x) \leq m-2\}$ and let $\langle \overline{r}_m, \ldots, \overline{r}_{n-1} \rangle$ be the ranked height sequence of $\overline{R} = \Gamma(\mathscr{A} | X(m-1))$ and let f be the restriction map, $f \colon \Gamma(\mathscr{A} | X(m-2)) \to \Gamma(\mathscr{A} | X(m-1))$, $r \to \overline{r}$. As in the hypothesis suppose $\langle \overline{r}_m, \ldots, \overline{r}_{n-1} \rangle$ is $\overline{R}$-regular. Then claim: (2) $\langle r_m, \ldots, r_{n-1} \rangle$ is $\Gamma(\mathscr{A} | X(m-2))$-regular. For consider the exact sequence of R-modules (R-action via restriction) $0 \to K \to \Gamma(\mathscr{A} | X(m-2)) \to \Gamma(\mathscr{A} | X(m-1)) \to 0$ where $K \cong \bigoplus\{A_x | \mathrm{rk}(x) = m-1\}$. By precisely the same argument as for the Claim, $\langle r_m, \ldots, r_{n-1} \rangle$ is K-regular and Lemma 6.3 yields the result (2). By an inductive assumption for integers smaller than $m-2$ the proof is complete.

COROLLARY 6.5. *Let Σ be a pure complex of dimension N and let $d = \operatorname{depth} \mathrm{SR}(F, \Sigma)_M$, M the homogeneous maximal ideal. Then $U = \langle r_{N-d+2}, \ldots, r_{N+1} \rangle$, the subsequence of the ranked height sequence for $\Gamma(A) \cong \mathrm{SR}(F, \Sigma)$, is $\Gamma(\mathscr{A})$-regular.*

Proof. Let Σ be a pure complex with $d = \operatorname{depth} \mathrm{SR}(F, \Sigma)_M$, M the homogeneous maximal ideal. Let $(\mathscr{A}, X)$ be the usual pair with

$$R = \Gamma(\mathscr{A}) \cong \mathrm{SR}(F, \Sigma), \qquad \overline{R} = \Gamma(\mathscr{A} | X(N-d+1)) \cong \mathrm{SR}(F, \Sigma^{d-1})$$

where

$$X(N-d+1) = X - \{\sigma \in \Sigma | \mathrm{rk}(\sigma) \leq N-d+1\} = X(\Sigma^{d-1}).$$

Let $r \to \overline{r}$ denote the restriction *map*, $R \to \overline{R}$. Let $m = N + d + 2$.

The hypotheses of Theorem 6.4 need to be satisfied: $(\mathscr{A}, X)$ is a basic pair and $A_\sigma = F[Z]_{Z \in \sigma}$ is CM for all $\sigma \in \Sigma$. To see that (b) holds take $r_i \in \{r_1, \ldots, r_{N+1}\}$ and let $\alpha \circ \rho_\sigma \colon R \to \Gamma(\mathscr{A}|\overline{X}_\sigma) \cong A_\sigma$ be the restriction map followed by the isomorphism α. Then by the Note after Lemma 3.2 $\langle r_1, \ldots, r_{N+1}\rangle$ can be chosen so that

$$\alpha \circ \rho_\sigma r_i = \begin{cases} 0, & \text{if } \operatorname{rk}(\sigma) > i, \\ \text{a homogeneous sum of monomials,} & \text{if } \operatorname{rk}(\sigma) \leq i. \end{cases}$$

It remains to show $(*)$ of the Theorem holds, i.e. $\langle \overline{r}_{N-d+2}, \ldots, \overline{r}_{N+1}\rangle$ is $\overline{R}$-regular. Note the sequence is regular following [2]. Here is another proof. By Theorem 3.7 and Corollary 3.8, Σ^{d-1} is $\mathrm{CF}(F)$. By Reisner's Theorem, $\mathrm{SR}(F, \Sigma^{d-1})$ is CM. By Theorem 1.5 and Lemma 1.7 (whereas restriction takes a ranked height sequence to likewise), $\langle \overline{r}_{N-d+2}, \ldots, \overline{r}_{N+1}\rangle$ is $\overline{R}$-regular.

Apply the conclusion of the Theorem with $m = N - d + 2$ and conclude $U = \langle r_{N-d+2}, \ldots, r_{N+1}\rangle$ is regular. The proof is complete.

Note that U is of length d so that by Corollary 3.8,

$$d \leq \operatorname{depth}_M \mathrm{SR}(F, \Sigma) = \operatorname{depth} \mathrm{SR}(F, \Sigma)_M = d$$

and it can be concluded (using the isomorphism $\Gamma(\mathscr{A}) \cong \mathrm{SR}(F, \Sigma)$) that U is a maximal regular sequence within the "homogeneous maximal ideal" of $\Gamma(\mathscr{A})$.

REMARK 6.6. In the body of the last proof, reference is made to Baclawski and Garsia [2]. By choice of $\langle r_1, \ldots, r_{N+1}\rangle$ in the note after Lemma 3.2, $\phi\colon \mathrm{SR}(F, \Sigma) \to \Gamma(\mathscr{A})$ has the property that $\phi(a_j) = r_j$ for $j = 1, \ldots, N + 1$, with each a_j homogeneous. There is an induced isomorphism of rings

$$\overline{\phi}\colon \mathrm{SR}(F, \Sigma^{d-1}) \to \overline{R} = \Gamma(\mathscr{A}|X(N - d + 1)).$$

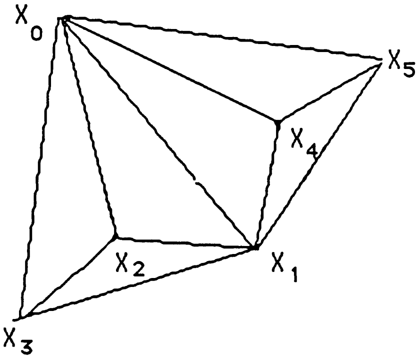
It happens that $\overline{\phi}(\overline{a}_j) = \overline{r}_j$ where for $j = N - d + 2, \ldots, N + 1$

$$\overline{a}_j = \sum_{\{x_{i(1)}, \ldots, x_{i(N-j+2)}\} \in \Sigma} X_{i(1)} \cdots X_{i(N-j+2)} + I_{\Sigma^{d-1}}.$$

In terms of [2], $\langle \overline{a}_{N-d+2}, \ldots, \overline{a}_{N+1}\rangle$ is a *frame* of $\mathrm{SR}(F, \Sigma^{d-1})$, i.e. $\dim \mathrm{SR}(F, \Sigma^{d-1})/(\overline{r}_{N-d+2}, \ldots, \overline{r}_{N+1}) = 0$. By a result of [2] (Prop. 2.3, p. 162) because $\mathrm{SR}(F, \Sigma^{d-1})$ is CM it follows that $\langle \overline{a}_{N-d+2}, \ldots, \overline{a}_{N+1}\rangle$ is regular. One then has a different proof that $\langle \overline{r}_{N-d+2}, \ldots, \overline{r}_{N+1}\rangle$ is $\overline{R}$-regular.

EXAMPLE 6.7. The following example (see 5.3) helps to justify the title of §6.

$|\Sigma|$:



Define $\Sigma = \Sigma^3$ so that $|\Sigma|$ is as above. Before it was demonstrated that Σ^2 is a maximal CM(F) skeleton for any F; hence $\operatorname{depth}_M \mathrm{SR}(F, \Sigma) = 3$. As in Remark 6.6, a maximal regular sequence of $\mathrm{SR}(F, \Sigma^2)$ is given by $\langle \overline{a}_2, \overline{a}_3, \overline{a}_4 \rangle$ where

$$\begin{aligned} \overline{a}_2 &= \sum_{\{x_i, x_j, x_k\} \in \Sigma} X_i X_j X_k + I_{\Sigma^2}, \\ \overline{a}_3 &= \sum_{\{x_i, x_j\} \in \Sigma} X_i X_j + I_{\Sigma^2}, \quad \text{and} \\ \overline{a}_4 &= \sum_{\{x\} \in \Sigma} X + I_{\Sigma^2}. \end{aligned}$$

Corollary 6.5 shows that $\langle a_2, a_3, a_4 \rangle$ is a maximal regular sequence of $\mathrm{SR}(F, \Sigma)$ where

$$a_2 = \sum_{\{x_i, x_j, x_k\} \in \Sigma} X_i X_j X_k + I_\Sigma, \qquad a_3 = \sum_{\{x_i, x_j\} \in \Sigma} X_i X_j + I_\Sigma,$$
$$a_4 = \sum_{\{x\} \in \Sigma} X + I_\Sigma.$$

One has of course that the natural projection $\nu\colon \mathrm{SR}(F, \Sigma) \to \mathrm{SR}(F, \Sigma^2)$ is defined so that for $i = 2, 3, 4$ that $\nu(a_i) = \overline{a}_i$.

LEMMA 6.7 (*DeConcini, et al.* **[4]**). *Suppose* $0 \to A \to B \to C \to 0$ *is an exact sequence of R-modules and* $\langle r_1, \ldots, r_n \rangle$ *is a sequence of ring elements such that* $r_1 C = 0$. *If* $\langle r_1, \ldots, r_n \rangle$ *is B-regular and* $\langle r_2, \ldots, r_n \rangle$ *is C-regular, then* $\langle r_1, \ldots, r_n \rangle$ *is A-regular.*

The lemma can be used to prove one implication of the following.

THEOREM 6.8. *Let* $(\mathscr{A}, X)$ *be a basic pair with* $\operatorname{rank} X = n, X$ *be a prelattice and* R *be CM and local with* $\dim R = n-1$. *Then* $R/P(2)$ *is CM if and only if* A_x *is CM for all* x *of rank* 1.

Proof. Assume A_x is CM all x of rank 1 along with the other hypotheses. X has a unique maximal element m of rank n (Cor. 2.3, p. 180 [**21**]) and $\operatorname{ht} P_m = n-1$ as in the definition of a basic pair. So P_m is the unique maximal ideal of R. Theorem 2.2 gives depth $R/P(2) = n-2$. Whereas $\dim R/P(2) = n-2$ by the equation $\operatorname{ht} P(2) + \dim R/P(2) = \dim R$ (see p. 108, Theorem 31, [**12**]) and the fact that $\operatorname{ht} P(2) = 1$, the argument is finished.

Now assume the hypotheses plus the condition that $R/P(2)$ is CM. Consider the exact sequence of R-modules, with R-action gotten by restriction, $0 \to P(2) \to \Gamma(\mathscr{A}) = R \to R/P(2) \to 0$. Since R is CM Theorem 1.5 and Lemma 1.7 imply that $\langle r_1, \ldots, r_{n-1}\rangle$ is R-regular. Note the R-module isomorphism $R/P(2) \cong \Gamma(\mathscr{A}|X(1))$, so $r_1(R/P(2)) = 0$. Also $\langle \bar{r}_2, \ldots, \bar{r}_{n-1}\rangle$ is the ranked height sequence of $\Gamma(\mathscr{A}|X(1))$ where restriction is given by $R \to \Gamma(\mathscr{A}|X(1))$, $r \to \bar{r}$. Since $\Gamma(\mathscr{A}|X(1))$ is a CM ring, as above $\langle \bar{r}_2, \ldots, \bar{r}_{n-1}\rangle$ is $\Gamma(\mathscr{A}|X(1))$-regular. But then considering R-action, $\langle r_2, \ldots, r_{n-1}\rangle$ is $R/P(2)$-regular. By Lemma 6.6, (1) $\langle r_1, \ldots, r_{n-1}\rangle$ is $P(2)$-regular.

Recalling $P(2) \cong \{\bigoplus A_x | \operatorname{rk}(x) = 1\}$, use (1) to conclude $\langle r_1, \ldots, r_{n-1}\rangle$ is A_x-regular for each x of rank 1. Considering the R-action on each A_x is given by restriction, then $\langle r_1(x), \ldots, r_{n-1}(x)\rangle$ is A_x-regular. Thus depth $A_x \geq n-1$. But $\dim A_x \leq n-1$ by the homomorphism $\rho_x \colon R \to \Gamma(\mathscr{A}|\overline{X}_x) \cong A_x$. Conclude for all x of rank 1 depth $A_x = \dim A_x = n-1$. A_x is therefore CM for all x of rank 1.

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Oakland University
Rochester, MI 48309-4401