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PSEUDOCONVEX CLASSES OF FUNCTIONS. II. AFFINE PSEUDOCONVEX CLASSES ON R^N

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A complete description of invariant pseudoconvex classes of functions on R^N which are closed with respect to addition of affine functions is given. Each such class is shown to be equal to its own bidual, and approximation results, including piecewise-smooth approximation and a counterexample to smooth approximation, are obtained. The results of the paper have applications to multivariate interpolation of normed spaces and to approximation of analytic multifunctions, which are given elsewhere.

Introduction. In this paper, which is a sequel to [**9**], we continue to explore pseudoconvex classes of functions, a notion developed to provide conceptual framework and technical background for the study of multivariate interpolation methods for families of normed and quasinormed spaces, which was undertaken in [**10**].

Here, we restrict our attention to those pseudoconvex classes on R^N which are preserved by addition of linear functions and by translations. They will be called, shortly, *affine pseudoconvex classes*; axioms (0.1)–(0.9), listed below, comprise their precise definition.

Since the most important examples of pseudoconvex classes are, in fact, affine, and in view of the clarify of the methods required to analyse the Euclidean case, it seems worthwhile to obtain detailed description of the structure of affine pseudoconvex classes on R^N. This is the purpose of this paper.

In §2 the operation of supremum-convolution from [7] is used to approximate functions of a translation-invariant pseudoconvex class by functions of the same class which have almost everywhere second-order derivatives in the Peano sense.

This makes it possible to assign to every affine pseudoconvex class a nonempty set consisting of those $N \times N$ symmetric matrices which correspond to the Hessian forms of functions of the given class. In §3 it is proved that the set so obtained is closed and preserved by addition of positive-definite matrices. It is shown that such sets of matrices are in one-to-one correspondence with affine pseudoconvex classes of functions on R^N (cf. Theorem 3.11).

Much of the difficulty in dealing with those pseudoconvex classes which are not closed with respect to addition stems from the fact that approximation by smooth functions within such class is, in general, impossible. A counterexample to smooth approximation is constructed in §6. However, approximation by "piecewise smooth" functions can be obtained, namely by functions which are locally equal to the maximum of several quadratic polynomials which belong to the given class (Theorem 4.1).

Among essential tools used in §§3 and 4 are the notion of the dual class of functions (to a given one) and the theorem that an affine pseudoconvex class is equal to its own bidual (Theorem 3.9). The latter fact is a consequence of the solution to the Dirichlet problem in abstract pseudoconvex classes, which is obtained in §1 (Theorem 1.8).

In §5 the general results of this paper are illustrated by and applied to the classical examples of q-convex, subharmonic, and q-plurisubharmonic functions. Furthermore, new pseudoconvex classes, which are invariant with respect to the group of complex linear maps, are found.

We will list now, for easy reference, axioms (0.1)–(0.9) which define an affine pseudoconvex class P on R^N.

(0.1) $P = \bigcup P(U)$, where U is an open subset of R^N, $P(U) \subset \operatorname{usc}(U) =$ the class of all upper semicontinuous functions on U with values in $[-\infty, +\infty)$.

(0.2) If $V \subset U \subset R^N$ and $u \in P(U)$, then $u|V \in P(V)$.

(0.3) If $(u_n) \subset P(U)$, $n = 1, 2, \ldots, u_n(x) \searrow u(x)$, $x \in U$, then $u \in P(U)$.

(0.4) If $(u_t)_{t \in T} \subset P(U)$, $u(x) = \sup_t u_t(x)$, and u is locally bounded on U, then its usc regularization u^* belongs to $P(U)$.

(0.5) If $u \in P(U)$ and C is a constant, then $(u + C) \in P(U)$.

(0.6) If $U \subset R^N$ is relatively compact, then $P(U)$ contains a bounded function.

(0.7) (Sheaf axiom) If $u \in \operatorname{usc}(U)$ and $U = \bigcup_t U_t$ (U_t open), then $u \in P(U)$, if and only if $u|U_t \in P(U_t)$, $t \in T$.

(0.8) If $u \in P(U)$ and $l(x)$ is an affine function, then $(u+l) \in P(U)$.

(0.9) If $u \in P(U)$, $y \in R^N$ and $u_y(x) = u(x-y)$, then $u_y \in P(U+y)$.

One can observe that the only difference between affine pseudoconvex classes and translation invariant pseudoconvex classes on R^N is in axiom (0.8). It raises a natural question whether the two notions are actually different, and we show in §7 that it is so (translation invariant classes being more general). This leads to a discussion of the localization and separation properties for an abstract pseudoconvex class which are weaker versions of (0.8).

Some auxiliary material is included in the Appendices.

1. Dirichlet problem and duality. We will show in this section (cf. Theorem 1.8 below) that a pseudoconvex class P is uniquely determined by its dual P^d, provided P satisfies some mild requirements, which hold, in particular, for all affine classes (cf. Theorem 3.9). With future uses in mind (namely **[11]**), we prove the results of this section in the more general context of pseudoconvex classes of functions, as defined in [**9**, Definition 1.4]. (A reader not interested in this generality can restrict his attention to the affine case.)

We recall that a pseudoconvex class P consists, in general, of functions on a locally compact space M and is defined by nine axioms, cf. [**9**, (1.1)–(1.9)] of which six are identical with (0.1)–(0.7) above (U, V being now subsets of M), while (0.8), (0.9) have the following counterparts:

(1.1) (*localization axiom*) if $K \subset M$ is compact, $u \in \text{usc}(K)$ and $\varepsilon > 0$, then there is $x_0 \in K$ and $\rho \in AP(\text{nbhd}\, K)$, such that $\sup_{x \in K} |\rho(x)| \leq \varepsilon$ and $(u + \rho)(x_0) > (u + \rho)(x)$, $x \in K \backslash \{x_0\}$, where $AP(V) = \{v \in \text{usc}(V) \colon (u + v) \in P(U \cap V) \text{ if } u \in P(U)\}$;

(1.2) (part (i) of the continuity axiom (1.9) in [**9**]), if $x^* \in M$, $K, L \subset M$, $\varphi \in C(K)$ and $\varepsilon > 0$ are given, where $x^* \in L \subset \text{Int}(K)$, and K, L are compact, then there is a neighborhood V of x with $V \subset K$, such that for every $u \in P(\text{nbhd}\, K)$, satisfying inequality $u(y) < \varphi(y)$, $y \in K$, and for every $x \in V$, there is $u_x \in P(\text{nbhd}\, L \cup V)$, such that

(1.3) $u_x(y) < \varphi(y)$, $y \in L \cup V$;

(1.4) $u_x(x) > u(x^*) - \varepsilon$.

Our treatment of Dirichlet problem, including the next definition, is inspired by Walsh **[13]**.

Definition 1.1. Let P be a pseudoconvex class of functions on a locally compact space M and let $G \subset M$ be an open, relatively compact set.

(a) Let $g \colon \overline{G} \to (-\infty, +\infty]$ be a function. Denote

$$E(P, g)(z) = \sup\{u(z) \colon u \in \text{usc}(\overline{G}), u|G \in P(G), u \leq g \text{ on } \overline{G}\}.$$

The function $E(P, g)$ will be called *the lower envelope of g relative to P*.

(b) If $f\colon \partial G \to R$, then $E(P, f)$ is understood as $E(P, g)$, where $g|\partial G = f$ and $g|G = +\infty$; $E(P, f)$ is called the Perron envelope of f.

REMARK 1.2. In the situation of Definition 1.1(a), if $g\colon \overline{G} \to R$ is continuous, then $u = E(P, g)$ is an usc function, such that $u|G \in P(G)$ and $u \leq g$ on $\overline{G}$. (Clearly, $u^*(z) \leq g(z)$, $z \in \overline{G}$ and $u^* \in \operatorname{usc}(\overline{G})$. By axiom (0.4), $u^*|G \in P(G)$. Thus $u = u^*$, as required.)

LEMMA 1.3. *In the situation of Definition* 1.1(a), *if $g\colon \overline{G} \to R$ is a continuous function and if the function $u(z) = E(P, g)(z)$ is continuous at every point of ∂G, then u is continuous on $\overline{G}$.*

Proof. By Remark 1.1, it remains to show that $u(z)$ is lsc (= lower semicontinuous) at every point $x^* \in G$. Fix $x^* \in G$ and $\varepsilon > 0$. Since u is continuous at all points of ∂G, one can find a compact $L \subset G$ and a continuous function $\varphi\colon \overline{G} \to R$, such that

$$u(x) < \varphi(x) < u(x) + \varepsilon, \qquad x \in \overline{G} \backslash L; \tag{1.5}$$

$$u(x) < \varphi(x) < g(x) + \varepsilon, \qquad x \in \overline{G}. \tag{1.6}$$

(Details omitted.) Without loss of generality, $x^* \in \operatorname{Int}(L)$. Choose further a compact $K \subset G$, such that $L \subset \operatorname{Int}(K)$. Applying axiom (1.2) above to the data x^*, K, L, φ we obtain a neighborhood V of x^* and functions $u_x \in P(\operatorname{nbhd} L \cup V)$ satisfying (1.3) and (1.4).

Let now $v_x(y) = \max(u_x(y) - \varepsilon, u(y))$ for $y \in \operatorname{Int}(K)$ and $v_x(y) = u(y)$ for $y \in \overline{G} \backslash L$, where $x \in V$. By (1.3), (1.5), (1.6), the definition is consistent and $v_x \in \operatorname{usc}(\overline{G})$, and by axioms (0.5), (0.7), $v_x|G \in P(G)$. Furthermore, $v_x(y) \leq g(y)$, $y \in \overline{G}$, by (1.3), (1.5), (1.6). Thus, $v_x \leq E(P, g) = u$ and, by (1.4), $u(x) \geq v_x(x) = u_x(x) - \varepsilon > u(x^*) - 2\varepsilon$, for $x \in V$, which shows that u is lsc at $x^* \in G$. □

DEFINITION 1.4. Let $G \subset M$ and P be a pseudoconvex class of functions on M. We say that G is weakly P-regular if $\overline{G}$ is compact and for an arbitrary point $x \in \partial G$, neighborhood U of x, and constants $E, \varepsilon > 0$, there is a function $v \in \operatorname{usc}(\overline{G}) \cap P(G)$, such that $u|\overline{G} \leq 0$, $v(x) > -\varepsilon$, $v|\overline{G} \backslash U < -E$, and $\lim_{y \to x} v(y) = v(x)$.

COROLLARY 1.5. *Let P be a pseudoconvex class of functions on M and $G \subset M$ be weakly P-regular. Let $g \in C(\overline{G})$ and $u = E(P, g)$. Then, $u|\partial G = g$, $u \in C(\overline{G})$ and $u|G \in P(G)$ (where $C(\overline{G})$ = the space of all continuous functions on $\overline{G}$).*

Proof. We prove first that u is continuous at each boundary point $x \in \partial G$. Fix $\varepsilon > 0$ and $x \in \partial G$. By the P-regularity of G and axiom (0.5), there is $v_x \in \text{usc}(\overline{G}) \cap P(G)$, such that $v_x(x) > g(x) - \varepsilon$, $v_x(y) \leq g(y)$, for $y \in \overline{G}$, and $\lim_{y \to x} v_x(y) = v_x(x)$. Choose a relative neighborhood V_x of x in $\overline{G}$, such that $v_x(y) > g(y) - \varepsilon$ for $y \in V_x$. Clearly, $u = E(P, g) \geq v_x$ on $\overline{G}$, and so $u(y) \geq g(y) - \varepsilon$ on V_x. Thus, $\liminf_{y \to x} u(y) \geq g(x) - \varepsilon$, for every $\varepsilon > 0$. On the other hand, $\limsup_{y \to x} u(y) \leq g(x)$, by Remark 1.2, and so $\lim_{y \to x} u(y) = g(x)$, $x \in \partial G$ and also $u|\partial G = g|\partial G$. Now, Lemma 1.3 is applicable, and so $u \in C(\overline{G})$. (Recall $u|G \in P(G)$, by Remark 1.2.) □

THEOREM 1.6. *Let P be a pseudoconvex class of functions on M and $G \subset M$ be weakly P-regular (cf. Definition 1.4). Let $u \in \text{usc}(\overline{G}) \cap P(G)$. Then, there is a sequence of functions $u_n \in C(\overline{G}) \cap P(G)$, such that $u_n(x) \searrow u(x)$, $x \in \overline{G}$.*

Proof. Choose a sequence $(g_n)_{n=1}^{\infty} \subset C(\overline{G})$, such that $g_n(x) \searrow u(x)$, $x \in \overline{G}$. Let $u_n(x) = E(P, g_n)(x)$, $x \in \overline{G}$. By Definition 1.1, $u_n(x) \geq u_{n+1}(x) \geq u(x)$, and by Remark 1.2, $u_n(x) \leq g_n(x)$, $x \in \overline{G}$. Thus, $u_n(x) \searrow u(x)$, $x \in \overline{G}$. The remaining properties of u_n's follow from Corollary 1.5. □

PROPOSITION 1.7. *Let P be a pseudoconvex class on M and $G \subset M$ be open and relatively compact. Let $g \in C(\overline{G})$ and assume that the function $u = E(P, g)$ is continuous at every point of ∂G. Denote $U = \{x \in G: u(x) < g(x)\}$. Then U is open in M and $(-u)|U \in P^d(U)$.*

Proof. For the definition of the dual class P^d, the reader is referred to [**9**, Definition 1.11]. Suppose $(-u)|U \notin P^d(U)$. Then, by [**9**, Lemma 2.8], there exist: a point $x \in U$, a neighborhood V of x with $V \subset U$ and $u_0 \in P(V)$, such that $(-u)(x) + u_0(x) = 0 > (-u)(y) + u_0(y)$, $y \in V \backslash \{x\}$; that is $u_0(x) = u(x)$ and $u_0(y) < u(y)$ for $y \in V \backslash \{x\}$. Choose a neighborhood B of x with $\overline{B} \subset V$. Since g and u are continuous (cf. Corollary 1.5), there is $\varepsilon > 0$, such that $u_0(y) + \varepsilon < g(y)$, for $y \in \overline{B}$, and $u_0(y) + \varepsilon < u(y)$ for $y \in \partial B$. Let $u_1(y) = \max(u_0(y) + \varepsilon, u(y))$ for $y \in B$, and $u_1(y) = u(y)$ for $y \in \overline{G} \backslash B$. By [**9**, Proposition 3.3], $u_1 \in P(G)$ and, clearly, $u_1 \in \text{usc}(\overline{G})$ and $u_1 \leq g$ on $\overline{G}$. Since $u = E(P, g)$, we get that $u_1 \leq u$ on $\overline{G}$, which contradicts the inequality $u_1(x) = u_0(x) + \varepsilon > u(x)$. □

THEOREM 1.8. *Let P be a pseudoconvex class of functions and F be a class of usc functions on M, such that $P^d \subset F^d$. Assume that M has* *a basis consisting of weakly P-regular neighborhoods. Then $F \subset P$. In particular, $P^{dd} = P$.*

Proof. Let $U \subset M$ and $u \in F(U)$. By the sheaf property (0.7) of P, it suffices to show that whenever G is a (weakly) P-regular neighborhood, such that $\overline{G} \subset U$, then $u|G \in P(G)$.

Fix such G and choose a sequence $(g_n) \subset C(\overline{G})$, such that $g_n(x) > u(x)$, $x \in \overline{G}$, and let $u_n = E(P, g_n)$. By Corollary 1.5, $u_n \in C(\overline{G})$, $u_n|G \in P(G)$ and $u_n|\partial G = g_n$. Fix n and consider $U_n = \{x \in G : u_n(x) < g_n(x)\}$. Observe that U_n is a relatively compact set and $u_n|\partial U_n = g_n$ (because $u_n|\partial G = g_n$). Define the function $v : \overline{U}_n \to [-\infty, +\infty)$ by $v(x) = (-u_n)(x) + u(x)$. By Proposition 1.7, $(-u_n)|U_n \in P^d(U_n) \subset F^d(U_n)$ (by the assumptions), and so $v|U_n \in F^d + F$. Thus, $v|U_n$ has the local maximum property on U_n (cf. [**9**, Definition 1.11]). Furthermore, $v \in \operatorname{usc}(\overline{U}_n)$ (because $-u_n$ is continuous on $\overline{U}_n$ and u is usc on $\overline{G}$), and so $v(x) \le \max v|\partial U_n = \max(u - u_n)|\partial U_n = \max(u - g_n)|\partial U_n \le 0$ for $x \in \overline{G}$, cf. [**9**, Corollary 4.4]. Thus, $u \le u_n$ on U_n. Since $u_n = g_n$ on $\overline{G} \backslash U_n$, we conclude that $u \le u_n \le g_n$ on $\overline{G}$. On the other hand, $u_n(x) \searrow u(x)$, $x \in G$ (for $g_n(x) \searrow u(x)$, $g_n(x) \ge u_n(x) \ge u_{n+1}(x)$, $x \in \overline{G}$) and $u_n|G \in P(G)$, and so $u|G \in P(G)$ by axiom (0.3). This proves that $F \subset P$.

If we let $F = P^{dd}$, then $F^d = P^d$, cf. [**9**, Proposition 2.4], and so $F = P^{dd} \subset P$, by the argument above. The opposite inclusion is obvious. □

The next corollary is an easy consequence of the last theorem.

COROLLARY 1.9. *Let P and P_1 be two pseudoconvex classes of functions on M, such that $P^d = P_1^d$. Assume that M has a basis consisting of weakly P-regular neighborhoods and a basis consisting of P_1-regular neighborhoods. Then $P = P_1$.*

2. Regularization by supremum-convolution. In this section we adapt the method of [7, §2] to approximate functions of a given pseudoconvex class by functions of the same class with lower-bounded Hessian. In fact, the method works in the wider context of translation invariant generalized pseudoconvex classes on R^N (and we will use this in §4). Recall that a *generalized pseudoconvex class of functions P* is defined by axioms (0.1)–(0.5) and (1.1), cf. [**9**, Definition 1.2]. Function $u : U \to R$, $U \subset R^N$, is said to have *lower bounded Hessian*, if for some $L \ge 0$ the function $x \to u(x) + \frac{1}{2}L|x|^2$ is locally convex, cf. [7, Definition 2.1]; then we write $u \in C_L^1(U)$ and denote $C_L^1(R^N) = C_L^1$.

If $u, g\colon R^N \to [-\infty, +\infty)$ and $\sup g < +\infty$, $\sup u < +\infty$, we define supremum-convolution of u and g as

$$u *_a g(x) = \sup\{u(y) + g(x-y)\colon y \in R^N\}. \tag{2.1}$$

If $u\colon U \to [-\infty, +\infty)$, $U \subset R^N$, then $u *_a g$ is understood as $\tilde{u} *_a g$, where $\tilde{u}|U = u$ and $\tilde{u}|(R^N\backslash U) = 0$.

This definition, which is a modification of [7, Definition 2.4] is very close to the definition of infimum-convolution of Moreau [**6**].

PROPOSITION 2.1. *Let* $u\colon U \to [-\infty, +\infty)$ *and* $g\colon R^N \to R$ *be bounded from the above. Assume that* $g \in C_L^1$, $L \geq 0$, *and that* $u(x_0) \neq -\infty$ *for some* $x_0 \in U$. *Then* $u *_a g$ *is finite-valued everywhere on* R^N, *continuous, and* $u *_a g \in C_L^1$.

Proof. Clearly, $(u *_a g)(x) \geq u(x_0) + g(x - x_0) > -\infty$ for every $x \in R^N$. Furthermore, $(u *_a g)(x) = \sup_y f^y(x)$, where $y \in \{x\colon \tilde{u}(x) \neq -\infty\}$ and $f^y(x) = u(y) + g(x-y)$. Since the family $\{f^y\} \subset C_L^1$, by [7, Proposition 2.3 (ii)], and is pointwise bounded from the above, $\sup_y f^y \in C_L^1$, by [7, Proposition 2.3 (iv)]. □

Let now $g_L(x) = -\frac{1}{2}L|x|^2$, $L \geq 0$, $x \in R^N$ and denote by $B^{+\infty}$ the space of all functions on R^N that are Borel-measurable and bounded from the above. For $u \in B^{+\infty}$ and $L \geq 0$ define the function $(R_L u)(x) = (u *_a g_L)(x), x \in R^N$. The properties of operators R_L listed in the next proposition follow easily from the above remarks (note that $g_L \in C_L^1$); cf. also [7, §2].

PROPOSITION 2.2. *For every* $L \geq 0$

$$R_L\colon B^{+\infty} \to C_L^1(R^N).$$

Furthermore,

(a) $(R_L u)(x) \geq (R_{L^1}, u)(x) \geq u(x)$, *if* $0 \leq L \leq L^1$, $x \in R^N$;

(b) $(R_L u)(x) \geq (R_L v)(x)$, *if* $u \geq v$ *on* R^N;

(c) $\lim_{L\to+\infty}(R_L u)(x^*) = u(x^*)$, *for every point* $x^* \in R^N$ *at which* u *is usc.*

LEMMA 2.3. *Assume that* P *is a class of functions on* R^N *satisfying conditions* (0.1), (0.2), (0.4), (0.5), (0.9). *Let* $u \in P(U) \cap L^\infty(U)$, *where* U *is an open subset of* R^N. *Let* $\delta > 0$. *Then*

$$(R_L u)|U_\delta \in P(U_\delta), \quad \textit{where } U_\delta = \{x \in U\colon \operatorname{dist}(x, \partial U) > \delta\}, \tag{2.2}$$

provided $L \geq L_0 = 4\delta^{-2}\|u\|_\infty$.

Proof. Since $(R_L u)(x) = \sup\{u(x-w) - \frac{1}{2}L|w|^2 \colon w \in R^N\}$, and since $(R_L u)(x) \geq u(x)$ and $u(x) \geq u(x-w) - \frac{1}{2}L|w|^2$ for $|w| \geq \delta$ and $L \geq L_0$, therefore

$$(2.3) \qquad (R_L u)|U_\delta = \sup\{f^w|U_\delta \colon |w| \leq \delta\},$$

where $f^w(x) = u(x-w) - \frac{1}{2}L|w|^2$ is defined in U. By (0.2), (0.5) and (0.9), $f^w|U_\delta \in P(U_\delta)$, and so, by (2.3) and (0.4), $R_L u|U_\delta \in P(U_\delta)$, seeing that $R_L u$ is continuous by Proposition 2.1. □

NOTATION 2.4. Denote the class of all functions with lower bounded Hessian by $C^{1,1}_{-\infty,\mathrm{loc}}$, that is

$$(2.4) \qquad C^{1,1}_{-\infty,\mathrm{loc}} = \bigcup_{L \geq 0} C^1_L.$$

COROLLARY 2.5. *Let P be a translation invariant, generalized pseudoconvex class of functions on R^N (i.e. conditions* (0.1)–(0.6) *and* (0.9) *hold). Then, the class $P \cap C^{1,1}_{-\infty,\mathrm{loc}}$ is dense in P in the following sense*:

> *for every $u \in P(U)$, U open in R^N, and for every compact $K \subset U$, there is a sequence of functions $(u_n)_{n=1}^\infty$, such that*
> (i) $u_n(x) \searrow u(x)$, $x \in K$,
> (ii) $u_n \in P(\mathrm{nbhd}\, K) \cap C^1_{L(n)}$, $L(n) \geq 0$, $n = 1, 2, \ldots$

Proof (*Sketch*). If $u \in L^\infty(U)$, then we can simply take $\delta = \frac{1}{2}\mathrm{dist}(K, \partial U)$ and let $u_n = R_n \tilde{u}$, for $n > L_0 = 4\delta^{-2}\|u\|_\infty$. By Proposition 2.2 and Lemma 2.3, conditions (i) and (ii) are clearly fulfilled.

In the general case, fix $K \subset U$ and assume without loss of generality (shrinking U if necessary) that U is relatively compact and $\sup u|U < +\infty$. By axiom (0.6), there is a bounded function $g \in P(U)$. Let $v_n(x) = \max(u(x), g(x) - n)$, $x \in U$, and 0 otherwise. Clearly, $v_n \in P(U) \cap L^\infty(U)$ and $v_n(x) \searrow u(x)$, $x \in U$. The approximations u_n will be of the form $u_n = R_{L(n)} v_n$, for suitably chosen constants $L(n)$, $n = 1, 2, \ldots$ Clearly, $u_n \geq u_{n+1}$ on U, if $L(n) \leq L(n+1)$, $n = 1, 2, \ldots$ and $u_n \in C^1_{L(n)}$. One can show easily that, if $L(n)$ grows rapidly enough, then $u_n \in P(\mathrm{nbhd}\, K)$ (use Lemma 2.3) and $u_n(x) \searrow u(x)$, $x \in K$ (cf. Proposition 2.2(c)). □

REMARK AND PROBLEM. It is obvious that the supremum-convolution is associative. One can also compute easily that $g_L *_a g_R = g_S$, where $S = \frac{1}{2}LR(L+R)^{-1}$. Thus, if we reparametrize the regularization operators R_L as follows

$$(T_t u)(x) = (R_{1/t}u)(x) = \sup\{u(x-y) - \tfrac{1}{2}t^{-1}|y|^2 : y \in R^N\},$$

then $\{T_t\}_{t>0}$ form a semigroup, $T_t T_{t'} = T_{t+t'}$. Furthermore,

$$\lim_{t\to 0} T_t(x) = u(x),$$

if u is usc at x. While it does not seem to have any direct relevance for the problems of this paper, it would be certainly interesting to determine the infinitesimal generator of this semigroup.

3. Description of affine pseudoconvex classes of functions on R^N in terms of second-order derivatives. As shown in the last section, any translation-invariant generalized pseudoconvex class of functions on R^N contains plenty of functions with lower-bounded Hessian (denoted $C^{1,1}_{-\infty,\mathrm{loc}}$), cf. Corollary 2.5. Due to a result of Alexandrov **[1]**, cf. also Buseman **[3]**, such a function must have almost everywhere second-order derivatives in the pointwise (Peano) sense; see [7, p. 311, p. 317] for more comments on this.

Definition 3.1. Let F be a translation invariant class of functions on R^N satisfying conditions (0.5), (0.8), such that $F \cap C^{1,1}_{-\infty,\mathrm{loc}}$ is dense in F in the sense of Corollary 2.5. Let Y' be the set of all $n \times n$ symmetric matrices A, such that there is $x \in R^N$, a neighborhood U of x and a function $u \in F(U) \cap C^{1,1}_{-\infty,\mathrm{loc}}$, such that u has the second-order (Peano) differential at x whose homogeneous quadratic part is equal to $x \to \frac{1}{2}(Ax, x)$. Denote by Y the closure of Y' (in the space of $N \times N$ matrices). We will call Y *the class of matrices associated to* F, or—in the situations in which the next proposition is applicable—*the order cone of symmetric matrices representing* F.

Remark. The definition applies in particular to affine pseudoconvex classes.

Proposition 3.2. *Let F be a translation invariant generalized pseudoconvex class of functions on R^N satisfying axiom* (0.8). *Let Y denote the class of symmetric matrices associated to F. Then*

(3.1) *whenever $A \in Y$ and B is a positive semidefinite symmetric $N \times N$ matrix, then $(A + B) \in Y$;*

(3.2) *whenever $A \in Y'$ and B is a positive semidefinite symmetric matrix, then $(A + B) \in Y'$,*

where Y' is as in the Definition 3.1.

REMARK 3.3. Conditions (3.1) and (3.2) mean that the sets Y' and Y are order-cones with respect to the partial order on the vector space of symmetric $N \times N$ matrices determined by the open convex cone of positive definite matrices; see Appendix A, Definition A.2 (i). It turns out that properties of order cones become more transparent, if they are treated in the abstract context of vector spaces, rather than matrices, and for that reason they are separated in the Appendix A.

The purpose of this section is to show that affine pseudoconvex classes of functions on R^N are in one-to-one correspondence with closed order cones of symmetric $N \times N$ matrices, cf. Theorem 3.11.

Proof of Proposition 3.2. (*Sketch*). Axiom (0.8) means that the class AF, which was defined in condition (1.1), cf. also [**9**, Definition 1.3], contains all affine functions. Since the class AF must satisfy condition (0.4) if F does, we conclude that every convex function (being the supremum of a family of linear ones) belongs to AF.

Thus, the function $u(x) + \frac{1}{2}(Bx, x)$ is of class $F \cap C^{1,1}_{-\infty,\mathrm{loc}}$, if u is and if B is positive semidefinite, which trivially implies (3.1), (3.2).□

To describe the dual pseudoconvex classes we need the notion of a dual order cone.

DEFINITION 3.4. Let Y be an order cone of symmetric $N \times N$ matrices. *The dual order cone to* Y, denoted by Y^D, is the set of all symmetric $N \times N$ matrices B, such that the set $Y + B = \{A + B \colon A \in Y\}$ does not contain a negative definite matrix.

Once again, the notion of the dual order cone is best studied in the abstract setting of ordered vector spaces, see Definition A.2 (iii) and the following propositions.

REMARK 3.5. The definition of Y^D makes sense for an arbitrary set of symmetric matrices, not necessarily an order cone. Still, it is easy to see that Y^D is an order cone, provided it is nonempty. This observation allows us to formulate the following lemma.

LEMMA 3.6. *Let F be a class of usc functions on R^N satisfying assumptions of Definition* 3.1 *and let Y denote the class of matrices associated to F. Assume that Y^D is nonempty. Then the dual class of functions $P = F^d$, in the sense of* [**9**, *Definition* 1.11], *is affine pseudoconvex* (*i.e. axioms* (0.1)–(0.9) *hold*) *and its associated class of matrices is Y^D. Furthermore, if u is a function with lower bounded Hessian, i.e. $u \in C^{1,1}_{-\infty,\mathrm{loc}}(U)$, $U \subset R^N$, then $u \in P(U)$, if and only if* (*Hess* u)$(x) \in Y^D$ *for a.a.* $x \in H$.

REMARK 3.7. If Y is a nonempty set of symmetric matrices, denote by F_Y the set of functions of the form $x \to l(x) + \frac{1}{2}(Ax, x)$, where $A \in Y$. It is obvious that F_Y satisfies the assumptions of the last lemma and that the class of matrices associated to F_Y is Y. Thus, if F is a class of functions satisfying conditions of Lemma 3.6, with associated class of matrices equal to Y, then the dual classes to F and F_Y are equal.

REMARK 3.8. Combining Lemma 3.6 with Proposition 2.2 and Lemma 2.3, we obtain that a bounded usc function $u: U \to R$ belongs to $P(U)$, if and only if for every compact $K \subset U$, there is $L_0(K) \geq 0$ such that $\mathrm{Hess}(R_L u)(x) \in Y^D$ for a.a. $x \in K$ and all $L \geq L_0(K)$. It is obvious how to modify this criterion if u is unbounded (using axiom (0.6)).

Proof of Lemma 3.6.

Assertion 1. If $g \in C^{1,1}_{-\infty,\mathrm{loc}}(U)$ and $(\mathrm{Hess}\, g)(x) \in Y^D$ for a.a. $x \in U$, then $g \in F^d(U)$.

By [9, Definition 1.11], we have to show that if $f \in F(V)$ and K is a nonempty compact subset of $U \cap V$, then

$$(3.3) \qquad \max(f+g)|K \leq \max(f+g)|\partial K.$$

By the assumptions of Lemma 3.6, there exist a sequence

$$(f_n) \subset C^{1,1}_{-\infty,\mathrm{loc}} \cap F(\mathrm{nbhd}\, K),$$

such that $f_n(x) \searrow f(x)$, $x \in K$. Then $\mathrm{Hess}(f_n + g)(x) \in Y + Y^D$, for a.a. x in a neighborhood of K. Thus, $\mathrm{Hess}(f_n + g)(x)$ has at least one non-negative eigenvalue for a.a. x near K (cf. Definition 3.4) and since $(f_n + g) \in C^{1,1}_{-\infty,\mathrm{loc}}$, we conclude by Theorem B.1 (in Appendix B) that $\max(f_n + g)|K \leq \max(f_n + g)|\partial K$, for all n. Since $(f_n + g)(x) \searrow (f+g)(x)$, $x \in K$, Eq. (3.3) follows.

Assertion 2. Let $u \in P(U)$ and $x^* \in U$. Assume that u has a second-order (Peano) differential at x^*. Then $(\mathrm{Hess}\, u)(x^*) \in Y^D$.

Denote $A = (\mathrm{Hess}\, u)(x^*)$; if $A \notin Y^D$, then there is $B' \in Y$ such that $A+B'$ is negative definite (Definition 3.4) and, by Definition 3.1, there is a function $f \in P(\mathrm{nbhd}\, y^*) \cap C^{1,1}_{-\infty,\mathrm{loc}}$, which has a second-order Peano differential at y^* and such that $A + B$ is a negative definite matrix, where $B = (\mathrm{Hess}\, f)(y^*)$. We can assume without loss of generality that $x^* = 0 = y^*$ (note, F is translation invariant). Let $l(x) = u(0) + f(0) + (\operatorname{grad} u(0), x) + (\operatorname{grad} f(0), x)$, and $f_1(x) = f(x) - l(x)$. By the construction and Peano differentiability of $u + f_1$, we obtain that $(u + f_1)(x) = \frac{1}{2}((A + B)x, x) + r(x)$, where $\lim_{x\to 0} r(x)|x|^{-2} = 0$. Hence, the function $u + f_1$ has a strict local maximum at $x = 0$ which contradicts the assumptions that $u \in F^d$ and $f_1 \in F$ (because $f - f_1$ is a linear function; cf. axiom (0.8)).

If we apply both assertions to the function

$$g_A(x) = \tfrac{1}{2}(Ax, x), \tag{3.4}$$

the following observation becomes obvious.

Assertion 3. If A is a symmetric matrix, then $g_A \in F^d$, if and only if $A \in Y^D$. Furthermore, Y^D is the class of matrices associated to $P = F^d$.

We will check now that $P = F^d$ is an affine pseudoconvex class. It is clear that P satisfies axioms (0.9) and (0.8), cf. [**9**, Remark 2.7]. By the last assertion, if Y^D is nonempty, with $A \in Y^D$, then P contains g_A which is a locally bounded function, and so P satisfies axiom (0.6). The remaining axioms follow now by [**9**, Lemmas 2.9 and 2.10]. □

THEOREM 3.9. *Every affine pseudoconvex class P is equal to its own bidual P^{dd}.*

Proof. By Theorem 1.8, it suffices to show that R^N has a basis consisting of P-regular neighborhoods. By Corollary 2.5, there is a function $u_0 \in (P \cap C^1_L)(U)$, where $U \subset R^N$, $L \geq 0$. We can assume without loss of generality that u_0 has a second-order Peano differential at $x = 0$. By the definition of the C^1_L class, the function $u_0(x) + \frac{1}{2}L|x|^2$ is convex. Then the function

$$v(x) = u_0(x) - u_0(0) - (\operatorname{grad} u_0(0), x) + (\tfrac{1}{2}L + 1)|x|^2,$$

is strictly convex and

$$v(0) = 0 < v(x), \qquad x \neq 0.$$

Denote $V_\varepsilon = \{x \colon v(x) < 0\}$. Clearly, for $0 < \varepsilon \leq \varepsilon_0$, with ε_0 small enough, V_ε are convex and form a basis of neighborhoods of 0. It suffices to show that V_ε are P-regular (cf. Definition 1.4).

Consider a point $x \in \partial V_\varepsilon$. Since v is a strictly convex function, V_ε is a strictly convex set, and so there is an affine function $l(y)$, such that

$$l(x) = 0, \qquad l(y) < 0 \quad \text{for } y \in V_\varepsilon. \tag{3.5}$$

Let now $v_n(y) = (v(y) - \varepsilon) + nl(y)$. Clearly, $v_n \in P(\text{nbhd}\,\overline{V}_\varepsilon)$, for $n > 0$, because $v \in P$ (recall that $(\frac{1}{2}L+1)|x|^2$, being a convex function, belongs to AP, cf. (1.1)). By (3.5), $v_n(x) = 0$ and $\lim_n v_n(y) = -\infty$ for $y \in \overline{V}_\varepsilon \backslash \{x\}$. Thus, V_ε is P-regular. □

COROLLARY 3.10. *Let P be an affine pseudoconvex class on R^N and let Y be the class of symmetric matrices associated to P. Then, $A \in Y$, if and only if $g_A \in P(R^N)$, where $g_A(x) = \frac{1}{2}(Ax, x)$.*

Proof. By Remark 3.3, Y is a closed order cone.

Case 1. *Y^D is nonempty.* Let $F = P^d$. By Lemma 3.6, F is an affine pseudoconvex class with the associated class of matrices $Y_F = Y^D$. By Theorem 3.9, $P = F^D$. The conclusion follows now from Assertion 3 in the proof of Lemma 3.6 applied to F. Namely, $g_A \in P = F^d$, if and only if $A \in (Y_F)^D = (Y^D)^D = Y$ (by Corollary A.8 (iii)).

Case 2. *Y^D is empty.* The statement follows from the next assertion.

Assertion. Y^D is empty, if and only if $P = \text{usc}$. The sufficiency is obvious. As for necessity, if $Y^D = \emptyset$, then, by Corollary A.8, the set Y' of Definition 3.1 is dense in the set of all symmetric matrices. Thus, for every point $x \in R^N$ and every $C > 0$ there is a function $g_{x,C} \in P \cap C^{1,1}_{-\infty,\text{loc}}(\text{nbhd}\,x)$, with a second-order Peano differential at x, such that $\text{Hess}\, g_{x,C}(x) \le -CI$. Let $u \in C^{(2)}(U)$; we will show that $u \in P(U)$. If $x \in U$, define

$$u_{x,C}(y) = u(x) + g_{x,C}(y) - g(x) + (\text{grad}\, u(x) - \text{grad}\, g_{x,C}(x), y - x),$$

for y near x. By (0.5), (0.8), $u_{x,C} \in P(\text{nbhd}\,x)$.

It is clear that for each $x \in U$ we can choose $C = C(x)$ and a neighborhood U_x of x, such that $u_{x,C}(x) = u(x) > u_{x,C}(y)$, $y \in U_x \backslash \{x\}$. By [9, Theorem 3.5], this implies that $u \in C^{(2)}(U)$. By axiom (0.3), this implies that $P(U) = \text{usc}(U)$. □

THEOREM 3.11. *Proper (i.e. different from the class of all usc functions) affine pseudoconvex classes of functions on R^N are in one-to-one correspondence with proper, closed order cones of real symmetric $N \times N$ matrices, given by the map $P \to Y$, where Y is the class of matrices associated by Y.*

Furthermore, $u \in P(U)$, if and only if for every open relatively compact subset $H \subset U$ there is a sequence $(u_n) \subset C^{1,1}_{-\infty,\text{loc}}(H)$, such that

$u_n(x) \searrow u(x)$, $x \in H$, and Hess $u_n(x) \in Y$ for a.a. $x \in H$. The dual class of matrices Y^D corresponds to the dual pseudoconvex class P^d.

Proof (*Sketch*). Let Y be a closed proper order cone of matrices. Let $Y_1 = Y^D$ and let F_Y be the set of functions defined in Remark 3.7. By this remark and Lemma 3.6, the dual class $P = (F_{Y_1})^d$ is an affine pseudoconvex class on R^N with associated class of matrices equal to $Y_1^D = Y^{DD} = Y$ (cf. Lemma A.8). Thus, the map

$$P \to Y \tag{3.6}$$

is onto.

If P is any affine pseudoconvex class with an associated class of matrices equal to Y, then, by Remark 3.7, $P^d = (F_Y)^d$, and by Theorem 3.9, $P = P^{dd} = (F_Y)^{dd}$. Thus P is uniquely determined by Y, and the map (3.6) is one-to-one.

The remaining statements follow by applying Lemma 3.6 to the representation $P = F^d$, where $F = (F_Y)^d$. (Note, that the order cone associated to F is Y^D, by Remark 3.7.) □

4. Approximation by piecewise functions in pseudoconvex classes. It is a natural question whether the regularization procedure, described in §2, can be improved so that the approximations u_n in Corollary 2.5 become C^∞-smooth. The answer is negative in general: in §6 we give an example of an affine pseudoconvex class P on R^N, such that $C^\infty \cap P$ is not dense in P in the sense of Corollary 2.5. The next theorem shows that piecewise smooth approximation is possible (and describes also more precisely how an affine pseudoconvex class P is determined by its associated class of matrices Y).

THEOREM 4.1. *Let P be an affine pseudoconvex class of functions on R^N (i.e. axioms* (0.1) *through* (0.9) *hold). Let $u \in P(U)$, where $U \subset R^N$ is open. Then, for every compact set $K \subset U$ there is a sequence $(u_n)_{n=1}$, such that*

(i) *$u_n \in C(U_n) \cap P(U_n)$, where U_n is a neighborhood of K;*
(ii) *$u_n(x) \searrow u(x)$, $x \in K$;*
(iii) *for every $x_0 \in U_n$ there are functions*

$$f_1, \dots, f_m \in (C^\infty \cap P)(\mathrm{nbhd}\, x_0),$$

such that $u_n \equiv \max(f_1, \dots, f_n)$ near x_0.

Moreover, functions f_j can be chosen in the form $l(x) + \frac{1}{2}(Ax, x)$, where $l(x)$ is an affine function and $A \in Y$ (= the order cone of matrices associated to P).

The theorem is a direct consequence of the technical Lemma 4.3 (another application of this lemma will be given in **[12]**).

NOTATION 4.2. If F is a class of continuous functions on (open subsets of) R^N, denote $\overline{F} = \bigcup \overline{F}(U)$, where $\overline{F}(U)$ = the closure of $F(U)$ in $C(U)$ with respect to the uniform convergence on compact subsets of U.

LEMMA 4.3. *Let P be an affine pseudoconvex class of functions on R^N and F be a class of continuous functions contained in P. Assume that F is invariant with respect to translations and*

(4.1) $F^d = P^d$;

(4.2) *if U is relatively compact, then $F(U)$ contains a bounded function*;

(4.3) $g + F(U) \subset \overline{F}(U)$, *provided U is open and $g(x)$ is an affine function on R^N.*

Then for every $u \in P(U)$ and a compact set $K \subset U$ there exist approximations u_n, $n = 1, 2, \ldots$, satisfying conditions analogous to (i)–(iii) *in Theorem* 4.1, *with $f_1, \ldots, f_n \in F(\text{nbhd}\, x_0)$. Moreover, functions u_n can be chosen from the class F_1, which is defined next.*

NOTATION 4.4. We define three classes F_1, F_2, F_3, constructed from F.

Let $U \subset R^N$ be open. Then we say that $u \in F_1(U)$, if there exist a locally finite covering $\{V_n\}_{n=1}^{\infty}$ of U and functions $v_n \in C(\overline{V}_n)$, such that $\{\overline{V}_n\}_n$ is a locally finite family of compact subsets of U and

$$v_n|\partial V_n < u|\partial V_n, \qquad v_n \leq u \text{ on } \overline{V}_n, \tag{4.4}$$

$$\bigcup_{n=1}^{\infty} \operatorname{Int}\{v_n = u\} = U, \tag{4.5}$$

and for every n there are functions $f_1, \ldots, f_{m(n)} \in F(\text{nbhd}\, \overline{V}_n)$, such that $v_n(x) = \max(f_1(x), \ldots, f_{m(n)}(x))$, $x \in \overline{V}_n$.

F_2 will denote the class of functions that are locally equal to the maximum of several functions of class F.

$F_3(U)$, $U \subset R^N$, consists of usc functions $u\colon U \to [-\infty, +\infty)$, such that for every compact subset $K \subset U$, there exist functions $u_n \in F_1(\text{nbhd}\, K)$, $n = 1, 2, \ldots$, such that $u_n(x) \searrow u(x)$, $x \in K$.

Outline of Proof of Lemma 4.3. The following inclusions, with exception for $F_2 \subset F_3$, are obvious.

$$F \subset F_2 \subset F_3 \subset P, \qquad F_1 \subset F_2. \tag{4.6}$$

Since $F^d = P^d$ by the assumptions, we conclude that

$$F_3^d = P^d. \tag{4.7}$$

Our main task will be to prove that F_3 is an affine pseudoconvex class (i.e. axioms (0.1) through (0.9) hold). An affine pseudoconvex class being equal to its own bidual (cf. Theorem 3.9), Eq. (4.7) implies then that $F_3 = P$, which is precisely what is required in Lemma 4.3.

To ease the handling of functions of the class F_1, we introduce the following terminology.

Terminology 4.5. Let $U \subset R^N$ be open and $\{v_n\}_{n=1}^{\infty}$ be a sequence of continuous functions $v_n \colon X_n \to R$. We say that $\{v_n\}$ is a *good family of functions relative to* U, if $\{X_n\}_{n=1}^{\infty}$ form a locally finite family of compact subsets of U, the sets $\mathrm{Int}(X_n)$, $n = 1, 2, \ldots$, form a covering of U, and for every n and every $x \in \partial X_n$, there exists an index m, such that $x \in \mathrm{Int}(X_m)$ and $v_m(x) > v_n(x)$.

The following observations are obvious.

REMARKS 4.6. (a) If v_n is a good family of functions relative to U, where $v_n \colon X_n \to R$, then the function

$$u(x) := \sup\{v_n(x) \colon x \in X_n\} \tag{4.8}$$

is continuous on U.

(b) $u \in F_1(U)$, if and only if there exist a good family of functions $\{v_n \colon X_n \to R\}_{n=1}^{\infty}$ relative to U, such that (4.8) holds and for every n there are functions $f_1, \ldots, f_{m(n)} \in F(\mathrm{nbhd}\, X_n)$, such that $v_n = \max(f_1, \ldots, f_{m(n)})|X_n$.

(c) If $\{v_n \colon X_n \to R\}_{n=1}^{\infty}$ is a good family of functions relative to U, then there is a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive real numbers, such that whenever $\{u_n \colon X_n \to R\}$ is a sequence of continuous functions with the property $\max_{X_n} |u_n - v_n| < \varepsilon_n$, $n = 1, 2, \ldots$, then $\{u_n\}_{n=1}^{\infty}$ is also a good family of functions relative to U. (Here the local finiteness of the covering $\{X_n\}_{n=1}^{\infty}$ is crucial.)

LEMMA 4.7. *Let $u \colon U \to R$ be a continuous function ($U \subset R^N$ is open). Let $\{v_x \colon V_x \to R\}_{x \in U}$ be a family of continuous functions, such that V_x is an open neighborhood of x, $V_x \subset U$, $v_x(x) = u(x)$ and $v_x \leq u|V_x$. Then for every $\delta, \varepsilon > 0$ there exist sequences of points $\{x(n)\}_{n=1}^{\infty}$, of relatively compact open neighborhoods $\{V_n\}_{n=1}^{\infty}$ and of affine functions $\{l_n\}_{n=1}$, such that $\overline{V}_n \subset V_{x(n)}$ and the functions $\{v_n + l_n|V_n\}_{n=1}^{\infty}$, where $v_n := v_{n(x)}|\overline{V}_n$, form a good family of functions* *relative to* U. *Furthermore, if we let*

$$v(x) = \sup\{(v_n + l_n)(x) \colon x \in \overline{V}_n\},$$

then

$$u(x) + \varepsilon|x|^2 \le v(x) \le u(x) + \varepsilon|x|^2 + \delta, \qquad x \in U.$$

Proof. Fix $\varepsilon > 0$, $\delta > 0$. For every $x \in U$ consider the function $u_x \colon V_x \to R$, defined by

$$u_x(y) = v_x(y) + \varepsilon|y|^2 - \varepsilon|y - x|^2, \qquad y \in V_x. \tag{4.9}$$

Then, $u_x(x) = u(x) + \varepsilon|x|^2$ and $u_x(y) \le u(y) + \varepsilon|y|^2 - \varepsilon|y - x|^2$, for $y \in V_x$. Choose $r(x)$ so that

$$0 < r(x) < \min(\operatorname{dist}(x, \partial G), \delta/\varepsilon), \tag{4.10}$$

and choose $\delta(x) \in (0, \varepsilon r(x)^2)$, $x \in U$. Then, the sets

$$H_x = \{y \in V_x \colon u_x(y) + \delta(x) > u(y) + \varepsilon|y|^2\}, \qquad x \in U,$$

form an open covering of U.

It is possible to choose a sequence of points $x(n) \in U$, $n = 1, 2, \ldots$, so that sets $\{H_{x(n)}\}_{n=1}^{\infty}$ form an open covering of U, while the closed balls $\{\overline{B}(x(n), r(x(n)))\}_{n=1}^{\infty}$ form a locally finite covering of U (and, clearly, $H_{x(n)} \subset B(x(n), r(x(n)))$). To see this, represent $U = \bigcup_{k=1}^{\infty} Z_k$, where

$$Z_k = \{x \in U \colon 2^{-k} \le \operatorname{dist}(x, \partial U) \le 2^{-k+1}, |x| \le k\}.$$

Sets Z_k being compact, we can choose, for each k, a finite number of points $x_{n_k}, \ldots, x_{n_{k+1}-1} \in Z_k$, so that the sets $\{H_{x(n)}\}$, $n_k \le n < n_{k+1}$, form a covering of Z_k. Then, the sets $\{\overline{B}(x(n), r(x(n)))\}_{n=1}^{\infty}$ form a locally finite covering of U. Indeed, let $x_0 \in Z_k$. If the open ball $B(x_0, 2^{-k-2})$ intersects $\overline{B}(y, r(y))$, then $\operatorname{dist}(y, \partial U) \ge 2^{-k-2}$, because $\operatorname{dist}(x_0, \partial U) \ge 2^{-k}$ and $r(y) < \operatorname{dist}(y, \partial U)$. Thus, every ball $\overline{B}(x, r(x))$ intersecting $B(x_0, 2^{-k-2})$ intersects $Z_1 \cup \cdots \cup Z_{k+2}$, and so the covering $\{\overline{B}(x(n), r(x(n)))\}_{n=1}^{\infty}$ is locally finite.

Let now $V_n = B(x(n), r(x(n)))$, $n = 1, 2, \ldots$; it follows that the functions $(u_{x(n)} + \delta(x(n)))|\overline{V}_n$ form a good family of functions relative to U (seeing that $\bigcup_n H_{x(n)} = U$ and $u_{x(n)}(y) + \delta(x(n)) > u(y) + \varepsilon|y|^2$ for $y \in \partial V_n$, by (4.9) and (4.10)). Define affine functions

$$l_n(y) = \delta(x(n)) + \varepsilon|x(n)|^2 + 2\varepsilon(x(n), y), \qquad y \in R^N.$$

Then, by (4.9), $u_{x(n)}(y) + \delta(x(n)) = v_{x(n)}(y) + l_n(y)$. With $v_n = v_{x(n)}|\overline{V}_n$, all the required properties are now obvious. □

Proof of Lemma 4.3. We prove four assertions first (cf. Notation 4.4).

Assertion 1. If $u \in F_1(U)$ and $l(x)$ is an affine function, then $u + l|U \in F_3(U)$.

Let $u(x) = \sup\{u_n(x) \colon x \in \overline{V}_n\}$, where $\{u_n \colon \overline{V}_n \to R\}_{n=1}^{\infty}$ is a good family of functions relative to U and $u_n = \max(f_1, \dots, f_{m(n)})$, where $f_i \in F(\operatorname{nbhd} \overline{V}_n)$. By property (4.3) of the class F, for any positive δ_n (to be specified later) there exist $g_i \in F(\operatorname{nbhd} \overline{V}_n)$, such that

$$f_i + l \le g_i \le (f_i + l) + \delta_n, \quad \text{on } \overline{V}_n, \text{ for } i = 1, \dots, m(n). \tag{4.11}$$

Let $v_n = \max(g_1, \dots, g_{m(n)})|\overline{V}_n$. With δ_n small enough, $v_n \colon \overline{V}_n \to R$, $n = 1, 2, \dots$, form a good family of functions relative to U (cf. Remark 4.6 (c)). Let $v(x) = \sup\{v_n(x) \colon x \in \overline{V}_n\}$. By Remark 4.6 (a), $v \in F_1(U)$, and by (4.11),

$$u(x) + l(x) \le v(x) \le u(x) + l(x) + \delta, \qquad x \in U,$$

where $\delta = \max \delta_n$ (assuming, without loss of generality, that $\delta_n \searrow 0$). It is clear now that there is a sequence of functions $v^k \in F_1(U)$, such that $v^k(x) \searrow u(x) + l(x)$, $x \in U$, and so $u + l|U \in F_3(U)$, as required.

Assertion 1 implies directly that

$$F_3(U) + l \in F_3(U), \quad \text{for any affine function.} \tag{4.12}$$

Assertion 2. $F_2 \subset F_3$. Precisely, given $u \in F_2(U)$, $U \subset R^N$, $\varepsilon > 0$, $\delta > 0$, there is a function $\rho \in F_1(U)$, such that

$$\varepsilon|x|^2 + u(x) \le \rho(x) \le u(x) + \delta + \varepsilon|x|^2, \qquad x \in U. \tag{4.13}$$

By the definition of $F_2(U)$ (cf. Notation 4.4), for every $x \in X$ there is a neighborhood V_x, such that $u|V_x = \max(f_1, \dots, f_{n(x)})$, $f_j \in F(V_x)$. We apply Lemma 4.7 to the family of functions $v_x = u|V_x$. Then, there are a covering $\bigcup_{n=1}^{\infty} V_n = U$ and affine functions l_n, such that $\{\overline{V}_n\}$ is locally finite in U, $\overline{V}_n$ is a compact subset of some $V_{x(n)}$ and, if $v_n := v_{x(n)}|\overline{V}_n$, then $\{v_n + l_n|\overline{V}_n\}_{n=1}^{\infty}$ is a good family of functions relative to U and

$$\varepsilon|x|^2 + u(x) \le v(x) \le u(x) + \tfrac{1}{2}\delta + \varepsilon|x|^2, \tag{4.14}$$

where $v(x) = \max\{v_n(x) + l_n(x) \colon x \in \overline{V}_n\}$. Since $(v_n + l_n)(x) = \max(f_1 + l_n, \dots, f_{m(n)} + l_n)(x)$, we can choose $\delta_n \in (0, \frac{1}{2}\delta)$ and functions $g_1, \dots, g_m \in F(\overline{V}_n)$, such that

$$v_n + l_n \le \max(g_1, \dots, g_{m(n)}) \le \delta_n + v_n + l_n, \quad \text{on } \overline{V}_n. \tag{4.15}$$

If $\delta_n \searrow 0$ quickly enough, then the functions $\{\max(g_1, \ldots, g_{m(n)})\colon \overline{V}_n \to R\}$ form a good family of functions relative to U (by Remark 4.6 (c)). Let now

$$\rho(x) = \max\{\rho_n(x)\colon x \in \overline{V}_n\},$$

where $\rho_n = \max(g_1, \ldots, g_{m(n)})|\overline{V}_n$. Then $\rho \in F_1(U)$, by Remark 4.6 (b), and (4.13) holds by (4.14) and (4.15).

Assertion 3. $F_1 \subset F_2$.

This follows directly from the definition of F_1 (the local finiteness of the covering $\overline{V}_n$ in the definition of a good family of functions is crucial here).

Assertion 4. Let $u \in C(U)$ and $u_x \in C(V_x) \cap F_3(V_x)$, for $x \in U$, where $x \in V_x \subset U$, with V_x open. Assume that $u_x(x) = u(x)$ and $u_x \le u|V_x$, $x \in U$. Then $u \in F_3(U)$. More precisely, for every $\delta, \varepsilon > 0$ there is $g \in F_1(U)$, such that

$$\varepsilon|x|^2 + u(x) \le g(x) \le u(x) + \delta + \varepsilon|x|^2, \qquad x \in U. \tag{4.16}$$

The setup is as in Lemma 4.7 and applying the latter we obtain a covering $\{V_n\}_{n=1}^{\infty}$ and functions $v_n\colon \overline{V}_n \to R$, l_n, $n = 1, 2, \ldots$, with required properties, so that $\{v_n + l_n|\overline{V}_n\}$ is a good family of functions and

$$\tfrac{1}{2}\varepsilon|x|^2 + u(x) \le v(x) \le u(x) + (\delta/3) + \tfrac{1}{2}\varepsilon|x|^2, \qquad x \in U, \tag{4.17}$$

where $v(x) = \sup\{v_n(x) + l_n(x)\colon x \in \overline{V}_n\}$, and $v_n \in F_3(\operatorname{nbhd}\overline{V}_n)$.

By (4.12), $v_n + l_n|\overline{V}_n \in F_3(\operatorname{nbhd}\overline{V}_n)$ and, by the definition of F_3, there is $\rho_n \in F_1(\operatorname{nbhd}\overline{V}_n)$, such that

$$(v_n + l_n)(x) \le \rho_n(x) \le \delta_n + (v_n + l_n)(x), \qquad x \in \overline{V}_n, \tag{4.18}$$

where $\delta_n > 0$ is a prescribed constant. If $\delta_n \searrow 0$ quickly enough and $\delta_n < (\delta/3)$, $n = 1, 2, \ldots$, then (by Remark 4.6 (c)), $\{\rho_n|\overline{V}_n\}_{n=1}^{\infty}$ is a good family of functions relative to U and, by (4.17) and (4.18), the function $\rho(x) = \max\{\rho_n(x)\colon x \in \overline{V}_n\}$, satisfies the inequality

$$\tfrac{1}{2}\varepsilon|x|^2 + u(x) \le \rho(x) \le u(x) + (2/3)\delta + \tfrac{1}{2}\varepsilon|x|^2, \qquad x \in U. \tag{4.19}$$

By Assertion 3, $\rho_n \in F_2(\operatorname{nbhd}\overline{V}_n)$. This and the fact that

$$\{\rho_n\colon \overline{V}_n \to R\}_{n=1}^{\infty}$$

is a good family of functions relative to U, implies easily that $\rho \in F_2(U)$. (Use the local finiteness of the covering $\{\overline{V}_n\}_{n=1}^{\infty}$.) Applying Assertion 2 to ρ, we find $g \in F_1(U)$, such that

$$\tfrac{1}{2}\varepsilon|x|^2 + \rho(x) \le g(x) \le \rho(x) + (\delta/3) + \tfrac{1}{2}\varepsilon|x|^2, \qquad x \in U.$$

This and (4.19) yields (4.16), which completes the proof of Assertion 4.

We can conclude now the proof of the lemma. With Assertions 2 and 3 the following inclusions are obvious:

$$(4.20) \qquad F \subset F_2 \subset F_3 \subset P; \qquad F_1 \subset F_2 \subset F_3.$$

We will check now that the class F_3 satisfies axioms (0.1) through (0.9). Axioms (0.1), (0.3), (0.6) and (0.9) follow directly from properties of F, and axioms (0.5) and (0.8) are implied by Assertion 1. As for (0.2), if $U \supset V$ and $u \in F_1(U)$, then $u \in F_2(U)$, by Assertion 3. Clearly, $u|V \in F_2(V)$, and so $u|V \in F_3(V)$, by Assertion 2. Hence, $F_3(U)|V \subset F_3(V)$.

Concerning axiom (0.4), observe first that, if $u_1, \dots, u_m \in F_1(U)$, then $\max(u_1, \dots, u_m) \in F_1(U)$. Now, in the situation of axiom (0.4), if $K \subset U$ is any compact subset of U and $\varphi\colon K \to R$ is an arbitrary continuous function, such that $u^*|K < \varphi$, then for every $x \in X$ there is $v_x \in F_1(\operatorname{nbhd} K)$, such that $v_x|K < \varphi$ and $v_x(y) > u^*(y)$ near x. (To construct v_x, note that by the definition of u^*, there exist: $\varepsilon > 0$, $h \in R^N$ and an index $t(x)$, such that $K+h \subset U$, $u_{t(x)}(x+h) > u^*(x) - \varepsilon$ and $u_{t(x)}(y+h) < \varphi(y) - \varepsilon$. Since the function $y \to u_{t(x)}(y+h) + \varepsilon$ is of class F_3 near K, and is majorized by φ, there is a function $v_x \in F_1(\operatorname{nbhd} K)$, such that $u_{t(x)}(y+h) + \varepsilon < v_x(y) < \varphi(y)$, $y \in K$. In particular, $v_x(x) > u^*(x)$, and so $v_x(y) > u^*(y)$ (near x.)

Covering K by a finite family of open sets $\{v_x > u^*\}$, where $x \in S$ (a finite set), we obtain that $u^*(y) < v(y) < \varphi(y)$, $y \in K$, where $v(y) = \max_{x \in S} v_x(y)$, $y \in K$, and $v \in F_1(\operatorname{nbhd} K)$. This implies that $u \in F_3(U)$.

It remains to check that F_3 satisfies the sheaf axiom (0.7). If the function u (as in (0.7)) is continuous, then $u \in F_3(U)$, by Assertion 4. If $u \in \operatorname{usc}(U)$ and is locally bounded, consider an arbitrary compact $K \subset U$ and choose a finite covering $\{U_1, \dots, U_n\}$ of K, such that $u|U_j \in F_3(U_j)$, $j = 1, \dots, n$, and $\overline{U}_j$ are compact subsets of U. Choose $\varepsilon > 0$, so that the sets

$$V_j = \{x \in U_j\colon \operatorname{dist}(x, \partial U_j) > \varepsilon\}, \qquad j = 1, 2, \dots, n,$$

form a covering of K. Note, that u is bounded on $\overline{U}_1 \cup \dots \cup \overline{U}_n$. Consider now regularizations $R_L\tilde{u}$, $L \geq 0$, as in Lemma 2.3, where $\tilde{u} = u$ on $U_1 \cup \dots \cup U_n$, and $\tilde{u} = 0$ otherwise. Since we already know that (0.1)–(0.6) and (0.9) hold for F_3, and since $\tilde{u}|U_j \in F_3(U_j)$, $j = 1, \dots, n$, Lemma 2.3 implies that there is L_0, such that for $L \geq L_0$,

$(R_L\tilde{u})|V_j \in F_3(V_j)$, $j = 1, \dots, n$. Since $R_L\tilde{u}$ is a continuous function, we conclude (by the preceding comments or Assertion 4) that $R_L\tilde{u}|(V_1 \cup \dots \cup V_n) \in F_3$. By Proposition 2.2 (c), $R_L\tilde{u}(x) \searrow u(x)$, $x \in V_1 \cup \dots \cup V_n$, we get $u \in F_3(\text{nbhd}\, K)$, $K \subset U$. Finally, if $u \in \text{usc}(U)$ with $u|U_j \in F_3(U_j)$, consider $K \subset U$, K compact and choose a bounded function $g \in F(\text{nbhd}\, K)$. Then, the functions $\max(u, g - n)$, $n = 1, 2, \dots$, are locally in F_3 and are locally bounded, and so, by the above argument, $\max(u, g - n) \in F_3$. Since $\max(u, g - n)(x) \searrow u(x)$, $x \in K$, we get $u \in F_3(\text{Int}\, K)$. Since K is an arbitrary compact set, $u \in F_3(U)$, as required.

Since $F_3^d = P^d$, cf. (4.7), and F_3 is an affine pseudoconvex class, $F_3 = P$ (cf. Theorem 3.9). □

For future reference, we can reformulate now Assertion 4 as follows.

COROLLARY 4.8. *Let F, P satisfy assumptions of Lemma* 4.3 *and $u\colon U \to R$, $U \subset R^N$, be a continuous function of class P. Then, for every $\varepsilon, \delta > 0$, there is a function $v \in F_1(U)$, such that*

$$\varepsilon|x|^2 + u(x) < v(x) < u(x) + \delta + \varepsilon|x|^2, \qquad x \in U.$$

5. Examples of affine pseudoconvex classes. In this section we return to classes of subharmonic, q-convex and q-plurisubharmonic functions, which were briefly discussed in [7, Examples 2.1–2.3], and apply the results of §§3 and 4 to obtain piecewise-smooth approximation theorems for these classes and to characterize them by their invariance properties. We also construct new examples of $\text{GL}(\mathbf{C}^n)$ invariant affine pseudoconvex classes on $\mathbf{C}^n$.

EXAMPLE 5.1. $P = \text{subh}$ is the class of all usc subharmonic functions defined on open subsets of R^N. It is clear, e.g. by Corollary 3.10, that its associated order cone is $Y = \{A\colon A^T = A, \text{tr}\, A \geq 0\}$. Then by Lemma A.6, $Y^D = Y$, and so by Theorem 3.11, $P^d = P$.

REMARK 5.2. An affine pseudoconvex class P is closed with respect to addition ($P + P \subset P$), if and only if so is its associated order cone Y of symmetric matrices (i.e. $Y + Y \subset Y$). (Obvious, by Definition 3.1 and Theorem 3.11.)

PROPOSITION 5.3. *Let P be an affine proper pseudoconvex class of functions on R^N and Y be its associated class of matrices. Then, the following conditions are equivalent*:

(i) $P + P \subset P$, $P^d = P$;

(ii) $P + P \subset P$, $P^d + P^d \subset P^d$;

(iii) *there is a positive definite symmetric matrix* (δ_{ij}), *such that* $A = (a_{ij}) \in Y$, *if* $\sum a_{ij}\delta_{ij} \geq 0$.

(iv) P *consists of all functions subharmonic with respect to the Laplace operator* $\sum_{ij} \delta_{ij}\partial^2 f/\partial x_i \partial x_j$ *(corresponding to some inner-product metric on* R^N*).*

Proof (Sketch). Implications (iii)$\Rightarrow$(i) and (i)$\Rightarrow$(ii) are obvious (cf. Lemma A.6, Remark 5.2).

(ii)$\Rightarrow$(iii). By Remark 5.2, $Y + Y \subset Y$ and $Y^D + Y^D \subset Y^D$. The latter implies, by Lemma A.6, that $[-\mathrm{Int}(Y)]^c$, and so $[\mathrm{Int}(Y)]^c$ as well, are semigroups. Then, the boundary $\partial Y = Y \cap [\mathrm{Int}(Y)]^c$ is a closed nowhere dense semigroup in $S = S(R^{N\times N})$ = the space of symmetric $N \times N$ matrices. We will show that $(-A) \in \partial Y$ whenever $A \in \partial Y$, which will imply that ∂Y is a subgroup of S. (Indeed, if $A \in \partial Y$ and $(-A) \notin \partial Y$, then either $-A \in \mathrm{Int}(Y)$ or $-A \in Y^c$. In the first case, consider $A_n \in \mathrm{Int}(Y)$, such that $A_n \to A$. For some n, both $A_n \in \mathrm{Int}(Y)$ and $(-A_n) \in \mathrm{Int}(Y)$, and since $\mathrm{Int}(Y) + \mathrm{Int}(Y) \subset \mathrm{Int}(Y)$, $0 \in \mathrm{Int}(Y)$, which implies that for some $\varepsilon > 0$, $-\varepsilon I \in \mathrm{Int}(Y)$. The latter being a semigroup, $-n\varepsilon I \in \mathrm{Int}(Y)$ for $n > 0$, and since Y is an order cone, $Y = S$, contrary to the properness. In the second case, consider $A_n \in Y^c$, such that $A_n \to A$. Then, by Lemma A.6, $-A_n \in \mathrm{Int}(Y^D)$. Since $-A \in Y^c$, $A \in \mathrm{Int}(Y^D)$ and $A_n \in \mathrm{Int}(Y^D)$ for large n, and so $O = (-A_n) + A_n \in \mathrm{Int}(Y^D)$, which, similarly as above, contradicts the properness of Y^D.)

Since ∂Y is a closed subgroup of S, it is isomorphic with $R^k \times Z^l$. Since $S\backslash\partial Y = \mathrm{Int}(Y) \cup [-\mathrm{Int}(Y^D)]$, it is a union of two disjoint connected open sets (an order cone is clearly connected), and so $\partial Y \cong R^k$ and is a hyperplane in S. Consequently, $Y = \{(a_{ij})\colon \sum \delta_{ij}a_{ij} \geq 0\}$, for some symmetric matrix (δ_{ij}). Furthermore, $0 \in Y$, and so Y contains the cone of positive-definite matrices, which forces (δ_{ij}) to be positive definite.

It is obvious that (iii) and (iv) are equivalent. □

EXAMPLE 5.4. Denote by conv_q the class of all q-convex functions on R^N, cf. e.g. [**9**, Example 2.3] for the definition. Then, (by Corollary 3.10), its associated class of matrices is

$$Y_q = \{A \in S(R^{N\times N})\colon A \text{ has at most } q \text{ negative eigenvalues}\}.$$

REMARK 5.5. q-convex functions can be approximated by piecewise linear q-convex functions in the following sense. If $u \in \mathrm{conv}_q(\mathrm{nbhd}\, K)$, where K is a compact set, then there exists a sequence $(u_n)_{n=1}$ of functions continuous near K, such that $u_n(x) \searrow u(x)$, $x \in K$, and u_n, $n = 1, 2, \ldots$, is locally representable (near $x \in K$) by $\max(f_1, \ldots, f_{m(x)})$, with $f_j = \min(l_0, l_1, \ldots, l_q)$, where l_k are affine functions on R^N. This observation is a corollary of Lemma 4.3. For the proof, let $F = \{\min(l_0, l_1, \ldots, l_q) \colon l_k \text{ affine}\}$, then conditions (4.2) and (4.3) of Lemma 4.3 are obvious, and condition (4.1), that $F^d = (\mathrm{conv}_q)^d$, follows easily from **[9]**, Definition 1.11 and Example 2.3.

PROPOSITION 5.6. *Every affine pseudoconvex class of functions on R^N, which is preserved by composition with nonsingular linear transformations, is identical with* conv_q, *for some* $0 \leq q \leq N - 1$.

Proof. Let P be a pseudoconvex class in question and Y its associated order cone of matrices. Applying Corollary 3.10 and Theorem 3.11, one obtains that P is preserved by composition with the linear map $x \to Wx \colon R^N \to R^N$, if and only if $\{W^T A W \colon A \in Y\} \subset Y$. Choose $A_0 \in Y$ to be the matrix with largest number of negative eigenvalues and denote this number by q. Then every (symmetric) matrix with exactly q negative eigenvalues is of the form $W^T A_0 W$ for some W nonsingular and belongs to Y. If a symmetric matrix B has less than q negative eigenvalues it can be represented (e.g. using the diagonal form) as $A_1 + A_2$, where A_1 has q negative eigenvalues and A_2 is positive semidefinite. Then $A_1 \in Y$ and $B = A_1 + A_2 \in Y$, because Y is an order cone. Thus, $Y = Y_q$, which concludes the proof (by Theorem 3.11 and Example 5.4). □

In the remainder of this section we will study examples of affine pseudoconvex classes on $\mathbf{C}^n = R^{2n}$. In this setting, it is more convenient to identify the order cone Y associated to P (as in Definition 3.1) with a class of real-homogeneous quadratic forms on $\mathbf{C}^n$, rather than with the class of $2n \times 2n$ real symmetric matrices. Every such form can be uniquely represented as

$$z \to 2(\overline{z}^T H z + \operatorname{Re} z^T A z), \tag{5.1}$$

where $H^* = H$ and $A^T = A$ are respectively Hermitian and symmetric complex $n \times n$ matrices. Thus, Y can be also identified with a set of such pairs (H, A).

EXAMPLE 5.7. Denote by P_q the class of all usc q-plurisubharmonic functions on (open subsets of) $\mathbf{C}^n$ (originally studied by Hunt and

Murray [**4**]; cf. also [**5**, §1] for the definition). Denote by $\tilde{Y}_q$ its associated class of quadratic forms. By [**7**, Theorem 4.1],

$$(5.2) \qquad (H, A) \in \tilde{Y}_q, \quad \text{if and only if } H \text{ has at most } q \text{ negative eigenvalues,}$$

where H, A are as in (5.1).

We will apply now Lemma 4.3 to get a partial generalization, to classes P_q, of the approximation result for the plurisubharmonic function due to Bremermann [**2**]; cf. also Gamelin and Sibony [**5**].

THEOREM 5.8. *Let u be a q-plurisubharmonic function in a neighborhood of a compact set K. Then, there is a sequence of continuous functions $u_n \in C(V_n)$, $V_n \supset K$, with V_n open, such that $u_n(z) \searrow u(z)$, $z \in K$, and every $z_0 \in V_n$ has a neighborhood V_0, such that $u|v_0$ is equal to the maximum of several functions of the form*

$$(5.3) \qquad z \to \min(\log|f_0(z)|, \log|f_1(z)|, \ldots, \log|f_q(z)|), \qquad z \in V_0,$$

where $f_i(z)$ are nonvanishing analytic functions in V_0. If, in addition, u is continuous and q-plurisubharmonic in a domain U, u_n can be chosen as continuous functions in $V_n = U$, so that $u_n(z) \searrow u(z)$, $z \in U$.

Proof. Denote by $F(V_0)$ the set of functions of the form (5.3) and let $F = \bigcup F(V_0)$, with V_0 open. In view of Lemma 4.3 it suffices to show that F satisfies conditions (4.1)–(4.3), of which (4.2) is obvious (let $f_0 = f_1 = \cdots = f_q \equiv 1$) and (4.3) can be seen by replacing in (5.3) f_j by $e^h f_j$, where $h(z)$ is an analytic function with $\operatorname{Re} h = g$.

It remains to check that F and P_q have the same dual class, $F^d = P_q^d$. Since $F \subset P_q$, by [**7**, Lemma 6.2] and $P_q^d = P_{n-q-1}$, by [**9**, Definition 1.11] and [**7**, Proposition 1.1 (i) and Theorem 5.1], we have to show that

$$F^d \subset P_{n-q-1}.$$

Let $u \in F^d(U)$. By [**8**, Definition 1.1 and Lemma 4.4], u is $(n-q-1)$-plurisubharmonic, if for every polynomial $p(z)$, for every $(n-q)$-dimensional complex plane L and for every ball B with $\overline{B} \subset U$ it holds

$$(5.4) \qquad \max(u + \operatorname{Re} p)|\overline{B} \cap L \le \max(u + \operatorname{Re} p)|(\partial B) \cap L.$$

Choose complex affine functions $l_1, \ldots, l_q$, such that $L = \{z : l_j(z) = 1, j = 1, \ldots, q\}$, and consider functions

$$v_n = \min(\log|e^p|, n\log|l_1|, \ldots, n\log|l_q|).$$

Clearly, $v_n \in F(\text{nbhd}\,\overline{B})$ (provided l_j are so chosen that $l_j(z) \neq 0$ for $z \in \overline{B}$), and so $\max(u + v_n)|\overline{B} \leq \max(v + v_n)|\partial B$, for $n = 1, 2, \ldots$ As $n \to +\infty$, this inequality yields (5.4). □

PROPOSITION 5.9. *Every proper pseudoconvex class on* $\mathbf{C}^n$ *which is preserved by composition with biholomorphic mappings and satisfies axiom* (0.8) *is identical with the class of all* q*-plurisubharmonic functions for some* q, $0 \leq q \leq n-1$.

COROLLARY 5.10. *In particular, the class of all plurisubharmonic functions on* $\mathbf{C}^n$ *is the unique proper pseudoconvex class on* $\mathbf{C}^n$ *which is closed with respect to addition, biholomorphically invariant and satisfies axiom* (0.8).

We conjecture that Proposition 5.9 remains true, if the axiom (0.8) is replaced by the weaker localization axiom (1.8) of **[9]**.

Proof of Proposition 5.9.

Assertion 1. If an affine pseudoconvex class P on $\mathbf{C}^n$ is preserved by composition with biholomorphic maps, then it is *complex* in the following sense

(5.5) whenever $u \in P(U)$ and v is pluriharmonic on V, then $u + v \in P(U \cap V)$.

In other words, we have to show that the class AP (as defined above in Condition (1.1)) contains all pluriharmonic functions. Clearly, AP is an affine pseudoconvex class itself and is preserved by composition with biholomorphic mappings. Let $v\colon V \to R$ be pluriharmonic and $z_0 \in V$. Since AP satisfies axiom (0.7), it suffices to show that $u \in P(\text{nbhd}\, z_0)$, $z_0 \in V$. If $\operatorname{grad} u(z_0) \neq 0$, there exists an analytic function f_0 near z_0, such that $\operatorname{Re} f_0 = u$ near z_0 and analytic functions $f_1, \ldots, f_n$, such that $\Phi(z) = (f_0(z), \ldots, f_{n-1}(z))$ is a biholomorphic map on a neighborhood of z_0. Since (0.8) holds, the function $l(z) \in \operatorname{Re} z_1$ is of class AP, and so $u = l \circ \Phi \in AP(\text{nbhd}\, z_0)$. If $\operatorname{grad} u(z_0) = 0$, consider any nonconstant affine function $l(z)$, then $\operatorname{grad}(u+l) \neq 0$ for z near z_0, and so, by the above argument, $(u+l) \in AP(\text{nbhd}\, z_0)$. Since AP satisfies axiom (0.8) as well, $u = (u + l) + (-l) \in AP(\text{nbhd}\, z_0)$. The assertion is established.

If A_0 is a complex symmetric matrix, then $v(z) = \operatorname{Re} z^T A_0 z$ is a pluriharmonic function; by adding it to the form (5.1) we can modify its antihermitian part at will, which gives the next assertion.

Assertion 2. If P is a translation invariant pseudoconvex class which is complex in the sense of Assertion 1, and Y is its associated order cone, then

$$(H, A) \in Y, \text{ if and only if } (H, 0) \in Y. \tag{5.6}$$

Denote by Z the set of complex Hermitian H, such that $(H, 0) \in Y$ and by Z_q the set of Hermitian matrices with no more than q negative eigenvalues. If W is a nonsingular matrix and H is the complex Hessian of u at some point, then W^*HW is the complex Hessian of $z \to u(Hz)$ (at some point), and so

$$H \in Z, \text{ if and only if } W^*HW \in Z, \text{ for } W \text{ nonsingular.} \tag{5.7}$$

In the same way as in the proof of Proposition 5.6, one shows that an order cone Z (relative to the cone of positive definite Hermitian matrices), which satisfies (5.7), must be equal to Z_q, for some $0 \leq q \leq n-1$, and so $Y = \tilde{Y}_q$, cf. (5.2), which concludes the proof. □

We will consider now $\mathrm{Gl}(\mathbf{C}^n)$-invariant affine pseudoconvex classes on $\mathbf{C}^n$. Except for trivial cases, they will not be biholomorphically invariant (and so will not be "complex" in the sense of conditions (5.5)). To avoid excessive complications of linear-algebraic nature, we discuss below classes P, which consist of plurisubharmonic functions. This amounts to the following assumption on the order cone Y associated to P:

$$\text{if } (H, A) \in Y, \text{ then } H \text{ is positive semi-definite.} \tag{5.8}$$

REMARK 5.11. An affine pseudoconvex class is $\mathrm{Gl}(\mathbf{C}^n)$ invariant, if and only if

$$(W^*HW, W^TAW) \in Y, \text{ whenever } (H, A) \in Y \text{ and } W \in \mathrm{Gl}(\mathbf{C}^n). \tag{5.9}$$

This is obvious by Corollary 3.10 and Theorem 3.11.

We have to review now basic facts on the structure of the forms (5.1) with positive definite H. (Their proofs can be easily produced by the reader, but we do not know the references.)

If H, A are complex $n \times n$ matrices and H is Hermitian positive definite and A is symmetric, then there exist non-zero vectors a_j, and non-negative numbers k_j, $j = 0, 1, 2, \ldots, n-1$, such that

$$Aa_j = k_j \overline{H} \overline{a}_j, \qquad j = 0, 1, \ldots, n-1, \tag{5.10}$$

$$\overline{a}_j^T H a_j \geq 0, \qquad j = 0, 1, \ldots, n-1, \tag{5.11}$$

$$a_j^T A a_s = \overline{a}_j^T H a_s = 0, \qquad j \neq s, \tag{5.12}$$

$$\operatorname{span}\{a_j\} = \mathbf{C}^n. \tag{5.13}$$

Thus, the forms $\overline{z}^T Hz$ and $z^T Az$ are simultaneously diagonalized by the basis $(a_j)_j$.

NOTATION 5.12. We can reorder the basis so that $k_0 \geq k_1 \geq \cdots \geq k_{n-1}$. Then the sequence $(k_0, k_1, \ldots, k_{n-1})$ is uniquely determined (which follows from the next proposition). We will call k_j the jth critical value of the pair (H, A) and denote $k_j = c_j(H, A)$.

PROPOSITION 5.13. *For* $j = 0, 1, \ldots, n-1$,

$$c_j(H, A) = \min_X \max_{z \in X \setminus \{0\}} (\operatorname{Re} z^T Az / \overline{z}^T Hz),$$

where X *varies through complex subspaces of* $\mathbf{C}^n$ *of codimension* j.

REMARK 5.14. By the last proposition, the form $z \to \overline{z}^T Hz + \operatorname{Re} z^T Az$, with H positive-definite, is positive semi-definite, if and only if $c_0(H, A) \leq 1$.

REMARK 5.15. If the form $z \to \overline{z}^T H_0 z + \operatorname{Re} z^T Az$ is positive semi-definite and H is Hermitian positive definite, then $c_j(H+H_0, A+A_0) \leq \max(1, c_j(H, A))$, $j = 0, 1, \ldots, n-1$.

Indeed, if $j \in \{0, 1, \ldots, n-1\}$ and $k_j = \max(c_j(H, A), 1)$, there is a subspace $X_j \subset \mathbf{C}^n$ of complex codimension j, such that

$$\operatorname{Re} z^T Az \leq k_j \overline{z}^T Hz, \qquad z \in X_j. \tag{5.14}$$

Since $z \to \overline{z}^T H_0 z + \operatorname{Re} z^T A_0 z$ is positive semi-definite, $\operatorname{Re} z^T A_0 z \leq \overline{z}^T H_0 z$, and so $\operatorname{Re} z^T A_0 z \leq k_j \overline{z}^T H_0 z$, $z \in \mathbf{C}^n$. By this and (5.14), $\operatorname{Re} z^T (A + A_0) z \leq k_j \overline{z}^T (H + H_0) z$, $z \in X_j$, and so $c_j(H, A) \leq k_j$, by Proposition 5.13.

We can describe now the next example which provides building blocks for the construction of classes Y satisfying (5.8) and (5.9).

EXAMPLE 5.16. For every sequence of real numbers $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$, such that $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{n-1} \geq 1$, define $Y_\alpha = Y_{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}}$ as the closure of the set of all pairs (H, A) (with $H^* = H$, $H > 0$, $A^T = A$), such that $c_j(H, A) \leq \alpha_j$, $j = 0, 1, \ldots, n-1$. By Remark 5.15, Y_α is stable with respect to the addition of positive semi-definite forms, and so is an order cone. It is also clear, e.g. by Proposition 5.13, that for every $W \in \mathrm{Gl}(\mathbf{C}^n)$

$$c_j(W^* HW, W^T AW) = c_j(H, A), \qquad j = 0, 1, \ldots, n-1, \tag{5.15}$$

and so Y_α satisfies (5.9).

PROPOSITION 5.17. *If a closed* $\mathrm{Gl}(\mathbf{C}^n)$ *invariant order cone* Y *contains a pair* (H, A) *with* H *positive-definite, then* $Y \supset Y_\alpha$*, where* $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{n-1})$ *and* $\alpha_j = \max(1, c_j(H, A))$.

Proof. Using relations (5.10)–(5.13), one can observe that every two pairs (H, A) and (H_1, A_1), with the same critical numbers $c_j(\cdot,\cdot)$, are equivalent, namely $(H_1, A_1) = (W^*HW, W^TAW)$, where W is the nonsingular transformation mapping diagonalizing vectors of one pair onto those of the other. Thus, the form

$$z \to \sum_{j=0}^{n-1} |z_j|^2 + \sum_{j=0}^{n-1} k_j \operatorname{Re} z_j^2 \tag{5.16}$$

belongs to Y, where $k_j = c_j(H, A)$, $j = 0, 1, \dots, n-1$. Let $0 < \beta_j < \max(k_j, 1)$, $j = 0, 1, \dots, n-1$. It remains to show that a form with critical numbers equal to $\beta_0, \dots, \beta_{n-1}$ belongs to Y.

Note, first, that for every j there are $h_j > 0$ and $s_j \in [0, 1]$, such that

$$(k_j + h_j s_j)/(1 + h_j) = \beta_j, \qquad j = 0, 1, \dots, n-1. \tag{5.17}$$

Then, the form

$$z \to \sum_{j=0}^{n-1} h_j |z_j|^2 + \sum_{j=0}^{n-1} h_j s_j \operatorname{Re} z_j^2 \tag{5.18}$$

is positive semidefinite. Y being an order cone, it contains the sum of the forms (5.16) and (5.18). By (5.17), the critical numbers of this sum are equal to $\beta_0, \beta_1, \dots, \beta_{n-1}$. □

REMARK 5.18. Every closed order cone, satisfying conditions (5.8) and (5.9), is of the following form: there is a set $S_0 \subset R_+^n$, such that $(H, A) \in Y$, if and only if $H^* = H$, $H > 0$ and $(c_j(H, A))_{j=0}^{n-1} \in S_0$. Set S_0 has properties (5.19) and (5.20), given next, and every subset $S_0 \subset R_+^n$ with these properties corresponds to some $\mathrm{Gl}(\mathbf{C}^n)$ invariant order cone.

$$\begin{aligned} &\text{If } \alpha = (\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \in S_0, \quad \text{then} \\ &\qquad \alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{n-1} \geq 0 \quad \text{and} \quad (\max(\alpha_j, 1))_{j=0}^{n-1} \in S_0. \end{aligned} \tag{5.19}$$

$$\begin{aligned} &\text{If } \alpha = (\alpha_j)_{j=0}^{n-1} \in S_0, \qquad \beta = (\beta_j)_{j=0}^{n-1}, \quad \text{where} \\ &\qquad \beta_0 \geq \beta_1 \geq \cdots \geq \beta_{n-1} \quad \text{and} \quad \beta \leq \alpha, \quad \text{then } \beta \in S_0. \end{aligned} \tag{5.20}$$

EXAMPLE 5.19. In the special case, when $\alpha = (K, K, \ldots, K)$, $+\infty \geq K \geq 1$, $(H, A) \in Y_\alpha$, if and only if

$$\operatorname{Re} z^T A z \leq K \bar{z}^T H z, \qquad z \in \mathbf{C}^n. \tag{5.21}$$

Denote this order cone Y_α by Y^K. Condition (5.21) makes it clear that Y^K is closed with respect to addition, $Y^K + Y^K \subset Y^K$, and so is a convex cone.

THEOREM 5.20. *Affine pseudoconvex classes on $\mathbf{C}^n$ which are closed with respect to addition and are* $\mathrm{Gl}(\mathbf{C}^n)$ *invariant form a monotone one parameter family, parametrized by $K \in [1, +\infty]$. Namely, for every $K \in [1, +\infty)$, there is exactly one such class which contains function $z \to |z|^2 + K \operatorname{Re} z^T z$, but does not contain any function $z \to |z|^2 + (K + \varepsilon) \operatorname{Re} z^T z$. The case $K = +\infty$ corresponds to the class of all q-plurisubharmonic functions.*

Proof. By Theorem 3.11, it suffices to show that every closed cone Y satisfying (5.8), (5.9) and $Y + Y = Y$ is equal to Y^K for some $K \in [1, +\infty]$. Let

$$K = \sup\{c_0(H, A) \colon (H, A) \in Y\}. \tag{5.22}$$

Clearly, $Y \subset Y^K$. In view of the previous arguments, in order to show that $Y^K \subset Y$ it is enough to check that for every $K_0 \in (0, K)$ the form

$$z \to |z|^2 + K_0 \operatorname{Re} z^T z \tag{5.23}$$

belongs to Y. By Proposition 5.17 and Eq. (5.22), if $0 < \varepsilon < \min(K_0, 1)$, then for each $j = 0, 1, \ldots, n-1$ the form $z \to |z_j|^2 + \varepsilon \sum_{i \neq j} |z_i|^2 + K_0 \operatorname{Re} z_j^2$ belongs to Y, and so Y contains their sum, which is equal to $z \to (1 + \varepsilon n)|z|^2 + K_0 \operatorname{Re} z^T z$. Letting $\varepsilon \to 0$, we obtain that the form (5.23) belongs to Y. □

6. A counterexample to smooth approximation in an affine pseudoconvex class. The next example and the following proposition show that Theorem 4.1 cannot be, in general, improved to yield smooth approximation. It is presently unknown which pseudoconvex classes (aside for those closed with respect to addition) allow for smooth approximation.

EXAMPLE 6.1. For an open set $U \subset \mathbf{C}^5$ define $P(U)$ as the set of all 2-plurisubharmonic functions $u(z, w)$ on U, where $z = (z_1, z_2, z_3, z_4)$, $w \in \mathbf{C}$, such that for every $z \in \mathbf{C}^4$, the function $w \to u(z, w)$ is locally convex. Let $P = \bigcup P(U)$, $U \subset \mathbf{C}^5$. Since the class of 2-plurisubharmonic functions is affine pseudoconvex (axioms (0.1)–(0.9) hold by **[4]**, cf. also **[8**, §1]), it is obvious that class P satisfies axioms (0.1)–(0.9) as well.

PROPOSITION 6.2. *For $(z,w) = (z_1, z_2, z_3, z_4, w) \in \mathbf{C}^5$, let*

$$u(z,w) = (\operatorname{Im} w)^2 + \max(-|z_1|^2 - |z_2|^2 - \operatorname{Re} w, -|z_3|^2 - |z_4|^2 + \operatorname{Re} w).$$

Then u belongs to the class P defined in Example 6.1. Moreover, it is not possible to find a neighborhood V of 0 in $\mathbf{C}^5$ and a sequence of functions $(u_n)_{n=1}^{\infty} \subset P(V) \cap C^{\infty}(V)$, such that

$$u_n(z,w) \searrow u(z,w), \qquad (z,w) \in V.$$

Proof. Let $u^1(z,w) = -|z_1|^2 - |z_2|^2 - \operatorname{Re} w + (\operatorname{Im} w)^2$ and $u^2(z,w) = -|z_3|^2 - |z_4|^2 + \operatorname{Re} w + (\operatorname{Im} w)^2$. Observe that each of these two functions is 2-plurisubharmonic (on $\mathbf{C}^5$) as a sum of a 2-plurisubharmonic function $(z,w) \to -|z_1|^2 - |z_2|^2$ or $(z,w) \to -|z_3|^2 - |z_4|^2$ and of a convex function $(z,w) \to \pm\operatorname{Re} w + (\operatorname{Im} w)^2$. Since both u^1 and u^2 are convex in direction w, u^1, u^2 are of class P. Since $u = \max(u^1, u^2)$, $u \in P$ by axiom (0.7). Let now $v(z) = \inf_w u(z,w)$. By direct computation

$$(6.1) \quad v(z) = \inf u(z,w) = -\tfrac{1}{2}|z|^2 = -\tfrac{1}{2}(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2).$$

Assertion. Let $B_1 \subset \mathbf{C}^n$ and $B_3 \subset B_2 \subset \mathbf{C}^q$ be open balls centered at 0, such that $\overline{B}_3 \subset B_2$. Assume that $u(z,w)$, $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_q)$ is a smooth r-plurisubharmonic function on a neighborhood of $\overline{B}_1 \times \overline{B}_2$. Assume further that for every $z \in \overline{B}_1$, the function $w \to u(z,w)$ is strongly convex (i.e. has positive-definite real Hessian) on a neighborhood of $\overline{B}_2$ and

$$(6.2) \qquad \max\{u(z,w) \colon w \in \overline{B}_3\} < \min\{u(z,\zeta) \colon \zeta \in \partial B_2\}.$$

Then, the function $v(z) = \min\{u(z,w) \colon w \in \overline{B}_2\}$ is $(q+r)$-plurisubharmonic on B_1.

This fact was obtained as a part of the proof of Theorem 5.1 in **[10]**, although, for $r \geq 1$, it was not formulated as a separate result. While the results in **[10**, §5] were formulated for the case when slice functions $w \to u(z,w)$ are strongly convex on $\mathbf{C}^q$, the proof given there remains valid under assumptions of the assertion.

Fix now $0 < \delta < 1$ and let $B_1 = \{z \in \mathbf{C}^4 \colon |z| < \delta\}$, $B_2 = \{w \in \mathbf{C} \colon |w| < \delta\}$ and $B_3 = \frac{1}{2}B_2$. Suppose now that there is a sequence $(u_n)_{n=1}^{\infty}$

of smooth functions of class P defined on a neighborhood of $\overline{B}_1 \times \overline{B}_2$, in $\mathbf{C}^5$, such that $u_n(z,w) \searrow u(z,w)$, $(z,w) \in \overline{B}_1 \times \overline{B}_2$. Without loss of generality, we can assume that functions u_n are strictly 2-plurisubharmonic and strongly convex in w on $\overline{B}_1 \times \overline{B}_2$. It is easy to see that for n large enough ($m \geq n_0$), functions u_n must satisfy condition 6.2. Hence, the assertion applies and we obtain the functions $v_n(z) = \min\{u_n(z,w) \colon w \in \overline{B}_2\}$ are 3-plurisubharmonic on B_1. Clearly, $v_n(z) \searrow v(z)$, $z \in \overline{B}_1$, and so $v(z)$ must be 3-plurisubharmonic in $B_1 \subset \mathbf{C}^4$, which contradicts (6.1). □

7. The localization axiom and separation properties. It is a natural question, whether affine pseudoconvex classes on R^N (as defined by axioms (0.1)–(0.9) above) are identical with those pseudoconvex classes on R^N (see [**9**, Definition 1.4] or §1 above) which are translation invariant in the sense of (0.9). The only axioms which are different for these two notions are axiom (0.8) and its abstract counterpart, the localization axiom (1.1). Thus, the problem is reduced to the question, whether, in the presence of axioms (0.1)–(0.7) and (0.9), the localization axiom (1.1) implies (0.8). The answer is negative, as the next example shows.

EXAMPLE 7.1. Write $x \leq y$ for $x, y \in R^N$, if $x_j \leq y_j$, $j = 1, 2, \ldots, N$. We will say that $f \colon U \to R$, $U \subset R^N$, is order preserving, if $f(x) \leq f(y)$ whenever $x \leq y$, $x, y \in U$. Define a class of functions P on R^N as $P = \bigcup P(U)$, where an usc function f belongs to $P(U)$, if and only if it is order preserving on each parallelepiped with sides parallel to the coordinate axis. It is clear that axioms (0.1)–(0.7) and (0.9) hold, while (0.8) fails. In contrast, the localization axiom (1.1) holds. To check the latter, it is convenient to look at it from a more general point of view. First, $P + P \subset P$, and so $P = AP$, cf. (1.1), therefore P has clearly the separation property (7.1), defined next. Then, the property (1.1) follows immediately from Proposition 7.2.

(7.1) (*Weak separation property*) For every compact set $K \subset M$ and for every $x, y \in K$ with $x \neq y$, there is a function $v \in AP(\mathrm{nbhd}\, K)$, such that $v(x) \neq v(y)$. (One of these values can be $-\infty$.)

PROPOSITION 7.2. (a) *If a class P of usc functions satisfies* (1.1), *then it has property* (7.1).

(b) *Let P be a translation invariant pseudoconvex class of functions on R^N satisfying conditions* (0.1)–(0.7). *Assume that*

$$(7.2) \qquad \textit{for every } r > 0 \textit{ and } v \in AP(U), \qquad rv \in AP(U).$$

Then, the weak separation property (7.1) *implies the localization axiom* (1.1).

Proof. (a) Consider $x, y \in K$, $x \neq y$, and suppose that $v(x) = v(y)$ for every $v \in AP(\text{nbhd}\, K)$. Let $u(z) = 0$, for $z = x, y$, and $u(z) = -\infty$, for $z \in K \backslash \{x, y\}$. Then, for every $v \in AP(\text{nbhd}\, K)$, $(u+v)(x) = (u+v)(y)$, while $(u+v)(z) = -\infty$ for $z = x, y$, which contradicts (1.1).

(b) It is clear that the class $P_1 = AP$ satisfies axioms (0.1) through (0.5), and (0.9); since AP contains constant functions, P_1 satisfies axiom (0.6) as well. Hence, Corollary 2.5 applies to P_1, and so the subclass $C(K) \cap AP(\text{nbhd}\, K)$ is dense in AP, in the sense of Corollary 2.5. Consequently, functions of the class $C(K) \cap AP(\text{nbhd}\, K)$ separate points of K.

The remainder of the proof is the modification of the argument used in the proof of [**9**, Lemma A.1], and so we use the notation and the results of this proof, adding only necessary changes.

Let $g \in \text{usc}(K)$, and $0 < \varepsilon < 1$. To prove (1.1), we will construct a function $v \in AP(\text{nbhd}\, K)$ and a point x_0, such that $\|v\|_\infty < \varepsilon$ and

$$(u+v)(x_0) > (u+v)(x), \qquad x \in K \backslash \{x_0\}. \tag{7.3}$$

Let X be a compact subset of R^N, such that $K \subset \text{Int}(X)$, and let $A =$ the uniform closure in $C(X)$ of the linear span of $C(X) \cap AP(\text{nbhd}\, X)$. As noted above, functions from $C(X) \cap AP(\text{nbhd}\, X)$ separate points of X, and so we can choose a sequence $(f_n) \subset AP(\text{nbhd}\, X) \cap C(X)$, such that $\text{Cl}(\text{span}\{f_n\}) = A$. By (7.2), we can assume without loss of generality that $\|f_n\| \leq \varepsilon 2^{-n-1}$, $n = 1, 2, \ldots$, so that

$$\sum_{n=1}^{\infty} \|f_n\| \leq \tfrac{1}{2}\varepsilon. \tag{7.4}$$

Let now $u(x) = g(x) + \sum_{n=1}^{\infty} f_n(x)$, for $x \in K$, and $u(x) = -\infty$, for $x \in X \backslash K$. It is clear that X, A, u and the sequence $(f_n)_{n=1}^{\infty}$ satisfy all the assumptions of [**9**, Lemma A.1] and of the arguments used in its proof. By the construction, employed in this proof, one can find $x_0 \in X$ and $(y_n) \in l^2$, such that

$$\sum_{n=1}^{\infty} y_n^2 < 4^{-1}\varepsilon^2, \tag{7.5}$$

$$u(x_0) + \rho(x_0) > u(x) + \rho(x), \qquad x \in X \backslash \{x_0\}, \tag{7.6}$$

where $\rho(x) = \sum_{n=1}^{\infty} y_n f_n(x)$. Necessarily, $x_0 \in K$ ($u = -\infty$ off K).

Let now

$$v(x) = \sum_{n=1}^{\infty} f_n(x) + \rho(x) = \sum_{n=1}^{\infty} (1 + y_n) f_n(x), \qquad x \in X.$$

Clearly, $v \in A$. Furthermore, $|y_n| < \varepsilon/2 < 1$, and so $(1 + y_n) f_n \in AP(\operatorname{nbhd} K)$, by (7.2), and since the series $\sum_{n=1}^{\infty}(1+y_n)f_n$ is uniformly convergent on X, $v \in AP(\operatorname{Int} X)$, cf. [9, Proposition 3.2], and so $v \in AP(\operatorname{nbhd} K)$. On the other hand, $u + \rho = g + v$, and so (7.6) implies (7.3). □

REMARK 7.3. In the above proof the translation invariance property (0.9) of P is used only to show that functions in $C(K) \cap AP(\operatorname{nbhd} K)$ separate points of K. Once this is known, the rest of the argument is valid for classes P of usc functions on a locally compact space M satisfying conditions (1.1)–(1.6) of [9].

Example 7.1 is somewhat pathological from our point of view, because it does not admit a topology basis consisting of P-regular neighborhoods (cf. Definition 1.4). To exclude situations like in Example 7.1, we consider the following form of separation property, which is, clearly, not shared by Example 7.1, and is stronger than the weak separation property (7.1).

(7.7) (*Separation property*). For every compact $K \subset M$ and $x, y \in K$, with $x \neq y$, there is $v \in AP(\operatorname{nbhd} K)$, such that $v(x) > v(y)$.

One might now update the initial question and ask whether, in the presence of axioms (0.1)–(0.7) and (0.9), the separation property (7.7) implies axiom (0.8). The next example shows that this attempt fails as well.

EXAMPLE 7.4. We define a class P of functions on R^N by first defining that a $C^{(2)}$-smooth function $f\colon U \to R$ belongs to $P(U)$, if $|\operatorname{grad} f(x)| \leq \operatorname{tr} \operatorname{Hess} f(x)$. Next, a usc function $u\colon U \to [-\infty, +\infty)$ belongs to P, if for every compact set $K \subset U$, there is a sequence $(u_n) \in C^{(2)}P(\operatorname{nbhd} K)$, such that $u_n(x) \searrow u(x)$, $x \in K$.

One can see easily that P is a convex cone, and so $AP = P$. Axiom (0.8) fails, while the separation property (7.7) holds. To see the latter, let $f_a(x) = \cos h(a, x)$, for $a, x \in R^N$. It is clear that functions f_a, $|a| \geq 1$, and their translates (note, that P is translation invariant) form a set rich enough from which separating function v in the property (7.7) can be always chosen. It remains to show that class P satisfies conditions (0.1)–(0.7), and so is a translation-invariant pseudoconvex class. We omit further details.

Problem 7.5. If P is a pseudoconvex class of functions on M, does the separation condition (7.7) imply the existence of a basis consisting of P-regular neighborhoods?

Comments. Example 7.4 indicates that there are natural instances of translation invariant pseudoconvex classes on R^N with separation property (7.7) and without property (0.8). A natural question is whether Theorem 3.11 can be generalized to this setting. It turns out that this can be done. We outline here some of the necessary modifications, omitting the details.

The associated class of matrices Y (cf. Definition 3.1) has to be replaced by the set of pairs (a, A), where a is a vector (corresponding to the gradient) and A is a matrix (corresponding to the Hessian at the same point). Class Y has the following properties:

(7.8) for every $a \in R^N$, there is $A \in R^{N\times N}$, such that $\{a, A\} \in Y$;

(7.9) if $(a, A) \in Y$ and B is a positive-definite matrix, then $(a, A+B) \in Y$.

With this understanding of Y and a natural definition of Y^D (as the set of (a, B), such that for every $(-a, A) \in Y$, $A+B$ is not negative definite), an obvious analog of Theorem 3.11 holds, with the following restriction. Namely, it is not known whether every closed set Y with properties (7.8) and (7.9) corresponds to some translation invariant pseudoconvex class with the separation property (7.7), and if not, what characterizes such sets Y. We omit further details.

Appendix A. *Order cones and dual order cones in finite dimensional spaces.* In **[11]** we have to consider modifications of Definition 3.1 above with different type of order cones consisting of quadratic forms. For this reason we study order cones in the abstract setting. Incidentally, the proofs become completely elementary. (Some of the results will be needed only in **[11]**.)

DEFINITION A.1. An ordered finite dimensional vector space is a pair (X, X_+), where X is a real vector space (of finite dimension) and X_+ is an open subset of X, such that

$$X_+ \cap X_- = \varnothing, \quad \text{where } X_- = -X_+ = \{-x : x \in X_+\}; \tag{A.1}$$

$$\text{if } x_1, x_2 \in X_+, \quad \text{then } (x_1 + x_2) \in X_+; \tag{A.2}$$

$$\text{if } x \in X_+ \quad \text{and} \quad r > 0, \quad \text{then } rx \in X_+. \tag{A.3}$$

Throughout the Appendix A, (X, X_+) will denote a finite dimensional ordered vector-space.

DEFINITION A.2. (i) A non-empty subset $Y \subset X$, where (X, X_+) is an ordered vector space, is called an *order cone*, if whenever $y \in Y$ and $x \in X_+$, then $(u + x) \in Y$.

(ii) An order cone Y is called proper, if $Y \neq X$.

(iii) If Y is a subset of X (order cone or not), denote

$$Y^D = \{x \in X : (x + Y) \cap X_- = \varnothing\}.$$

We will call Y^D *the dual order cone* (*to* Y).

This terminology is justified by the next proposition.

We note that, in general, an order cone does not have to be a cone in the sense of linear structure, i.e. condition (A.3) may fail (although it still holds in the natural examples, cf. §4 above), and typically, it is not a convex set (cf. the same §4).

PROPOSITION A.3. *If Y^D is nonempty, where $Y \subset X$, then it is a proper, closed order cone.*

Proof. Since Y^D is the intersection of the family of closed sets $\{x \in X : (x + y) \in X \backslash X_-\}$, where $y \in Y$, it is closed.

If $x \in Y^D$ and $x_1 = x + x_+$, where $x_+ \in X_+$, then $x_1 + Y = x + (x_+ + Y) \subset x + Y$, by Definition A.2 (i), and so $(x_1 + Y) \cap X_- = \varnothing$, because $(x + Y) \cap X_- = \varnothing$. Thus, $x_1 \in Y^D$, and so Y^D is an order cone.

To see that Y^D is proper, choose $y_0 \in Y$ and $x_+ \in X_+$. Let $x = -y_0 - x_+$. Then $(x + Y) \cap X_-$ is nonempty (contains $-x_0$), and so $x \in Y^D$. □

The next observation follows directly from Definition A.2.

PROPOSITION A.4. *The union of a family of order cones is an order cone. The intersection of a family of order cones is an order cone, provided it is nonempty.*

PROPOSITION A.5. *Let Y be an order cone in (X, X_+). Then*

(i) *the topological closure $\overline{Y}$ of Y is equal to*

$$\{x \in X : (x + X_+) \subset Y\}, \tag{A.4}$$

(*which might be called the order closure of Y*);

(ii) *the* (*topological*) *interior of Y*, $\mathrm{Int}(Y)$ *is equal to*

$$Y + X_+ = \bigcup_y (y + X_+); \tag{A.5}$$

(iii) *both* $\overline{Y}$ *and* $\operatorname{Int}(Y)$ *are order cones*;
(iv) $\overline{Y} = \operatorname{Cl}(\operatorname{Int}(Y))$;
(v) $\operatorname{Int}(\overline{Y}) = \operatorname{Int}(Y)$.

Proof. (i) Let $Y_1 = \{x \in X : (x + X_+) \subset Y\}$. Choose $x_0 \in X_+$. If $x \in Y_1$, then $(x + n^{-1}x_0) \in Y$ for $n > 0$, and so $x \in \overline{Y}$. Thus $Y_1 \subset \overline{Y}$. Conversely, if $y \in \overline{Y}$, then there is a sequence $(y_n) \subset Y$, such that $\lim_n y_n = y$ and $y_n + X_+ \subset Y$ for every n. Fix an arbitrary $x_0 \in X_+$. Then $(x_0 + y - y_n) \to x_0 \in X_+$, and since X_+ is open, $(x_0 + y - y_n) \in X_+$ for some n. Then, $y + x_0 = y_n + (x_0 + y - y_n) \in y_n + X_+ \subset Y$, for every $x_0 \in X_+$. We have actually proved

$$\text{(A.6)} \qquad \text{if } y \in \overline{Y}, \quad \text{then } y + X_+ \subset \operatorname{Int}(Y).$$

Thus, $\overline{Y} \subset Y_1$, so $\overline{Y} = Y_1$.

(ii) Denote $Y_2 = \bigcup_{y \in Y}(y + X_+)$. By Definition A.2, $Y_2 \subset Y$, and since Y_2 is open, $Y_2 \subset \operatorname{Int}(Y)$. Conversely, let $y \in \operatorname{Int}(Y)$ and $x_0 \in X_+$. There is $\varepsilon > 0$, such that $(y - \varepsilon x_0) \in \operatorname{Int}(Y)$. Then, $y = (y - \varepsilon x_0) + \varepsilon x_0 \in \operatorname{Int}(Y) + \varepsilon x_0 \subset Y_2$. Thus, $\operatorname{Int}(Y) \subset Y_2$, and so $Y_2 = \operatorname{Int}(Y)$.

(iii) The set (A.5) is an order cone by Proposition A.4, and the set (A.4) is an order cone by Definition A.2.

(iv) By relation (A.6), $\overline{Y}$ is contained in the order closure of $\operatorname{Int}(Y)$, and so in $\operatorname{Cl}(\operatorname{Int}(Y))$, by (i). The opposite inclusion is trivial.

(v) Inclusion $\operatorname{Int}(Y) \subset \operatorname{Int}(\overline{Y})$ is trivial; as for the opposite inclusion, $\operatorname{Int}(\overline{Y}) = \overline{Y} + X_+ \subset \operatorname{Int}(Y)$, by (ii) and (A.6) respectively. Thus, $\operatorname{Int}(Y) = \operatorname{Int}(\overline{Y})$. □

LEMMA A.6. *Let* Y *be an order cone in* (X, X_+). *Then*

$$Y^D = \operatorname{Cl}[(-Y)^c] = [-\operatorname{Int}(Y)]^c$$

(*where* "c" *denotes the complement of a set in* X).

Proof. Let

$$Y_1 = X \backslash (-Y), \qquad Y_2 = X \backslash [-\operatorname{Int}(Y)],$$

where $-Y = \{-y : y \in Y\}$. Clearly,

$$\text{(A.7)} \qquad \overline{Y}_1 \subset Y_2.$$

Suppose, that $Y_2 + Y$ intersects X_-, i.e. there are $x \in Y_2$, $y \in Y$, and $x_+ \in X_+$, such that $x + y = -x_+$. Then, $x = -(y + x_+)$. Since $(y + x_+) \in y + X_+ \subset \operatorname{Int}(Y)$, by Proposition A.2 (ii), we get

$x \in [-\mathrm{Int}(Y)]$, contrary to the assumption $x \in Y_2$. Hence, $(Y_2 + Y) \cap X_- = \varnothing$, i.e.

$$Y_2 \subset Y^D. \tag{A.8}$$

To show that $Y^D \subset \overline{Y}_1$, it suffices to check, by Proposition A.5 (i) that if $x \in Y^D$, then $x + X_+ \subset Y_1 = (-Y)^c$, i.e. $(x + X_+) \cap (-Y) = \varnothing$. Suppose, the latter relation fails. Then, there is $x_+ \in X_+$ and $y \in Y$, such that $x + x_+ = -y$, i.e. $x + y = -x_+$. This is impossible, because $x \in Y^D$ and $y \in Y$. Thus, $Y^D \subset \overline{Y}_1$, which, together with inclusions (A.7), (A.8) proves the lemma. □

REMARK A.7. The lemma is surprising in that the formula $Y^D = \mathrm{Cl}[(-Y)^c]$ does not depend on the order structure at all, although both definitions of an order cone and of its dual depend on the order structure. Thus, if Y is an order cone relative to (X, X_+), and X_0 is another open convex linear cone, such that $X_0 \subset X_+$, then the dual order cones to Y relative to the order structures (X, X_+) and (X, X_0) are the same.

LEMMA A.8. *Let Y be an order cone. Then*
(i) *Y is proper, if and only if Y^D is nonempty;*
(ii) $(\overline{Y})^D = Y^D$;
(iii) $(Y^D)^D = \overline{Y}$; *in particular, if Y is closed, then $Y^{DD} = Y$.*

Proof. (i) Since (by Lemma A.6)

$$Y^D = [-\mathrm{Int}(Y)]^c, \tag{A.9}$$

the dual cone Y^D is empty, if and only if $\mathrm{Int}(Y) = X$, i.e. Y is improper.

(ii) By (A.9), $(\overline{Y})^D = [-\mathrm{Int}(\overline{Y})]^c$; since $\mathrm{Int}(Y) = \mathrm{Int}(\overline{Y})$ (by Proposition A.5 (v)), therefore $Y^D = (\overline{Y})^D$.

(iii) By (A.9), $(Y^D)^c = -\mathrm{Int}(Y)$, and so

$$[-(Y^D)]^c = \mathrm{Int}(Y). \tag{A.10}$$

Then, by Lemma A.6, $(Y^D)^D = \mathrm{Cl}[-(Y^D)]^c = \mathrm{Cl}[\mathrm{Int}(Y)] = Y$, by (A.10) and Proposition A.5 (v). □

The next proposition is obvious (use Definition A.2 and Lemma A.6).

PROPOSITION A.9. *If (X^1, X^1_+) and (X^2, X^2_+) are ordered vector spaces, $t\colon X^1 \to X^2$ is an order isomorphism (i.e. $t(X^1_+) = X^2_+$) and Y is an order cone in X^1, then $t(Y)$ is an order cone in X^2, and $[t(Y)]^D = t(Y^D)$.*

LEMMA A.10. *If Y_1 and Y_2 are order cones in (X, X_+), then* $\mathrm{Cl}(Y_1 \cap Y_2) = \overline{Y}_1 \cap \overline{Y}_2$. *In particular,* $\mathrm{Cl}[\mathrm{Int}(Y_1) \cap \mathrm{Int}(Y_2)] = \overline{Y}_1 \cap \overline{Y}_2$.

Proof. The inclusion $\overline{Y_1 \cap Y_2} \subset \overline{Y}_1 \cap \overline{Y}_2$. Conversely, if $y \in \overline{Y}_1 \cap \overline{Y}_2$, then $y + X_+ \subset Y_1$, $y + X_+ \subset Y_2$, by Proposition A.5 (i), and so $y + X_+ \subset Y_1 \cap Y_2$, which implies $y \in \mathrm{Cl}(Y_1 \cap Y_2)$ (by the same Proposition A.5 (i)). Thus, $\mathrm{Cl}(Y_1 \cap Y_2) = \overline{Y}_1 \cap \overline{Y}_2$.

Applying this identity to order cones $\mathrm{Int}(Y_j)$, $j = 1, 2$, we get $\mathrm{Cl}[\mathrm{Int}(Y_1) \cap \mathrm{Int}(Y_2)] = \mathrm{Cl}[\mathrm{Int}(Y_1)] \cap \mathrm{Cl}[\mathrm{Int}(Y_2)] = \overline{Y}_1 \cap \overline{Y}_2$ (by Proposition A.5 (v)). □

COROLLARY A.11. *If Y_1, Y_2 are order cones in (X, X_+), then*
(i) $(Y_1 \cap Y_2)^D = Y_1^D \cup Y_2^D$,
(ii) $(Y_1 \cup Y_2)^D = Y_1^D \cap Y_2^D$.

Proof. (i) Let $Y = Y_1 \cap Y_2$. Then

$$\begin{aligned} Y^D &= \mathrm{Cl}[(-Y)^c] \quad \text{(by Lemma A.6)} \\ &= \mathrm{Cl}[-(Y_1 \cap Y_2)]^c = \mathrm{Cl}[(-Y_1) \cap (-Y_2)]^c = \mathrm{Cl}[(-Y_1)^c \cup (-Y_2)^c] \\ &= \mathrm{Cl}[(-Y_1)^c] \cup \mathrm{Cl}[(-Y_2)^c] = Y_1^D \cup Y_2^D. \end{aligned}$$

(ii) Since Y_1^D, Y_2^D are order cones (by Proposition A.3), we can substitute them in (i) for Y_1, Y_2 respectively. Then

$$\begin{aligned} (Y_1^D \cap Y_2^D)^D &= (Y_1^D)^D \cup (Y_2^D)^D = \overline{Y}_1 \cup \overline{Y}_2 \qquad \text{(by Lemma A.8 (iii))} \\ &= \overline{Y_1 \cup Y_2} = ((Y_1 \cup Y_2)^D)^D \qquad \text{(by Lemma A.8 (iii))} \end{aligned}$$

and so $Y_1^D \cap Y_2^D = (Y_1 \cup Y_2)^D$, since both sets are closed order cones with equal dual cones. □

Appendix B. *A criterion for the local maximum property.*

THEOREM B.1. *Let $K \subset R^N$ be a compact set and u be a function with a locally bounded real Hessian on* $\mathrm{Int}(K)$ *and usc on K. Assume that Hess $u(x)$ has at least one nonnegative eigenvalue for a.a. $x \in$* $\mathrm{Int}(K)$. *Then* $\max_{x \in K} u \leq \max_{x \in \partial K} u$.

Proof (*Sketch*). The result is practically contained in [7, Proof of Theorem 4.1], but is not formulated there. Suppose, $\max u|K > \max u|\partial K$. By [**8**, Lemma 4.5], there are: an affine function l on R^N, $x_0 \in \mathrm{Int}(K)$ and $\varepsilon > 0$, such that $(u + l)(x_0) = 0$, $(u + l)(x) \leq -\varepsilon|x - x_0|^2$, $x \in K$. Letting $u_1(z) = (u + l)(x_0 + z)$, we obtain $u_1(0) = 0$, $u_1(z) \leq -\varepsilon|z|^2$, for $|z| < r$ (where $r < \mathrm{dist}(x_0, \partial K)$), and

$u_1 \in C^1_L(B(0,r))$, for some $L \geq 0$. It was shown in [7, p. 319, lines 11–24] that function u_1 with these properties does not exist. (The argument in [7] was spelled out for $\mathbf{C}^n$, but the complex structure was irrelevant there.) $\square$

Note. The author has learned recently that L. Bungart has obtained piecewise smooth approximations for the class of q-plurisubharmonic functions. See L. Bungart: *Piecewise smooth approximations to q-plurisubharmonic functions*, (preprint).

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