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DEGREES AND FORMAL DEGREES FOR DIVISION ALGEBRAS AND GL_n OVER A p-ADIC FIELD

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We compute in the tame case, the degrees of the irreducible representations of a division algebra and the formal degrees of the discrete series of $\mathrm{GL}(n)$ over a p-adic field and compare them.

1. Introduction. Let F be a p-adic field of characteristic zero, and let $G = \mathrm{GL}_n(F)$. *Throughout this paper, we assume that* $(n, p) = 1$ (*the tame case*). The discrete series of G consists of (equivalence classes of) irreducible, unitary representations of G whose matrix coefficients are square integrable $(\mathrm{mod}\, Z)$, where Z is the center of G. The discrete series splits into two distinct classes ([**HC2**], [**J**]):

(1) Supercuspidal representations: irreducible unitary representations whose matrix coefficients are compactly supported $(\mathrm{mod}\, Z)$;

(2) Generalized special representations: irreducible unitary representations whose matrix coefficients are square integrable $(\mathrm{mod}\, Z)$, and which are subrepresentations of representations induced from a proper parabolic subgroup of G.

The supercuspidal representations of G were constructed by Howe [**H2**]. The first proof of the fact that all supercuspidal representations of G are contained in Howe's construction was given by Moy [**M**]. The generalized special representations of G were characterized by Bernstein-Zelevinsky ([**BZ**], [**Z**]). We note that the Bernstein-Zelevinsky construction uses the supercuspidal representations of $\mathrm{GL}_m(F)$ where $m|n$ $(m < n)$. Since $(m, p) = 1$ in the present case, the requisite supercuspidal representations can be obtained from Howe's construction.

The key to the study of the supercuspidal representations of G is the notion, due to Howe [**H2**], of an admissible character of an extension of degree n over F. In fact, the supercuspidal representations of G are parametrized by (conjugacy classes of) admissible characters of extensions of degree n over F, and generalized special representations are parametrized by (conjugacy classes of) admissible characters of extensions of degree m over F where $m|n$, $m < n$. (See **[M]** for additional details.)

Now, let D_n be a division algebra of dimension n^2 over F, and let $D^\times = D_n^\times$, be the multiplicative group of D_n. The irreducible representations of $D^\times$ were constructed as induced representations by Corwin **[Co]** and Howe **[H1]**. In these constructions, the inducing representations are obtained from (conjugacy classes of) admissible characters of extensions of degree m over F where $m|n$ (including $m = n$).

The proof by Moy **[M]** that Howe's representations exhaust the supercuspidal representations of G uses the abstract matching theorem. The abstract matching theorem was proved by Deligne-Kazhdan-Vigneras **[DKV]** and Rogawski **[R]**. Recall that, if E/F is an extension of degree n, then $E^\times$ can be embedded in both G and $D^\times$. In fact, any compact (mod center) Cartan subgroup of G (and $D^\times$) is isomorphic to $E^\times$ for some extension of degree n.

THEOREM 1.1 (*Abstract Matching Theorem*, **[DKV]**, **[R]**). *There is a bijection $\pi' \leftrightarrow \pi$ between irreducible representations of $D^\times$ and the discrete series of representations of G with the following properties*:

(1) *If $\theta_{\pi'}$ and θ_π are the characters of π' and π respectively, and γ is a regular element in a compact* (mod *center*) *Cartan subgroup $E^\times$, then*

$$\theta_{\pi'}(\gamma) = (-1)^{n-1}\theta_\pi(\gamma).$$

(2) *If the formal degree of the Steinberg representation* **[B]** *is normalized to be equal to one, then*

$$d(\pi') = d(\pi),$$

where $d(\pi')$ is the ordinary degree of the finite-dimensional representation π' and $d(\pi)$ is the formal degree of the infinite-dimensional representation π;

(3) *If $\varepsilon(\pi', \psi)$, $\varepsilon(\pi, \psi)$ are the ε-factors of π' and π respectively, then $\varepsilon(\pi', \psi) = (-1)^{n-1}\varepsilon(\pi, \psi)$. Here, ψ is a suitably chosen additive character on F.*

REMARKS 1.2. (1) Moy's proof **[M]** that the supercuspidal representations constructed by Howe and the generalized special representations constructed by Bernstein-Zelevinsky exhaust the discrete series of $\mathrm{GL}_n(F)$ uses the abstract matching theorem in an essential way. Thus, it is only after we use the abstract matching theorem that we can assert that the concrete matching by admissible characters is actually bijection.

(2) The abstract matching theorem gives no indication as to which representations of $D^\times$ correspond to the two distinct types of discrete series representations of G.

(3) Recently, Howe-Moy **[HM2]** have given a proof of the completeness of Howe's construction without the use of Theorem 1.1.

To sharpen our focus, we introduce the following distinction. If E/F is an extension of degree m, $m|n$, $m < n$, and θ is an admissible character of $E^\times$, we say that θ is *subadmissible* (for n). Thus, the term *admissible character* will be used only for extensions E/F of degree n. The conjugacy classes of admissible and subadmissible characters parametrize the irreducible representations of $D^\times$. As indicated above, the supercuspidal representations of G correspond to admissible characters, and the generalized special representations of G correspond to subadmissible characters. Thus, it is natural to conjecture that, if π'_θ is the irreducible representation of $D^\times$ corresponding to an admissible (resp. subadmissible) character, then the discrete series representation π of G which corresponds to π'_θ by the abstract matching theorem is supercuspidal (resp. generalized special).

This last assertion is indeed the case, and it is the purpose of this paper to give a proof using the degrees of the representations. To this end, we consider the following sets:

$$(1.3)\qquad \begin{aligned} A'_1 &= \{\pi'_\theta \in (D^\times)^\frown | \theta \text{ is admissible}\}; \\ A'_2 &= \{\pi'_\theta \in (D^\times)^\frown | \theta \text{ is subadmissible}\}. \end{aligned}$$

Here π'_θ is the representation of $D^\times$ constructed from θ by Corwin and Howe, and $(D^\times)^\frown$ is the unitary dual of $D^\times$. In a similar fashion, we define

$$(1.4)\qquad \begin{aligned} A_1 &= \{\pi_\theta \in \hat{G}_d | \theta \text{ is admissible}\}; \\ A_2 &= \{\pi_\theta \in \hat{G}_d | \theta \text{ is subadmissible}\}. \end{aligned}$$

In this case, we have the supercuspidal representations (resp. generalized special representations) constructed by Howe (resp. Bernstein-Zelevinsky). $\hat{G}_d$ denotes the discrete series in the unitary dual of G.

Now, letting $d(\pi)$ denote the ordinary or formal degree of a representation, we set

$$(1.5)\qquad \Delta'_1 = \{d(\pi'_\theta) | \pi'_\theta \in A'_1\}; \qquad \Delta'_2 = \{d(\pi'_\theta) | \pi'_\theta \in A'_2\};$$

$$(1.6)\qquad \Delta_1 = \{d(\pi_\theta) | \pi_\theta \in A_1\}; \qquad \Delta_2 = \{d(\pi_\theta) | \pi_\theta \in A_2\}.$$

If we assume that $d(\text{Steinberg}) = 1$, then (2) in the abstract matching theorem implies that $\Delta_1' \cup \Delta_2' = \Delta_1 \cup \Delta_2$. We show in Theorem 4.1 that

$$(1.7) \qquad \Delta_1' \cap \Delta_2' = \Delta_1 \cap \Delta_2 = \varnothing, \quad \Delta_1' = \Delta_1, \quad \text{and} \quad \Delta_2' = \Delta_2.$$

Since the trivial representation of $D^\times$ is in A_2', it follows that, under the abstract matching, representations in A_1' correspond to supercuspidal representations of G and representations in A_2' correspond to generalized special representations of G. It is interesting to note that the conductors of the representations π_θ and π_θ' appear naturally in the expressions for the formal degrees. This will be discussed in §4.

One of the more important consequences of (1.7) is worth observing here. Using the standard Frobenius formula for induced characters, we are able to give explicit formulas for the characters of the representations $\pi_\theta' \in (D^\times)^\wedge$. It follows from (1) of the abstract matching theorem that these are (up to a sign) explicit formulas for the characters of the discrete series of G on the elliptic set. The distinction provided by the formal degrees tells us which of these are supercuspidal characters and which are generalized special characters. In turn, this allows us to analyze the differences between the two different classes of characters. This analysis is carried out in **[CS]**.

In the case $n = p$, Carayol **[C]** has determined the formal degrees of the supercuspidal representations of G and the degrees of the corresponding representations of $D^\times$. He has also observed the relationship between the formal degree and the conductor of a representation. Waldspurger **[W]** has computed the formal degrees of the discrete series of G with a normalization which differs from ours. His techniques for obtaining these formulas are also different, but there are significant points of contact between some aspects of our computations and those of Waldspurger. In §4, we will give more detail about the relationship between our work and that of Carayol and Waldspurger.

In §2, we compute the formal degrees of the supercuspidal and generalized special representations of G. While the formal degrees of the supercuspidal representations are computed directly from their construction as induced representations in §2.1 and §2.2, the formal degrees of the generalized special representations are derived in §2.3 and §2.4 using the Hecke algebra isomorphisms proved in Howe-Moy **[HM2]**. This requires a discussion of the minimal K-types associated to generalized special representations.

Section 3 contains the calculation of the degrees of the irreducible representations of $D^\times$. Again, the degrees are computed from the inducing construction.

Finally, in §4, we prove the statement of (1.7). In addition, we make several observations concerning the relationship between degrees and characters, the appearance of the conductor in the expression for the degree of a representation, and the comparison of the formal degree of a generalized special representation with the formal degree of the associated supercuspidal representation. It is worth noting here, that our development hinges to a great extent on the fact that $(n, p) = 1$. However, if the formal degree of a generalized special representation is divided by an appropriate power of the associated supercuspidal representation, the resulting expression does not depend on the admissible character which parametrizes these representations. There is hope that such an expression pertains in the case when $p|n$.

Some of the results in this paper were announced in **[S1]**. We adopt the usual notation: $\mathscr{O}_F$ is the ring of integers in F, $\mathscr{P}_F$ the maximal ideal in $\mathscr{O}_F$, and $\tilde{\omega}_F$ a prime element in $\mathscr{P}_F$. The F-conductor of a multiplicative character ϕ on $F^\times$ will be denoted by $f_F(\phi)$.

2. Formal degrees for the discrete series of GL_n. In this section, we compute the formal degrees of the supercuspidal and generalized special representations of $G = \mathrm{GL}_n(F)$. As mentioned in the Introduction, the formal degrees of the supercuspidal representations are computed directly from their construction as induced representations, while the formal degrees of the generalized special representations are computed by using isomorphisms of certain Hecke algebras. It turns out that the actual computations are remarkably similar for the two cases.

2.1. *Degrees of the inducing representations.* Let E/F be an extension of degree n ($(n, p) = 1$), and let θ be an admissible character of $E^\times/F$ (**[H2]**, **[M]**). The irreducible supercuspidal representations of G may be parametrized by (conjugacy classes of) admissible characters of extensions of degree n over F. In fact, given θ, one constructs a compact (mod center) open subgroup K_θ of G and an irreducible representation σ_θ of K_θ such that

$$\pi_\theta = \mathrm{Ind}_{K_\theta}^G \sigma_\theta \tag{2.1.1}$$

is an irreducible supercuspidal representation of G. Moreover, all irreducible supercuspidal representations can be constructed in this way (**[H2]**, **[M]**).

Given an admissible character θ of $E^\times/F$, the construction of K_θ and σ_θ proceeds as follows. According to Howe **[H2]**, there is a unique tower of fields

$$E = E_t \supset E_{t-1} \supset \cdots \supset E_1 \supset E_0 = F, \tag{2.1.2}$$

and characters χ, $\phi_1, \dots, \phi_t$ of $F^\times, E_1^\times, \dots, E_t^\times$ respectively such that $\theta = (\chi \circ N_{E/F})(\phi_1 \circ N_{E/E_1}) \cdots (\phi_t)$. Each character ϕ_k is generic over E_{k-1} (see **[H2]**, **[M]**). For simplicity, we abuse the notation and write $\phi_k = \phi_k \circ N_{E/E_k}$, so that

$$\theta = \chi \cdot \phi_1 \cdot \cdots \cdot \phi_t. \tag{2.1.3}$$

This is the *Howe factorization* of θ. It is unique in the sense that the conductorial exponents of the characters are unique, and $f_E(\phi_1) > f_E(\phi_2) > \cdots > f_E(\phi_t) \geq 1$.

We set

$$n_k = [E\colon E_k], \quad e_k = e(E/E_k), \quad f_k = f(E/E_k), \qquad k = 0, \dots, t. \tag{2.1.4}$$

In particular, $n_0 = n$, $n_t = 1$, $e_0 = e(E/F) = e$, and $f_0 = f(E/F) = f$. If $j_k = f_E(\phi_k)$, the E-conductor of ϕ_k, we define integers i_k, $k = 1, 2, \dots, t$, as follows. For $k = 1, 2, \dots, t-1$,

$$i_k = \begin{cases} j_k/2, & j_k \text{ even}, \\ (j_k - 1)/2, & j_k \text{ odd}. \end{cases} \tag{2.1.5}$$

If $f_E(\phi_t) = j_t > 1$, define i_t as above, and, if $f_E(\phi_t) = j_t = 1$, set $i_t = 1$.

REMARK 2.1.6. (1) When $j_t = 1$, E/E_{t-1} is unramified **[H2]**.

(2) The relationship between the E-conductor of ϕ_k $(= \phi_k \circ N_{E/E_k})$ and the E_k-conductor of ϕ_k is $f_E(\phi_k) - 1 = e_k(f_{E_k}(\phi_k) - 1)$.

Now, writing $\mathscr{O}_{E_k} = \mathscr{O}_k$, and $\mathscr{P}_{E_k} = \mathscr{P}_k$, we define

$$\mathscr{B}_k = \begin{bmatrix} M_{f_k}(\mathscr{P}_k) & M_{f_k}(\mathscr{O}_k) & \cdots & M_{f_k}(\mathscr{O}_k) \\ M_{f_k}(\mathscr{P}_k) & M_{f_k}(\mathscr{P}_k) & \cdots & M_{f_k}(\mathscr{O}_k) \\ \vdots & & & \vdots \\ M_{f_k}(\mathscr{P}_k) & M_{f_k}(\mathscr{P}_k) & \cdots & M_{f_k}(\mathscr{P}_k) \end{bmatrix}, \qquad k = 0, \dots, t-1, \tag{2.1.7}$$

where there are e_k blocks in each row and column. We regard $1 + \mathscr{B}_k^h$ as a subgroup of G for any positive integer h (see **[M]**).

The inducing subgroup for π_θ is then defined as

$$K_\theta = E^\times(1 + \mathscr{B}_{t-1}^{i_t})(1 + \mathscr{B}_{t-2}^{i_{t-1}}) \cdots (1 + \mathscr{B}_1^{i_2})(1 + \mathscr{B}_0^{i_1}), \tag{2.1.8}$$

if $f_E(\phi_t) = j_t > 1$, and

$$K_\theta = E^\times K_{t-1}(1+\mathscr{P}_{t-1})(1+\mathscr{P}_{t-2}^{i_{t-1}})\cdots(1+\mathscr{P}_1^{i_2})(1+\mathscr{P}_0^{i_1}), \tag{2.1.9}$$

if $f_E(\phi_t) = j_t = 1$, where $K_{t-1} = \mathrm{GL}_{n_{t-1}}(\mathscr{O}_{t-1})$.

The inducing representation may be written as a tensor product

$$\sigma_\theta = \kappa_t \otimes \kappa_{t-1} \otimes \cdot \otimes \kappa_1 \otimes \chi, \tag{2.1.10}$$

where χ is a character of $F^\times$ which can be removed by a twist for the purpose of computing formal degrees. From the construction of κ_k ([**H2**], [**M**]), we have, for $1 \le k < t$,

$$\begin{aligned} &\deg(\kappa_k) = 1, \quad j_k \text{ even},\\ &\deg(\kappa_k) = [(1+\mathscr{P}_{k-1}^{i_k}):(1+\mathscr{P}_k^{i_k})(1+\mathscr{P}_{k-1}^{i_{k+1}})]^{1/2}, \quad j_k \text{ odd}. \end{aligned} \tag{2.1.11}$$

If $j_t > 1$, the above formulas are still valid for $\deg(\kappa_t)$, and, if $j_t = 1$,

$$\det(\kappa_t) = \prod_{l=1}^{f_{t-1}-1} (q_{t-1}^j - 1), \tag{2.1.12}$$

where $q_{t-1} = q^{f/f_{t-1}}$.

We now compute $\deg(\sigma_\theta)$ from the above data. We set $q_k = q^{f/f_k}$, $1 \le k \le t$, so that $(q_k^{f_k^2})^{e_k} = (q^{ff_k})^{e_k} = q^{fn_k}$.

Lemma 2.1.13. $[(1+\mathscr{P}_{k-1}^{i_k}):(1+\mathscr{P}_k^{i_k})(1+\mathscr{P}_{k-1}^{i_k+1})] = q^{fn_{k-1}}/q^{fn_k}$.

Proof.

$$\begin{aligned} &[(1+\mathscr{P}_{k-1}^{i_k}):(1+\mathscr{P}_k^{i_k})(1+\mathscr{P}_{k-1}^{i_k+1})]\\ &\quad = \frac{[(1+\mathscr{P}_{k-1}^{i_k}):(1+\mathscr{P}_{k-1}^{i_k+1})]}{[(1+\mathscr{P}_k^{i_k}):(1+\mathscr{P}_k^{i_k+1})]} = (q_{k-1}^{f_{k-1}^2})^{e_{k-1}}/(q_k^{f_k^2})^{e_k}. \end{aligned}$$

Lemma 2.1.14. *If θ is an admissible character for $E^\times/F$ ($[E:F]=n$), and σ_θ is the representation of K_θ given by* (2.1.10), *then*

(1) $\deg(\sigma_\theta) = q^{\beta(\theta)}$, *where*

$$\beta(\theta) = (f/2)\sum_{k:\ j_k \text{ odd}} (n_{k-1} - n_k), \qquad f_E(\phi_t) > 1,$$

(2) $\deg(\sigma_\theta) = q^{\beta(\theta)}[\prod_{j=1}^{f_{t-1}-1}(q_{t-1}^j - 1)]$, *where*

$$\beta(\theta) = \sum_{k:\ j_k \text{ odd}} (n_{k-1} - n_k), \qquad f_E(\phi_t) = 1.$$

Proof. This is an immediate consequence of (2.1.11), (2.1.12) and Lemma 2.1.13.

2.2. *Normalization of volumes and formal degrees of supercuspidal representations.* In order to use the abstract matching theorem for purposes of comparison between representation of $D^\times$ and G, we must normalize measures so that the formal degree of the Steinberg representation is equal to one (Theorem 1.1). We begin by recalling the basic formula for formal degrees (**[HC1]**, p. 5). If π is a representation of G which is square integrable $(\bmod Z)$, and Z_0 is a cocompact subgroup of Z (i.e. Z/Z_0 is compact), then

$$\int_{G/Z_0} |(v|\pi(x)v)|^2\,d\dot{x} = \deg(\pi, G/Z_0)^{-1}, \tag{2.2.1}$$

where v is a unit vector in the space of π, and $d\dot{x}$ is a Haar measure on G/Z_0. The formal degree $\deg(\pi, G/Z_0)$, in fact, depends on the normalization of $d\dot{x}$.

In the case of the Steinberg representation, it is well known (**[R]**) that

$$\deg(\mathrm{St}, G/Z)\operatorname{vol}(KZ/Z) = \frac{1}{n}\prod_{k=1}^{n-1}(q^k - 1). \tag{2.2.2}$$

It should be observed that, in imposing this normalization, we are simultaneously normalizing Haar measures on G and Z so that $\operatorname{vol}_G(K)/\operatorname{vol}_Z(K\cap Z) = \operatorname{vol}_{G/Z}(KZ/Z)$.

For our purposes, it is convenient to get an analogue of (2.2.2) for any cocompact subgroup Z_0 of Z. To this end, we impose the normalizations

$$\text{(2.2.3)}\qquad \begin{aligned} &\text{(i)} && \operatorname{vol}_G(K) = \frac{1}{n}\prod_{k=1}^{n-1}(q^k - 1);\\ &\text{(ii)} && \operatorname{vol}_Z(K\cap Z) = 1;\\ &\text{(iii)} && \operatorname{vol}_{Z_0}(K\cap Z_0) = 1. \end{aligned}$$

We then have

$$\deg(\mathrm{St}, G/Z_0)\operatorname{vol}(KZ_0/Z_0) = \frac{1}{n}\prod_{k=1}^{n-1}(q^k - 1). \tag{2.2.2$'$}$$

In particular, if Z_0 is the discrete subgroup of G given by $Z_0 = \langle\tilde{\omega}_F I_{n\times n}\rangle$, then (2.2.3), (iii), gives counting measure on Z_0. This will be used below in determining the formal degrees of the generalized special representations.

We now turn to the formal degrees of the supercuspidal representations of G. If K_θ and σ_θ are defined as in (2.1.8) (or (2.1.9)) and (2.1.10), and π_θ is given as the irreducible supercuspidal representation induced from σ_θ, then it is easy to see ([**S2**]) that a non-trivial matrix coefficient of σ_θ may be extended to G by defining it to be zero on the complement of K_θ, thus yielding a matrix coefficient of π_θ. It follows that $\deg(\pi_\theta, G/Z) = \deg(\sigma_\theta)/\operatorname{vol}(ZK_\theta/Z)$. So, to complete our calculation for $\deg(\pi_\theta, G/Z)$, we must determine $\operatorname{vol}(ZK_\theta/Z)$ relative to the normalization (2.2.2).

Define

$$K^{(e)} \left\{ \begin{bmatrix} \mathrm{GL}_f(\mathscr{O}_F) & & \mathscr{O}_F \\ & \ddots & \\ \mathscr{P}_F & & \mathrm{GL}_f(\mathscr{O}_F) \end{bmatrix} \right\},$$

and let $Z^{(e)}$ be the subgroup generated by

$$z^{(e)} = \begin{bmatrix} 0 & I_f & 0 & & \\ & & & 0 & \\ 0 & 0 & I_f & \ddots & \\ & & & & I_f \\ \tilde{\omega}_F I_f & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $f = f(E/F)$ and there are $e = e(E/F)$ blocks in each row and column. Then $Z^{(e)}$ normalizes $K^{(e)}$ and K_θ is a subgroup of $Z^{(e)}K^{(e)}$ for any admissible character θ.

We now have

$$\text{(2.2.4)} \qquad \operatorname{vol}(ZK_\theta/Z) = \operatorname{vol}(Z^{(e)}K^{(e)}/Z)[Z^{(e)}K^{(e)} \colon K_\theta]^{-1}.$$

Moreover, $Z^{(e)}K^{(e)} \cap ZK = ZK^{(e)}$, and

$$\begin{aligned} \text{(2.2.5)} \quad & \operatorname{vol}(Z^{(e)}K^{(e)}/Z) \\ & = [Z^{(e)}K^{(e)} \colon ZK^{(e)}][ZK \colon ZK^{(e)}]^{-1} \operatorname{vol}(ZK/Z). \end{aligned}$$

Thus, we must compute the three indices in (2.2.4) and (2.2.5).

LEMMA 2.2.6. (1) *If* $f_E(\phi_t) > 1$, *then*

$$[Z^{(e)}K^{(e)} \colon K_\theta] = |\mathrm{GL}_f(q)|^e (q^f - 1)^{-1} q^{\alpha f},$$

where $\alpha = \sum_{k=1}^t i_k(n_{k-1} - n_k) - n + 1$.

(2) *If* $f_E(\phi_t) = 1$, *then*

$$[Z^{(e)}K^{(e)}\colon K_\theta] = |\operatorname{GL}_f(q)|^e |\operatorname{GL}_{f_{t-1}}(q_{t-1})|^{-1} q^{\alpha f},$$

where $\alpha = \sum_{k=1}^{t} i_k(n_{k-1} - n_k) - n + 1$.

Proof. (1) We have $[Z^{(e)}K^{(e)}\colon K_\theta] = [K^{(e)}\colon \mathscr{O}_E^\times(1+\mathscr{L}_{t-1}^{i_t})\cdots(1+\mathscr{L}_0^{i_1})]$. Since $1+\mathscr{L}_{k-1}^{i_k}$ is normal in $K^{(e)}$, this last index is equal to

$$[K^{(e)}\colon (1+\mathscr{L}_0^{i_1})][\mathscr{O}_E^\times\colon 1+\mathscr{L}_t^{i_t}]^{-1}\left\{\prod_{k=1}^{t-1}[\mathscr{L}_k^{i_{k+1}}\colon \mathscr{L}_k^{i_k}]\right\}^{-1}.$$

Now, the following facts lead to the stated formula for $[Z^{(e)}K^{(e)}\colon K_\theta]$. First,

$$[K^{(e)}\colon (1+\mathscr{L}_0^{i_1})] = [K^{(e)}\colon (1+\mathscr{L}_0)][\mathscr{L}_0\colon \mathscr{L}_0^{i_1}] = |\operatorname{GL}_f(q)|^e q^{fn(i_1-1)}.$$

Second,

$$[\mathscr{O}_E^\times\colon 1+\mathscr{L}_t^{i_t}] = [\mathscr{O}_E^\times\colon 1+\mathscr{L}_t][\mathscr{L}_t\colon \mathscr{L}_t^{i_t}] = (q^f-1)(q^f)^{i_t-1}.$$

And, finally,

$$[\mathscr{L}_k^{i_{k+1}}\colon \mathscr{L}_k^{k_k}] = q^{fn_k(i_k-i_{k+1})}, \qquad k = 1, \dots, t-1.$$

(2) When $f_E(\phi_t) = 1$, we have E_t/E_{t-1} unramified, $n_{t-1} = f_{t-1}$, and $i_t = 1$. Also, $[\mathscr{O}_E^\times\colon 1+\mathscr{L}_t^{i_t}]$ is replaced by $[K_{t-1}\colon 1+\mathscr{L}_{t-1}] = |\operatorname{GL}_{f_{t-1}}(q_{t-1})|$.

In both cases, the transformation from

$$\sum n_k(i_k - i_{k+1}) \quad \text{to} \quad \sum i_k(n_{k-1} - n_k)$$

should be noted.

We now turn to (2.2.5).

LEMMA 2.2.7. (1) $[Z^{(e)}K^{(e)}\colon ZK^{(e)}] = e$.
(2) $[ZK\colon ZK^{(e)}] = |\operatorname{GL}_n(q)|/|\operatorname{GL}_f(q)|^e q^{(n^2-nf)/2}$.

Proof. (1) $Z^{(e)}$ has order $e \bmod Z$.

(2) Let K_1 be the first congruence subgroup of K. Then K_1 is a subgroup of $K^{(e)}$ and

$$[ZK\colon ZK^{(e)}] = [K\colon K_1]^{-1} = |\operatorname{GL}_n(q)|/|\operatorname{GL}_f(q)|^e|(\mathscr{O}_F/\mathscr{P}_F)^{f^2}|^{e(e-1)/2}.$$

But $f^2e(e-1)/2 = (n^2-nf)/2$.

We are now in a position to give an explicit formula for $\deg(\pi_\theta, G/Z)$.

THEOREM 2.2.8. *Let π_θ be the irreducible supercuspidal representation induced from σ_θ where θ is an admissible character of $E^\times/F([E\colon F] = n)$. Let $\{n_k, i_k\}$ be the data from the Howe factorization of θ given in* (2.1.4) *and* (2.1.5). *Then, if* $\operatorname{vol}(KZ/Z)$ *is given by* (2.2.2),

$$\deg(\pi_\theta, G/Z) = [f(q^n - 1)/(q^{n/e} - 1)]q^{(f/2)(\alpha(\theta)+2-n-e)},$$

where $\alpha(\theta) = \sum_{k=1}^{t} j_k(n_{k-1} - n_k)$.

Proof. As observed above, we have

$$\deg(\pi_\theta, G/Z) = \deg(\sigma_\theta)/\operatorname{vol}(ZK_\theta/Z).$$

The result follows from (2.1.5), Lemma 2.1.14, Lemma 2.2.6, Lemma 2.2.7, and some elementary arithmetic.

2.3. *Hecke algebra isomorphisms.* We now consider the generalized special representations. Let E/F be an extension of degree m, $m|n$, $m < n$, and let θ be a subadmissible character of $E^\times/F$. As in (2.1.2), there is a unique tower of fields

$$E = E_t \supset E_{t-1} \supset \cdots \supset E_1 \supset E_0 = F, \tag{2.3.1}$$

and the associated Howe factorization

$$\theta = \phi_t\phi_{t-1}\cdots\phi_1. \tag{2.3.2}$$

REMARK 2.3.3. Here we use the same conventions as above, that is, ϕ_k is used to denote $\phi_k \circ N_{E/E_k}$, and the character χ of $F^\times$ which appears in the Howe factorization of θ is twisted away for purposes of computing the formal degrees.

Let $n = am$, and extend E to an extension E'/F such that

$$[E'\colon E] = a \text{ and } E'/E \text{ is totally ramified.} \tag{2.3.4}$$

Thus, $[E'\colon F] = n = am$. Moreover, if $e = e(E/F)$, $f = f(E/F)$, and n_k, e_k, f_k are defined as in (2.1.4), we set

$$\begin{aligned} e' &= e(E'/F) = ea, \quad f' = f(E'/F) = f, \quad n'_k = [E'\colon E_k] = an_k, \\ e'_k &= e(E'/E_k) = ae_k, \quad f'_k = f(E'/E_k) = f_k. \end{aligned} \tag{2.3.5}$$

Note that $n'_0 = n$, and $n'_t = a$.

Define $\theta' = \theta \circ N_{E'/E}$. Then, we can write

$$\theta' = \phi'_t\phi'_{t-1}\cdots\phi'_1, \tag{2.3.6}$$

where $\phi'_k = \phi_k \circ N_{E'/E}$ $(= \phi_k \circ N_{E/E_k} \circ N_{E'/E}$, see Remark 2.3.3). If $f_E(\phi_k) = j_k$, then $f_{E'}(\phi'_k) = j'_k$, where $f_{E'}(\phi'_k) - 1 = a(f_E(\phi_k) - 1)$, that is,

$$j'_k - 1 = a(j_k - 1). \tag{2.3.7}$$

In analogy with (2.1.5), we set

$$i'_k = \begin{cases} j'_k/2, & j'_k \text{ even,} \\ (j'_k - 1)/2, & j'_k \text{ odd, } k = 1, 2, \ldots, t-1. \end{cases} \tag{2.3.8}$$

If $f_{E'}(\phi'_t) = j'_t > 1$, define i'_t as above. If $f_{E'}(\phi'_t) = j'_t = 1$, set $i'_t = 1$. We note from (2.3.7) that $j'_t = 1$ if and only if $j_t = 1$.

For the Hecke algebra isomorphisms to which we referred at the beginning of §2, we must define subgroups of G which are analogous to those in §2.1. Thus, we define $\mathscr{E}_k$ (in $M_n(F)$) as in (2.1.7), with e'_k and f'_k $(= f_k)$ replacing e_k and f_k respectively. Let $G_a = \mathrm{GL}_a(E)$, and let B_a be the Iwahori subgroup of G_a, considered as a subgroup of G. If $j'_t > 1$, we set

$$J_\theta = B_a(1 + \mathscr{E}_{t-1}^{i'_t}) \cdots (1 + \mathscr{E}_1^{i'_2})(1 + \mathscr{E}_0^{i'_1}). \tag{2.3.9}$$

If $j'_t = j_t = 1$, we write $h = n_{t-1} = [E\colon E_{t-1}]$, and let P_{t-1} be the $(h, h, \ldots, h)$ (a times) parahoric subgroup of $\mathrm{GL}_{ah}(E_{t-1})$. Then, if $j'_t = j_t = 1$, we define

$$J_\theta = P_{t-1}(1 + \mathscr{E}_{t-2}^{i'_{t-1}}) \cdots (1 + \mathscr{E}_1^{i'_2})(1 + \mathscr{E}_0^{i'_1}). \tag{2.3.10}$$

If π_θ is the generalized special representation constructed from θ, we write $(\Omega_\theta, J_\theta)$ for the minimal K-type associated to π_θ (**[HM2]**). The representation Ω_θ is constructed in a manner which is very similar to the construction of the inducing representations σ_θ for supercuspidal representations (see (2.1.10) ff). In particular, if $j'_t > 1$, $\Omega_\theta = (\phi'_t \circ \det) \otimes \kappa_{t-1} \otimes \cdots \otimes \kappa_1$, where $\phi'_t \circ \det$ is a one dimensional representation on B_a, and κ_k is defined as in **[M]**. If $j'_t = 1$, we consider ϕ_t as a character on the anisotropic Cartan subgroup of $\mathrm{GL}_h(q_{t-1})$ where $q_{t-1} = q^{f/f_{t-1}}$. Let $\overline{\kappa}_{t-1}$ be the cuspidal representation of $\mathrm{GL}_h(q_{t-1})$ associated to $\phi_t([G])$. We then let κ_{t-1} be $\otimes \overline{\kappa}_{t-1}$ (a times) inflated to P_{t-1}, and set $\Omega_\theta = \kappa_{t-1} \otimes \kappa_{t-2} \otimes \cdots \otimes \kappa_1$, where κ_k, $1 \le k \le t-2$, is defined as in **[M]**. Note that $\deg(\kappa_{t-1}) = [\prod_{k=1}^{h-1}(q_{t-1}^k - 1)]^a$.

The following lemma is the analogue of Lemma 2.1.14 for the case of generalized special representations.

LEMMA 2.3.11. *If θ is a subadmissible character for $E^\times/F$, ($[E\colon F] = m$, $m|n$, $m < n$), and $(\Omega_\theta, J_\theta)$ is the minimal K-type associated to θ, then*

(1) $\deg(\Omega_\theta) = q^{\gamma(\theta)}$, *where* $\gamma(\theta) = (f/2)\sum_{k\colon j'_k \text{ odd}}(n'_{k-1} - n'_k)$, $j'_t > 1$,

(2)

$$\deg(\Omega_\theta) = q^{\gamma(\theta)}\left[\prod_{j=1}^{h-1}(q_{t-1}^{j} - 1)\right]^a,$$

$$\text{where } \gamma(\theta) = f(/2)\sum_{k\colon j'_k \text{ odd};k<t}(n'_{k-1} - n'_k),\ j'_t = 1.$$

(*Here, as in* §2.1, $q_{t-1} = q^{f/f_{t-1}}$.)

Before stating the basic theorem on Hecke algebra isomorphisms, we make a simple observation. Set $T = \langle\tilde{\omega}_F I_{n\times n}\rangle$ and $T_a = \langle\tilde{\omega}_E I_{a\times a}\rangle$. We can choose the prime elements $\tilde{\omega}_F$ and $\tilde{\omega}_E$ so that, under the above embedding of G_a into G, T is a subgroup of T_a. It is clear that

$$[T_a\colon T] = e. \tag{2.3.12}$$

THEOREM 2.3.13 ([**HM2**]). *The Hecke algebras $\mathscr{H}(G/T, J_\theta T/T, \Omega_\theta)$ and $\mathscr{H}(G_a/T, B_aT/T, 1)$ are isomorphic. This isomorphism carries discrete series to discrete series and preserves Plancherel measure. In particular, the generalized special representation π_θ of G corresponds to the Steinberg representation of G_a, and* ([**HM2**]), (5.2))

$$\begin{aligned}\deg(\pi_\theta, G/T)\operatorname{vol}(J_\theta T/T)\\ = \deg(\Omega_\theta)\deg(\mathrm{St}, G_a/T)\operatorname{vol}(B_aT/T).\end{aligned} \tag{2.3.14}$$

Our goal is to determine the values of the factors in the formula (2.3.14). First of all, the degree of the minimal K-type Ω_θ is given in Lemma 2.3.11. Second, from (2.2.1) and (2.3.12), it follows that

$$\begin{aligned}&\deg(\mathrm{St}, G_a/T)\operatorname{vol}(B_aT/T)\\ &\quad= e^{-1}\deg(\mathrm{St}, G_a/T_a)\operatorname{vol}(K_aT_a/T_a)\\ &\quad\quad\times[\operatorname{vol}(B_aT/T)/\operatorname{vol}(K_aT_a/T_a)],\end{aligned} \tag{2.3.15}$$

where $K_a = \mathrm{GL}_a(\mathscr{O}_E)$. From the normalizations given by (2.2.3), we obtain

$$\operatorname{vol}(B_aT/T)/\operatorname{vol}(K_aT_a/T_a) = \operatorname{vol}(B_a)/\operatorname{vol}(K_a). \tag{2.3.16}$$

Thus, to determine $\deg(\pi_\theta, G/T)$ from (2.3.14), we must compute the volumes $\mathrm{vol}(J_\theta T/T) = \mathrm{vol}(J_\theta)$, and $\mathrm{vol}(B_a)/\mathrm{vol}(K_a)$ relative to the normalization of Haar measures given by (2.2.3) and (2.2.2′).

2.4. *Formal degree of generalized special representations.* The computation of $\mathrm{vol}(J_\theta)$ is similar to those contained in Lemma 2.2.6 and Lemma 2.2.7. In the present case, J_θ is compact, whereas, in §2.2, the subgroup K_θ is compact mod Z. Here, we define

$$K^{(ea)} = \left\{ \begin{bmatrix} \mathrm{GL}_f(\mathcal{O}_F) & & \mathcal{O}_F \\ & \ddots & \\ \mathcal{P}_F & & \mathrm{GL}_f(\mathcal{O}_F) \end{bmatrix} \right\},$$

where we have ea copies $\mathrm{GL}_f(\mathcal{O}_F)$ along the diagonal ($aef = am = n$).

In analogy with (2.2.4) and (2.2.5), we have

$$[K\colon J_\theta] = [K\colon K^{(ae)}][K^{(ae)}\colon J_\theta], \tag{2.4.1}$$

and

$$\mathrm{vol}(J_\theta) = \mathrm{vol}(K)/[K\colon J_\theta]. \tag{2.4.2}$$

LEMMA 2.4.3. (1) *If* $j'_t > 1$, *then*

$$[K^{(ae)}\colon J_\theta] = |\,\mathrm{GL}_f(q)|^{ae}(q^f-1)^{-a}q^{\gamma f},$$

where $\gamma = \sum_{k=1}^{t} i'_k(n'_{k-1} - n'_k) - n + a$.

(2) *If* $j'_t = 1$, *then*

$$[K^{(ae)}\colon J_\theta] = |\,\mathrm{GL}_f(q)|^{ae}|\,\mathrm{GL}_h(q_{t-1})|^{-a}q^{\gamma f},$$

where $\gamma = \sum_{k=1}^{t} i'_k(n'_{k-1} - n'_k) - n + a$, *and* $h = n_{t-1} = [E\colon E_{t-1}] = f_{t-1}$.

Proof. As expected, the proof is similar to that of Lemma 2.2.6.

(1) If $j'_t > 1$, then, since B_a and $(1+\mathcal{B}_k^{i'_{k+1}})$ are normal in J_θ, $k = 0, \dots, t-1$,

$$[K^{(ae)}\colon J_\theta] = [K^{(ae)}\colon 1+\mathcal{B}_0^{i'_1}][B_a\colon 1+\mathcal{B}_t^{i'_t}]^{-1}\left\{\prod_{k=1}^{t-1}[\mathcal{B}_k^{i'_{k+1}}\colon \mathcal{B}_k^{i'_k}]\right\}^{-1}.$$

The proof now proceeds as in Lemma 2.2.6 with the observation that

$$[B_a\colon 1+\mathcal{B}_t^{i'_t}] = [B_a\colon 1+\mathcal{B}_t][\mathcal{B}_t\colon \mathcal{B}_t^{i'_t}] = (q^f-1)^a q^{fa(i'_t-1)}.$$

(2) If $j'_t = 1$, $[B_a\colon 1+\mathcal{B}_t^{i'_t}]$ is replaced by

$$[p_{t-1}\colon 1+\mathcal{B}_{b_{t-1}}^{i'_{t-1}}] = [P_{t-1}\colon 1+\mathcal{B}_{t-1}][\mathcal{B}_{t-1}\colon \mathcal{B}_{t-1}^{i'_{t-1}}] = |\,\mathrm{GL}_h(q_{t-1})|^a q_{t-1}^{h^2a(i'_{t-1}-1)}.$$

We need three more observations before giving the formula for $\deg(\pi_\theta, G/T)$. First, from (2.2.2′), we have

$$\deg(\mathrm{St}, G_a/T_a)\operatorname{vol}(K_aT_a/T_a) = \frac{1}{a}\prod_{k=1}^{a-1}(q^{fk}-1). \tag{2.4.4}$$

Second, an easy calculation yields

$$\begin{aligned}\operatorname{vol}(B_a)/\operatorname{vol}(K_a) &= [K_a\colon B_a]^{-1} \\ &= (q^f-1)^a(q^{fa}-1)^{-1}\left\{\prod_{k=1}^{a-1}(q^{fk}-1)\right\}^{-1}.\end{aligned} \tag{2.4.5}$$

Finally, from (2.2.3)

$$\operatorname{vol}(K)/[K\colon K^{(ae)}] = [|\operatorname{GL}_f(q)|^{ae}q^{(n/2)(1-f)}]/[n(q^n-1)] \tag{2.4.6}$$

since $[K\colon K^{(ae)}] = |\operatorname{Gl}_n(q)|\,|\operatorname{GL}_f(q)|^{-a}q^{-f^2[ae(ae-1)/2]}$ (see the proof of Lemma 2.2.7(2)).

THEOREM 2.4.7. *Let E/F be an extension of degree m, $m|n$, $m < n$, and write $n = ma$. Let θ be a subadmissible character on $E^\times/G$, and let π_θ be the generalized special representation constructed from θ. Let $e = e(E/F)$, $f = f(E/F)$, and $\{n_k, j_k\}$ be the data from the Howe factorization of θ given in* (2.1.4) *and* (2.1.5). *Then*

$$\deg(\pi_\theta, G/T) = [f(q^n-1)/(q^{n/e}-1)]q^{(af/2)(a\alpha(\theta)+a+1-am-e)},$$

where $\alpha(\theta) = \sum_{k=1}^{t} j_k(n_{k-1}-n_k)$.

Proof. We have

$$\begin{aligned}\deg(\pi_\theta, G/T) = e^{-1}&[\operatorname{vol}(J_\theta)]^{-1}\deg(\Omega_\theta)\deg(\mathrm{St}, G_a/T_a) \\ &\times \operatorname{vol}(K_aT_a/T_a)[\operatorname{vol}(B_a)/\operatorname{vol}(K_a)]\end{aligned}$$

from (2.3.14), (2.3.15) and (2.3.16). The values of the various factors in the above expression are given in Lemma 2.3.11, Lemma 2.4.3, (2.4.4), (2.4.5) and (2.4.6). Some elementary arithmetic gives us

$$\deg(\pi_\theta, G/T) = [(n/ea)(q^n-1)/(q^{fa}-1)]q^{(f/2)(\alpha'(\theta)+2a-ea-n)},$$

where $\alpha'(\theta) = \sum_{k=1}^{t} j'_k(n'_{k-1}-n'_k)$.

The final formula is obtained by using (2.3.5) and (2.3.7).

REMARK 2.4.8. When $a = 1$, then $m = n$ and the formula in Theorem 2.4.7 reduces to the formula in Theorem 2.2.8. Note that the normalizations of measures given in (2.2.3) shows that, in the calculations leading to the formal degrees of both the supercuspidal and generalized special representations, Haar measure on G is given by (2.2.3), (i).

3. Degrees for the representations of $D_n^\times$. This section contains the calculation of the degrees of the irreducible representations of $D^\times = D_n^\times$. Since $D^\times$ is compact mod Z_D (Z_D the center of D, $Z_D \simeq F^\times$), these representations are finite dimensional. Moreover, as pointed out in the Introduction, the irreducible representations of $D^\times$ may be constructed as induced representations, and the inducing representations are determined by admissible or subadmissible characters of extensions E/F of degree m, where $m|n$.

Let $\mathcal{O}_D$ be the integers in D, $\tilde{\omega}_D$ a prime element in D, and set $\mathcal{P}_D^r = \tilde{\omega}_D^r \mathcal{O}_D$, $r \geq 1$. Let F_n be an unramified extension of degree n over F which is embedded in D. The residue class field $\overline{F}_n$ of F_n is also the residue class field of D, and $|\overline{F}_n| = q^n$. We may choose $\tilde{\omega}_D$ so that $\tilde{\omega}_D^n$ is a prime element of F. We set $K_0 = \mathcal{O}_D^\times$, and $K_h = 1 + \mathcal{P}_D^h$, $h \geq 1$.

Now, let E be an extension of degree m over F, where $m|n$, and, as in §2, write $e = e(E/F)$ and $f = f(E/F)$. Contrary to the situation in §2, there is no need here to separate the cases $m = n$ and $m < n$. We write

$$n = ma, \tag{3.1}$$

noting that we may have $a = 1$. Let θ be an admissible or subadmissible character of $E^\times/F$ with Howe factorization given by (2.1.2) and (2.1.3) (or (2.3.1) and (2.3.2)). As usual, we twist away the character χ for the purpose of computing degrees.

Using the notation in (2.3.5) without referring to an auxiliary extension E', we set

$$e_k' = ae(E/E_k), \quad f_k' = f(E/E_k), \quad n_k' = a[E\colon E_k] = an_k. \tag{3.2}$$

In particular $n_0 = m$.

For the E-conductor of ϕ_k, we write $f_E(\phi_k) = j_k$, and define $f_D(\phi_k) = j_k'$, where

$$j_k' - 1 = af(j_k - 1). \tag{3.3}$$

As in (2.3.8), we set, for $1 \leq k \leq t$,

$$i_k' = \begin{cases} j_k'/2, & j_k' \text{ even}, \\ (j_k' - 1)/2, & j_k' \text{ odd}. \end{cases} \tag{3.4}$$

We set $j_{t+1}' = 1$ and $i_{t+1}' = 0$.

Let π_θ' be the representation of $D^\times$ corresponding to θ. In order to compute $\deg(\pi_\theta')$, we recall a few facts about its construction. We embed E (and hence E_k) in D, and let D_k be the division algebra

$$D_k = \{x \in D | xy = yx \text{ for all } y \in E_k\}, \qquad 0 \leq k \leq t. \tag{3.5}$$

Note that $D_0 = D$. Now, define

$$H_\theta = D_t^\times (K_{i'_t} \cap D_{t-1}^\times)(K_{i'_{t-1}} \cap D_{t-2}^\times) \cdots (K_{i'_2} \cap D_1^\times)(K_{i'_1}). \tag{3.6}$$

Then ([**Co**], [**H1**], [**M**]), there is an irreducible representation σ'_θ of H_θ such that

$$\pi'_\theta = \operatorname{Ind}_{H_\theta}^{D^\times} \sigma'_\theta. \tag{3.7}$$

From [**Co**], [**M**], we can write

$$\sigma'_\theta = \kappa'_t \otimes \kappa'_{t-1} \otimes \cdots \otimes \kappa'_1. \tag{3.8}$$

For the remainder of this section, we write $\overline{f}_k = f(E_k/F)$, $k = 0, 1, \ldots, t$. This should not be confused with the notation $f_k = f(E/E_k)$ as used in §2. Note that $\overline{f}_{k-1} | \overline{f}_k$.

LEMMA 3.9 ([**Co**], [**M**]). *Let* σ'_θ *be the representation of* H_θ *given in* (3.8). *Then, for* $1 \leq k \leq t$,

$$\deg(\kappa'_k) = \begin{cases} 1 & \text{if } j'_k \text{ is even,} \\ q^{\alpha(k)/2} & \text{if } j'_k \text{ is odd,} \end{cases}$$

where

$$\alpha(k) = \begin{cases} 0 & \text{if } \overline{f}_{k-1} \nmid i'_k, \\ f e'_{k-1} & \text{if } \overline{f}_{k-1} \mid i'_k, \quad \overline{f}_k \nmid i'_k, \\ f(e'_{k-1} - e'_k) & \text{if } \overline{f}_k \mid i'_k. \end{cases}$$

Now, for the degree of π'_θ, we have

$$\deg(\pi'_\theta) = [D^\times : H_\theta] \deg(\sigma'_\theta), \quad \text{and} \tag{3.10}$$

$$[D^\times : H_\theta] = [D^\times : H_\theta K_0][H_\theta K_0 : H_\theta K_1][H_\theta K_1 : H_\theta]. \tag{3.11}$$

We calculate each of the three indices in (3.11) separately.

LEMMA 3.12. $[D^\times : H_\theta K_0] = f$.

Proof. First note that $[D^\times : H_\theta K_0] = [D^\times / K_0 : H_\theta K_0 / K_0]$. Let ν be the usual valuation on $D^\times$. Then $\nu : D^\times \to \mathbf{Z}$ has kernel K_0. Under this map, $\nu(H_\theta)$ is generated by $\nu(\tilde{\omega}_t)$, where $\tilde{\omega}_t$ is a prime in D_t. Since $\tilde{\omega}_t$ must commute with the unramified piece of E_t, $\nu(\tilde{\omega}_t) = f$.

LEMMA 3.13. $[H_\theta K_0 \colon H_\theta K_1] = (q^n - 1)/(q^{n/e} - 1)$.

Proof. We have $[H_\theta K_0 \colon H_\theta K_1] = [H_\theta K_0/\langle\tilde{\omega}_t\rangle K_1 \colon H_\theta K_1/\langle\tilde{\omega}_t\rangle K_1]$. Now $H_\theta K_0/\langle\tilde{\omega}_t\rangle K_1$ is isomorphic to $\overline{F}_n^\times$, and $H_\theta K_1/\langle\tilde{\omega}_t\rangle K_1$ is isomorphic to the multiplicative group of the residue class field of D_t. But, D_t is a division algebra of index a over E_t, and the residue class field of $E_t = E$ has order q^f.

Before calculating the remaining index, $[H_\theta K_1 \colon H_\theta]$, we establish some notation. Define, for $j \geq 0$,

$$\beta_j = \begin{cases} n - fe'_k & \text{if } i'_{k+1} \leq j < i'_k \text{ and } \overline{f}_k | j; \\ n & \text{if } i'_{k+1} \leq j < i'_k \text{ and } \overline{f}_k \nmid j. \end{cases} \tag{3.14}$$

Note that $i'_{k+1} < i'_k$ from the properties of the Howe factorization; and that $\beta_0 = n - af$.

LEMMA 3.15. *Let β_j be defined as above. Then* $[H_\theta K_1 \colon H_\theta] = q^{\sum_{j=1}^{i'_t - 1} \beta_j}$.

Proof. Since $K_{i'_t} \subset H_\theta$, we have

$$[H_\theta K_1 \colon H_\theta] = \prod_{j=1}^{i'_t - 1} [H_\theta K_j \colon H_\theta K_{j+1}].$$

Suppose $i'_{k+1} \leq j < i'_k$. We have

$$[H_\theta K_j \colon H_\theta K_{j+1}] = [K_j/K_{j+1}(H_\theta \cap K_{j+1}) \colon 1].$$

For $j \geq 1$, $K_j/K_{j+1} \cong \mathscr{P}_D^j/\mathscr{P}_D^{j+1} \cong \overline{F}_n$, and the elements of $H_\theta \cap K_j$ correspond, under this isomorphism, to

$$\{\delta \in \overline{F}_n | \delta\tilde{\omega}_D^j \in D_k \ (\operatorname{mod} \mathscr{P}_D^{j+1})\}.$$

The number of such elements is equal to $q^{fe'_k}$ if $\overline{f}_k | j$ and 1 if $\overline{f}_k \nmid j$. Hence, if $i'_{k+1} \leq j < i'_k$, we get

$$[H_\theta K_j \colon H_\theta K_{j+1}] = \begin{cases} q^{n - fe'_k} & \text{if } \overline{f}_k | j, \\ q^n & \text{if } \overline{f}_k \nmid j. \end{cases}$$

COROLLARY 3.16. *Let E/F be an extension of degree m, $m|n$, and let θ be an admissible or subadmissible character of $E^\times/F$. Let π'_θ be the irreducible representations of $D^\times$ associated to θ. Then*

$$\deg(\pi'_\theta) = [f(q^n - 1)/(q^{n/e} - 1)] q^{(1/2)\sum_{k=1}^{t} \alpha(k)} q^{\sum_{j=1}^{i'_t - 1} \beta_j}. \tag{3.17}$$

For the computations below, it is convenient to define

$$\alpha_1(k) = \begin{cases} 0 & \text{if } j'_{k+1} \text{ is even or } \overline{f}_k \nmid i'_{k+1}, \\ fe'_k & \text{otherwise;} \end{cases} \tag{3.18}$$
$$\alpha_2(k) = \begin{cases} 0 & \text{if } j'_k \text{ is even or } \overline{f}_k \nmid i'_k \\ fe'_k & \text{otherwise.} \end{cases}$$

Then, we have $\alpha(k) = \alpha_1(k-1) - \alpha_2(k)$, $1 \le k \le t$. We observe that $\alpha_1(t) = n/e$, and that $\alpha_1(0)$ is defined.

Set

$$\gamma(k) = (1/2)(\alpha_1(k) - \alpha_2(k)) + \sum_{j=i'_{k+1}}^{i'_k - 1} \beta_j, \qquad 1 \le k \le t. \tag{3.19}$$

To compute $\gamma(k)$, we consider four cases. Note that $\overline{f}_k | j'_k - 1$ always.

(I) j'_{k+1} *even,* j'_k *even.* Here, $\alpha_1(k) = \alpha_2(k) = 0$, and $\overline{f}_{k-1}$ is odd. The number of multiples of $\overline{f}_k$ in $[i'_{k+1}, i'_k - 1)$ is $(1/2\overline{f}_k)(j'_k - j'_{k+1})$. So

$$\begin{aligned}\gamma(k) &= n(i'_k - i'_{k+1}) - (fe'_k)(1/2\overline{f}_k)(j'_k - j'_{k+1}) \\ &= (1/2)(j'_k - j'_{k+1})(n - n'_k).\end{aligned}$$

(II) j'_{k+1} *odd,* j'_k *even.* Then $\overline{f}_k$ is odd, so that $\overline{f}_k | i'_{k+1}, \overline{f}_k \nmid i'_k$. This gives $\alpha_1(k) = fe'_k, \alpha_2(k) = 0$. The number of multiples of $\overline{f}_k$ in $[j'_{k+1}, j'_k - 1)$ is $(1/\overline{f}_k)(j'_k - j'_{k+1})$, an odd number, $\overline{f}_k$ also divides $j'_{k+1} - 1 = 2i'_{k+1}$. Thus, the number of multiples of $\overline{f}_k$ in $[j'_{k+1} - 1, j'_k - 1)$ is $1 + (1/\overline{f}_k)(j'_k - j'_{k+1})$, and the number of multiples of $\overline{f}_k$ in $[i'_{k+1}, i'_k - 1)$ is $(1/2\overline{f}_k)(j'_k - j'_{k+1}) + 1/2$. We then have

$$\begin{aligned}\gamma(k) &= n(i'_k - i'_{k+1}) - (fe'_k)(1/2\overline{f}_k)(j'_k - j'_{k+1}) - (1/2)fe'_k + (1/2)fe'_k \\ &= (1/2)(j'_k - j'_{k+1})(n - n'_k) + (n/2).\end{aligned}$$

(III) j'_{k+1} *even,* j'_k *odd.* Here $\overline{f}_{k+1}$ is odd, so that $\overline{f}_k$ is odd. Therefore, $\overline{f}_k | i'_k, \overline{f}_k \nmid i'_{k+1}, \alpha_1(k) = 0$, $\alpha_2(k) = fe'_k$. The reasoning is similar to that in (II) above, but here we lose a multiple of $\overline{f}_k$. We get $\gamma(k) = (1/2)(j'_k - j'_{k+1})(n - n'_k) - (n/2)$.

(IV) j'_{k+1} *odd,* j'_k *odd.* We must consider subcases here.

If $\overline{f}_k | i'_k$ and $\overline{f}_k | i'_{k+1}$, then $\alpha_1(k) = \alpha_2(k) = fe'_k$, and the number of multiples of $\overline{f}_k$ in the relevant interval is $(1/2\overline{f}_k)(j'_k - j'_{k+1})$. This gives $\gamma(k) = (1/2)(j'_k - j'_{k+1})(n - n'_k)$.

If $\overline{f}_k | i'_k$, but $\overline{f}_k \nmid i'_{k+1}$, then $\alpha_1(k) = 0$ and $\alpha_2(k) = fe'_k$, and the number of multiples of $\overline{f}_k$ in the relevant interval is $(1/2\overline{f}_k)(j'_k - j'_{k+1}) - (1/2)$. Thus,

$$\begin{aligned}\gamma(k) &= (1/2)(j'_k - j'_{k+1})(n - n'_k) + (1/2)fe'_k - (1/2)fe'_k \\ &= (1/2)(j'_k - j'_{k+1})(n - n'_k).\end{aligned}$$

The cases $\overline{f}_k \nmid i'_k$, $\overline{f}_k | i'_{k+1}$, and $\overline{f}_k \nmid i'_k$, $\overline{f}_k \nmid i'_{k+1}$ are similar, and give $\gamma(k) = 1/2(j'_k - j'_{k+1})(n - n'_k)$.

We now sum the $\gamma(k)$.

LEMMA 3.20. (a) *If j'_1 is even, then*

$$\sum_{k=1}^{t} \gamma(k) = (1/2)\left[\sum_{k=1}^{t} (j'_k - j'_{k+1})(n - n'_k) + n\right].$$

(b) *If j'_1 is odd, then*

$$\sum_{k=1}^{t} \gamma(k) = (1/2)\left[\sum_{k=1}^{t} (j'_k - j'_{k+1})(n - n'_k)\right].$$

Proof. (a) Here, case (II) occurs once more than case (III).

(b) If j'_1 is odd, case (II) and case (III) balance out.

We now have, from (3.17), (3.18) and Lemma 3.20,

$$\deg(\pi'_\theta) = [f(q^n - 1)/(q^{n/e} - 1)]q^{[\sum_{k=1}^{t} \gamma(k)] + \frac{1}{2}[\alpha_1(0) - \alpha_1(t)] - \beta_0}. \tag{3.21}$$

Note that $\alpha_1(0) = 0$ if j'_1 is even, and $\alpha_1(0) = n$ if j'_1 is odd. Also, $\alpha_1(t) = af$, and $\beta_0 = n - af$ in all cases. It follows that

$$\begin{aligned}&\sum_{k=1}^{t} \gamma(k) + \frac{1}{2}[\alpha_1(0) - \alpha_1(t)] - \beta_0 \\ &\qquad = \frac{1}{2}\left[\sum_{k=1}^{t} (j'_k - j'_{k+1})(n - n'_k) - n + af\right],\end{aligned} \tag{3.22}$$

and that

$$\deg(\pi'_\theta) = [f(q^{n-1})/(q^{n/e} - 1)]q^{1/2[\sum_{k=1}^{t} (j'_k - j'_{k+1})(n - n'_k) - n + af]} \tag{3.23}$$

Now, from (3.2) and (3.3), we see that

$$\begin{aligned}\sum_{k=1}^{t}(j'_k - j'_{k+1})(n - n'_k) &= n(j'_1 - j'_{t+1}) - \sum_{k=1}^{t} n'_k(j'_k - j'_{k+1})\\ &= nj'_1 - n - n'_0 j'_1 + n'_t + \sum_{k=1}^{t} j'_k(n'_{k-1} - n'_k)\\ &= -n + a + \sum_{k=1}^{t}[1 + af(j_k - 1)](an_{k-1} - an_k)\\ &= -n + a + a(n_0 - n_t) - a^2 f(n_0 - n_t) + (a^2 f)\sum_{k=1}^{t} j_k(n_{k-1} - n_k)\\ &= -a^2 fm + a^2 f + (a^2 f)\sum_{k=1}^{t} j_k(n_{k-1} - n_k).\end{aligned}$$

Thus, the exponent of q in (3.23) is

$$(3.24)\qquad (af/2)\left[a\sum_{k=1}^{t} j_k(n_{k-1} - n_k) + a + 1 - am - e\right].$$

THEOREM 3.25. *Let E/F be an extension of degree m, $m|n$, and write $n = ma$ (here, we may have $m = n$ and $a = 1$). Let θ be an admissible or subadmissible character of $E^\times/F$ and let π'_θ be the irreducible representation of $D^\times$ constructed from θ. Let $e = e(E/F)$, $f = f(E/F)$, and let $\{n_k, j_k\}$ be the data from the Howe factorization of θ. Then*

$$\deg(\pi_\theta) = [f(q^n - 1)/(q^{n/e} - 1)]q^{(af/2)[a\alpha(\theta)+a+1-am-e]}$$

where $\alpha(\theta) = \sum_{k=1}^{t} j_k(n_{k-1} - n_k)$.

4. Comparison of degrees. We are now in a position to wrap things up in fine fashion. In particular, we prove the statements in (1.7) and derive some consequences. We begin with a simple lemma.

LEMMA 4.1. *If a, b, c, d are positive integers such that $a|B$ and $c|d$, and q is a power of a prime, then $a/(q^b - 1) = c/(q^d - 1)$ if and only if $a = c$ and $b = d$.*

Proof. If $b = d$, then $a = c$. So, assume $b > d$. It is easy to see that $(q^b - 1, q^d - 1) = q^r - 1$ where $r = (b, d)$. Let $q^b - 1 = (q^r - 1)b'$,

$q^d - 1 = (q^r - 1)d'$. Then $ad' = b'c$ and $(b', d') = 1$. Thus, $b'|a$. Since $r < b/2$, $(q^r - 1)^2 < q^b - 1$. It follows that $q^b - 1 < (b')^2 \leq a^2 \leq b^2$. But, if $q \geq 3$, $q^b - 1 > b^2$ for all positive integers b. If $q = 2$, then $q^b - 1 > b^2$ when $b > 4$. We are left with $q = 2$, $4 \geq b \geq 1$, and an enumeration of cases shows that there can be no solution with these restrictions.

Disjunction of formal degrees.

THEOREM 4.2. *Let $d(\pi'_\theta)$ and $d(\pi_\theta)$ be the degrees and formal degrees given in Theorem 2.2.8, Theorem 2.4.7 and Theorem 3.25. Let Δ'_1, Δ'_2, Δ_1 and Δ_2 be the sets defined in (1.5) and (1.6). Then $\Delta'_1 \cap \Delta'_2 = \Delta_1 \cap \Delta_2 = \Delta'_1 \cap \Delta_2 = \Delta_1 \cap \Delta'_2 = \varnothing$. Thus, $\Delta'_1 = \Delta_1$ and $\Delta'_2 = \Delta_2$.*

Proof. This is an immediate consequence of Lemma 4.1 and the fact that $\Delta'_1 \cup \Delta'_2 = \Delta_1 \cup \Delta_2$.

REMARK 4.3. The disjunction of formal degrees provided by Theorem 4.2 uses only the factor in the degrees which is prime to p, that is, $f(q^n - 1)/(q^{n/e} - 1)$. This factor depends only on the field E and not on the particular admissible or subadmissible character. We expect that Theorem 4.2 is true in an appropriate sense when $p|n$.

Comparison of characters.

THEOREM 4.4. (1) *If θ is an admissible character of $E^\times/F$ ($[E:F] = n$) and π'_θ is the representation of $D^\times$ parametrized by θ, then the representation π of G corresponding to π'_θ under the abstract matching theorem is supercuspidal. Moreover, if $\Theta_{\pi'_\theta}$ is the character of π'_θ, then $\Theta_\pi = (-1)^{n-1}\Theta_{\pi'_\theta}$ is a supercuspidal character on the elliptic set in G.*

(2) *If θ is a subadmissible character (for n) of $E^\times/F$, $[E: F] = m$, $m|n$, $m < n$, and π'_θ is the representation of $D^\times$ parametrized by θ, then the representation π of G corresponding to π'_θ under the abstract matching theorem is a generalized special representation of G. Moreover, if $\Theta_{\pi'_\theta}$ is the character of π'_θ, then $\Theta_\pi = (-1)^{n-1}\Theta_{\pi'_\theta}$ is a generalized special character on the elliptic set in G.*

Proof. This follows from Theorem (1.1) since the trivial representation of $D^\times$ is in A'_2 and the Steinberg representation of G is in A_2 (see (1.3) and (1.14)).

REMARK 4.5. (1) On the elliptic set in G, the character of a discrete series representation π is equal to $\pm d(\pi)$ near 1. It follows from Theorem 4.4 that supercuspidal characters can be distinguished from generalized special characters by their values near 1 on the elliptic set.

(2) Theorem 4.4 tells us that representations of $D^\times$ in A_1' correspond to supercuspidal representations of G, and representations in A_2' correspond to generalized special representations of G. In fact, Remark 4.3 allows us to refine this correspondence. Thus, if $\pi_\theta' \in A_1'$ (resp. A_2') and π is the corresponding representation in A_1 (resp. A_2), then π is parametrized by an admissible (resp. subadmissible) character of a field E/F which has the same ramification index and residue class degree as the field associated to θ.

(3) In general, the concrete matching by admissible characters is not the same as that given by Theorem 1.1 (1). It would be of some interest to determine the exact relation between these two matchings (see **[M]** for additional details).

Dependence on θ.

THEOREM 4.6. *Let E/F be an extension of degree m, and let θ be an admissible character of $E^\times/F$. Let π_θ be the supercuspidal representation of $\mathrm{GL}_m(F)$ determined by θ, and let π_θ^a be the generalized special representation of $\mathrm{GL}_n(F)$ corresponding to π_θ, where $n = ma$. Then*

$$d(\pi_\theta)^{a^2}/d(\pi_\theta^a) = f^{a^2-1}[(q^m-1)^{a^2}(q^{fa}-1)/(q^f-1)^{a^2}(q^n-1)]q^{fa(a-1)(1-e)/2}.$$

In particular, this quotient is independent of θ. This formula may also be found in Waldspurger **[W]**, Theorem VII.3.2.

REMARK 4.7. It is reasonable to expect an expression similar to that above in the case $p|n$ even though the parametrization by admissible characters does not work.

CONDUCTORS. When $E = E_1$ (2.1.2) (the very cuspidal case), Carayol **[C]** has shown that the formal degree of π_θ determines its conductor and, conversely, the conductor of π_θ determines its formal degree. In the general case, it follows from Moy **[M]** that

$$\text{(4.8)} \qquad \operatorname{cond}(\pi_\theta) = af(j_1-1)+n,$$

where $[E\colon F] = m$, $n = ma$ and $f = f(E/F)$. Thus, comparing (4.8) with the expressions for the formal degrees in Theorem 2.2.8 and

Theorem 2.4.7, we see that there is no direct relationship between the conductor of π_θ and the formal degree of π_θ. We do note however that, if the data from the Howe factorization is known, in particular $j_1 = f_E(\theta)$, then the formula for the conductor (4.8) is an immediate consequence.

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