THE BOUNDARY BEHAVIOUR OF HARMONIC UNIVALENT MAPS

YUSUF ABU-MUHANNA AND ABDALLAH LYZZAIK

Let D denote the open unit disc in the complex plane and $f = h + \overline{g}$ a complex-valued, harmonic, univalent and orientation preserving map in D, where h and g are analytic in D. We show that $g, h \in H^\lambda$ and $f \in h^\lambda$ for some $\lambda > 0$, where H^λ (h^λ) is the Hardy space of order λ for analytic (harmonic) functions. We also study the correspondence under f between ∂D (boundary of D) and the prime ends of $f(D)$.

1. Introduction. Let D denote the open unit disk and S_H denote the class of all complex valued, harmonic, orientation-preserving, univalent functions f in D normalized by

$$(1) \qquad\qquad f(0) = 0 \quad \text{and} \quad f_z(0) = 1.$$

Each $f \in S_H$ can be expressed as

$$f = h + \overline{g}$$

where $h = z + \sum_{n=2}^{\infty} a_n z^n$ and $g = \sum_{n=1}^{\infty} b_n z^n$ are analytic in D. Clunie and Sheil-Small, [3], studied S_H together with some geometric subclasses of S_H. They proved, among other results, that S_H is normal with respect to the topology of uniform convergence on compact subsets of D. In this paper, we study the aspect of boundary behaviour of functions in S_H. We point out that, although some of our results are stated for $f \in S_H$, conditions (1) are not needed.

It is known that, when $f \in S_H$ is also analytic, the length of the image of the radius $[0, e^{i\theta}]$, $\int_0^1 |f'(re^{i\theta})|\, dr$, is finite for all θ except for a set of logarithmic capacity zero [8, p. 341]. It is also known that $f \in H^\lambda$, $0 < \lambda < \frac{1}{2}$, where H^λ is the Hardy space of analytic functions of order λ [8, p. 127], [5, p. 61]. In §2, we prove that the length of the image of the radius $[0, e^{i\theta}]$ under $f \in S_H$, $\int_0^1 |\frac{\partial f}{\partial r}(re^{i\theta})|$, is finite for almost all θ. We also prove that h, $g \in H^\lambda$ and $f \in h^\lambda$ for some λ small, where h^λ is the Hardy space of harmonic functions of order λ [4, p. 2]. Furthermore, as a corollary, we conclude that the radial limits $\hat{h}(e^{i\theta}) = \lim_{r\to 1^-} h(re^{i\theta})$, $\hat{g}(e^{i\theta}) = \lim_{r\to 1^-} g(re^{i\theta})$ and $\hat{f}(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$ exist for almost all θ [4, pp. 15–17].

One of the many differences between harmonic univalent maps and conformal maps is that a harmonic univalent map can be constant on an interval of the boundary of D, ∂D. In §3, we show that if $f \in S_H$ and f is constant on an interval then the interval "corresponds" to a prime-end satisfying a wedge condition (see Theorem 3). In addition, we obtain some corollaries.

In §4, we study the correspondence, under $f \in S_H$, between points of ∂D and the prime-ends of $f(D)$ (see Theorems 6, 7).

2. Radial limits. First, we have the following two lemmas.

LEMMA 1. *Let $f = h + \overline{g} \in S_H$. Then $\log h'$ is a Bloch function, that is,*

$$(2) \qquad \left|\frac{h''(z)}{h'(z)}\right| \leq \frac{c}{1-r} \qquad (z = re^{i\theta}),$$

for some absolute constant c.

Proof. For fixed $\zeta \in D$ consider the function

$$F(z) = \frac{f(\frac{z+\zeta}{1+\overline{\zeta}z}) - f(\zeta)}{(1-|\zeta|^2)h'(\zeta)}.$$

Then $F \in S_H$, and we can write

$$F(z) = H(z) + \overline{G}(z),$$

where

$$H(z) = z + \sum_{n=2}^{\infty} A_n z^n.$$

By direct computation we obtain

$$2A = (1 - |\zeta|^2)\frac{h''(\zeta)}{h'(\zeta)} - 2\overline{\zeta}.$$

Since $A = \sup_{S_H} |a_2|$ is finite [3],

$$|1 - |\zeta|^2|\left|\frac{h''(\zeta)}{h'(\zeta)}\right| \leq 2A + 2 = c,$$

and the proof is complete since ζ is arbitrary.

LEMMA 2. *Let h be an analytic and locally univalent function in D satisfying inequality (2). If $\alpha > 0$, then*

$$\lim_{r \to 1^-} (1-r)^{\alpha} h'(re^{i\theta}) = 0$$

for almost all θ.

This is a result of J. Clunie and T. MacGregor [2], [7, p. 73]. As a consequence of Lemma 2 we have the following theorem.

THEOREM 1. *Let $f = h + \overline{g} \in S_H$. Then the integrals*

$$\int_0^1 |h'(re^{i\theta})| \, dr, \quad \int_0^1 |g'(re^{i\theta})| \, dr, \quad and \quad \int_0^1 \left| \frac{\partial f}{\partial r} \right| dr$$

converge for almost all θ, and the boundary function $\hat{f}(e^{i\theta})$ exists almost everywhere.

Proof. Choose $0 < \alpha < 1$. Then by Lemma 2,

$$\lim_{r \to 1^-} (1 - r)^\alpha h'(re^{i\theta}) = 0$$

for almost all θ. Hence we have

$$|h'(re^{i\theta})| \leq \frac{k_\theta}{(1 - r)^\alpha}$$

for some constant k_θ, which gives at once $\int_0^1 |h'(re^{i\theta})| \, dr < \infty$ for almost all θ. This yields directly the convergence of the other integrals since $|g'| < |h'|$ because $f \in S_H$, and

$$\frac{\partial f}{\partial r}(re^{i\theta}) = e^{i\theta} h'(re^{i\theta}) + \overline{e^{i\theta} g'(re^{i\theta})}.$$

Now observe that $\int_0^1 |\frac{\partial f}{\partial r}(r, \theta)| \, dr < \infty$ means that the image curve under f of the radius: $re^{i\theta}$, $0 \leq r < 1$ is rectifiable. This evidently implies that $\hat{f}(e^{i\theta})$ exists a.e. and the proof is complete.

Our next result states that every normalized harmonic univalent map together with its analytic and coanalytic parts belong to some Hardy spaces which also imply that $\hat{f}(e^{i\theta})$ exists almost everywhere. In particular, we have:

THEOREM 2. *Let $f = h + \overline{g} \in S_H$. Then $h, g \in H^\lambda$ and $f \in h^\lambda$ for every λ, $0 < \lambda < 1/c^2$, where c is the absolute constant of Lemma 1.*

Proof. First we show that $h \in H^\lambda$ for all $\lambda < 1/c^2$. For this purpose we show that for $\lambda > 0$ the inequality:

(3) $$I_\lambda(r, h') = \frac{1}{2\pi} \int_{-\pi}^\pi |h'(re^{i\theta})|^\lambda \, d\theta$$

$$\leq 2(1 + \lambda^2 c^2)(1 - r)^{-\beta} \qquad (0 \leq r \leq 1)$$

where $\beta = \frac{1}{2\pi}(\sqrt{1 + 4c^2\lambda^2} - 1) < c^2\lambda^2$ and c is the constant of Lemma 1.

The proof of (3) is an adaptation of that of Lemma 5.3 in [8, pp. 128–129] from which it follows that for $r_0 < r < 1$ and $\lambda > 0$,

$$(4) \quad I_\lambda(r, h') \le u(r) = K_1(c, r_0, \lambda)(1 - r)^{-\beta} + K_2(c, r_0, \lambda)(1 - r)^{\beta+1}$$

where

$$(5) \qquad K_1(c, r_0, \lambda) = \frac{1 + \beta}{1 + 2\beta}I_\lambda(r_0, h')(1 - r_0)^\beta$$
$$+ \frac{1}{1 + 2\beta}I'_\lambda(r_0, h')(1 - r_0)^{\beta+1},$$

and

$$(6) \qquad K_2(c, r_0, \lambda) = \frac{\beta}{1 + 2\beta}I_\lambda(r_0, h')(1 - r_0)^{-\beta-1}$$
$$- \frac{1}{1 + 2\beta}I'_\lambda(r_0, h')(1 - r_0)^{-\beta}.$$

Using Lemma 5.1 [8, pp. 125–126] and Lemma 1, we obtain

$$\frac{d}{dr}[rI'_\lambda(r, h')] = \frac{\lambda^2 r}{2\pi}\int_0^{2\pi}|h'(re^{i\theta})|^\lambda\left|\frac{h''(re^{i\theta})}{h'(re^{i\theta})}\right|^2 d\theta$$
$$\le \frac{c^2\lambda^2 r}{(1 - r)^2}I_\lambda(r, h').$$

Then

$$rI'_\lambda(r, h') \le c^2\lambda^2\int_0^r \frac{t}{(1 - t)^2}I_\lambda(t, h')\,dt$$
$$\le c^2\lambda^2 I_\lambda(r, h')\int_0^r \frac{t}{(1 - t)^2}\,dt$$

since $I_\lambda(t, h')$ is a nondecreasing function of t. Hence

$$0 \le I'_\lambda(r_0, h') \le \frac{c^2\lambda^2}{1 - r_0}I_\lambda(r_0, h').$$

Using this inequality in (5) and (6) we infer

$$0 \le K_1(c, r_0, \lambda) \le \frac{1 + \beta + c^2\lambda^2}{1 + 2\beta}I_\lambda(r_0, h')(1 - r_0)^\beta,$$

and

$$0 \le K_2(c, r_0, \lambda) \le \frac{\beta + c^2\lambda^2}{1 + 2\beta}I_\lambda(r_0, h')(1 - r_0)^{-\beta-1}.$$

Consequently by using (4) for $r_0 \leq r < 1$ we obtain

$$I_\lambda(r, h') \leq \left[\frac{1 + \beta + c^2\lambda^2}{1 + 2\beta} I_\lambda(r_0, h')(1 - r_0)^\beta \right.$$

$$\left. + \frac{\beta + c^2\lambda^2}{1 + 2\beta} I_\lambda(r_0, h')(1 - r_0)^{-\beta-1}(1 - r)^{2\beta+1} \right] (1 - r)^{-\beta}$$

$$\leq 2 \left[\frac{1 + \beta + c^2\lambda^2}{1 + 2\beta} I_\lambda(r_0, h')(1 - r_0)^{-\beta-1} \right] (1 - r)^{-\beta}.$$

Since r_0 is arbitrary we can let r_0 tend to zero. This proves our claim because

$$1 = h'(0) = \lim_{r_0 \to 0} I_\lambda(r_0, h')$$

and

$$\frac{1 + \beta + c^2\lambda^2}{1 + 2\beta}$$

attains its maximum value for $\beta \geq 0$ at $\beta = 0$.

Now we prove that $h \in H^\lambda$ for $0 < \lambda < 1/c^2$. The tools of proof are the *maximal function* of h defined by

$$h^*(r, \theta) = \int_0^r \max_{s \leq t} |h'(se^{i\theta})| \, dt$$

which clearly majorises h, and the inequality

$$(7) \quad \int_{-\pi}^{\pi} [h^*(r, \theta)]^\lambda \, d\theta \leq 2(1 - r)^\lambda \int_{-\pi}^{\pi} \left(\frac{\partial h^*}{\partial r} \right)^\lambda d\theta$$

$$+ 2\lambda \int_0^r (1 - t)^{\lambda-1} \, dt \int_{-\pi}^{\pi} \left(\frac{\partial h^*}{\partial t} \right)^\lambda d\theta.$$

Both the maximal function and inequality (7) are due to Hardy and Littlewood [6, pp. 414–415]. So also the fact that there is an absolute constant, B_λ, depending only on λ, such that

$$\int_{-\pi}^{\pi} \left(\frac{\partial h^*}{\partial r} \right)^\lambda d\theta = \int_{-\pi}^{\pi} \left(\max_{s \leq r} |h'(se^{i\theta})| \right)^\lambda d\theta$$

$$\leq B_\lambda \int_{-\pi}^{\pi} |h'(re^{i\theta})|^\lambda \, d\theta$$

$$\leq 2(1 + c^2\lambda^2) B_\lambda (1 - r)^{-c^2\lambda^2}$$

by using (3) and the fact $\beta < c^2\lambda^2$. Then from (7) we obtain

$$\int_{-\pi}^{\pi} [h^*(re^{i\theta})]^\lambda \, d\theta \leq 2(1 + c^2\lambda^2) B_\lambda (1 - r)^{\lambda-\lambda^2c^2}$$

$$+ 4\lambda(1 + c^2\lambda^2) B_\lambda \int_0^r (1 - t)^{\lambda-c^2\lambda^2-1} \, dt.$$

If $0 < \lambda < 1/c^2$, then $\lambda - \lambda c^2 > 0$ and consequently

$$\int_{-\pi}^{\pi} [h^*(re^{i\theta})]^\lambda \, d\theta = 4\frac{1 + c^2\lambda^2}{1 - c^2\lambda} B_\lambda + O[(1 - r)^{\lambda - c^2\lambda^2}]$$

as $r \to 1^-$. Therefore $h \in H^\lambda$ for all λ, $0 < \lambda < 1/c^2$.

In the same manner, we conclude that $g \in H^\lambda$ for the specific values of λ since $|g'| < |h'|$. Now $h, g \in H^\lambda$ yield directly that $f = h + \overline{g} \in h^\lambda$. This completes the proof.

From the above proof we immediately conclude:

COROLLARY 1. *Let h be analytic in D. If $\log h'$ is a Bloch function, then $h \in H^\lambda$ for some λ.*

3. $\hat{f}$ constant on some arc of ∂D. The purpose of this section is to study the boundary behaviour of a function $f \in S_H$ along a closed subarc of ∂D on which f extends continuously to a constant function there.

First, we need the following:

DEFINITION 1. Let D be a simply connected domain in $\mathbb{C}$ with at least two boundary points, P an accessible prime end, and (C_n) a null-chain of P. We say that D satisfies the *wedge condition* at P if there exists a closed circular sector whose vertex belongs to the impression, $I(P)$, otherwise it lies in D, and whose sides intersect C_n for all large n.

We also need the functions F and ϕ associated with $f \in S_H$ such that

(8) $$\phi = F^{-1} \circ f$$

where F is a conformal map from D to $f(D)$ satisfying $F(0) = 0$. Hence ϕ is an automorphism of D with radial limits existing almost everywhere.

The main theorem of this section can be stated as follows.

THEOREM 3. *Let $f = h + \overline{g} \in S_H$. Suppose that f extends continuously to a nondegenerate subinterval, J, of ∂D such that $f(z) = w_0$ for all $z \in J$. Then $\phi(J) = \lambda$, $|\lambda| = 1$; and if P is the prime end of $f(D)$ that corresponds to λ, under F, then $f(D)$ satisfies the wedge condition at P.*

For a definition of a prime-end see [8, pp. 271–277].

The proof of the theorem makes use of the following interesting lemma.

LEMMA 3. *Let $f = h + \overline{g}$ be continuous on $\overline{D}$, where h and g are analytic in D. Suppose that $f(z) = w_0$ for all z belonging to a non-degenerate subinterval $J = \{e^{i\theta} : \theta_1 \leq \theta \leq \theta_2\}$ of ∂D. Then*

(a) *$f_\theta(z) \to 0$ as $z \to \zeta$ for every $\zeta \in \mathrm{Int}(J)$.*
(b) *h and g extend analytically to $\mathrm{Int}(J)$, and*
(c) *$zh'(z) = \overline{zg'(z)}$ for all $z \in \mathrm{Int}(J)$.*

Proof. (a) Since f is bounded, we can write

$$(9) \qquad f(z) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\left(\frac{1 + ze^{-it}}{1 - ze^{-it}}\right) f(e^{it})\, dt, \qquad z \in D.$$

Then

$$f_\theta(z) = -\frac{w_0}{\pi} \mathrm{Im}\left[\int_{\theta_1}^{\theta_2} \frac{ze^{-it}}{(1 - ze^{-it})^2}\, dt\right]$$
$$-\frac{1}{\pi} \int_{\theta_1}^{2\pi+\theta_1} \mathrm{Im}\left[\frac{ze^{-it}}{(1 - ze^{-it})^2}\right] f(e^{it})\, dt$$
$$= \frac{w_0}{\pi} \mathrm{Re}\left[\frac{1}{1 - ze^{-i\theta_2}} - \frac{1}{1 - ze^{-i\theta_1}}\right]$$
$$-\frac{1}{\pi} \int_{\theta_2}^{2\pi+\theta_1} \mathrm{Im}\left[\frac{ze^{-it}}{(1 - ze^{-it})^2}\right] f(e^{it})\, dt,$$

which gives

$$|f_\theta(z)| \leq \frac{|w_0|}{\pi} \left|\mathrm{Re}\left[\frac{1}{1 - ze^{-i\theta_2}} - \frac{1}{1 - ze^{-i\theta_1}}\right]\right|$$
$$+ \frac{\|f\|_\infty}{\pi} \int_{\theta_2}^{2\pi+\theta_1} \left|\mathrm{Im}\left[\frac{ze^{-it}}{(1 - ze^{-it})^2}\right]\right|\, dt.$$

If $\zeta \in \mathrm{Int}(J)$, then for $\varepsilon > 0$ there is $\delta > 0$ such that

$$\left|\mathrm{Im}\left[\frac{ze^{-it}}{(1 - ze^{-it})^2}\right]\right| < \varepsilon, \qquad \theta_2 \leq t \leq 2\pi + \theta_1, \quad \text{and}$$
$$\left|\mathrm{Re}\left[\frac{1}{1 - ze^{-i\theta_2}} - \frac{1}{1 - ze^{-i\theta_1}}\right]\right| < \varepsilon,$$

for all z, $|z - \zeta| < \delta$. Hence

$$|f_\theta(z)| \leq \left(\frac{|w_0|}{\pi} + 2\|f\|_\infty\right)\varepsilon,$$

and the proof of (a) is complete.

(b) It follows from (9) that

$$h'(z) = f_z(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-it}}{(1 - ze^{-it})^2} f(e^{it})\, dt$$

$$= \frac{w_0}{2\pi} \int_{\theta_1}^{\theta_2} \frac{e^{-it}}{(1 - ze^{-it})^2}\, dt + \frac{1}{2\pi} \int_{\theta_2}^{2\pi+\theta_1} \frac{e^{-it}}{(1 - ze^{-it})^2} f(e^{it})\, dt$$

$$= \frac{w_0}{2} \left[\frac{e^{-i\theta_1} - e^{-i\theta_2}}{(1 - ze^{-i\theta_1})(1 - ze^{-i\theta_2})} \right]$$

$$+ \frac{1}{2\pi} \int_{\theta_2}^{2\pi+\theta_1} \frac{e^{-it}}{(1 - ze^{-it})^2} f(e^{it})\, dt.$$

Clearly, each of the latter terms is an analytic function in

$$\mathbb{C} \backslash (\partial D \backslash \operatorname{Int}(J)).$$

Hence h' is an analytic function there, and so h extends to an analytic function in any simply connected domain containing $D \in \operatorname{Int}(J)$.

As for g we have

$$g'(z) = \overline{f}_{\overline{z}}(z) = \frac{\overline{w}_0}{2\pi i} \left[\frac{e^{-i\theta_1} - e^{i\theta_2}}{(1 - ze^{-i\theta_1})(1 - ze^{-i\theta_2})} \right]$$

$$+ \frac{1}{2\pi} \int_{\theta_2}^{2\pi+\theta_1} \frac{e^{-it}}{(1 - ze^{-it})^2} \overline{f}(e^{it})\, dt,$$

and arguing as above for h, we conclude (b).

(c) This follows immediately from (a) and (b), and the fact that $f_\theta(z) = i(zh'(z) - \overline{zg'(z)})$, $z \in D$.

Proof of Theorem. First we claim that there is no loss of generality in assuming that f satisfies the hypotheses of Lemma 3. For otherwise we can replace f by a function W which we construct as follows. Let γ be a crosscut of D meeting J at its endpoints, and let G be the subdomain of D bounded by $J \cup \gamma$. Suppose without loss of generality that $0 \in G$, and let $\psi \colon D \to G$ be a conformal map satisfying $\psi(0) = 0$. Then the function defined by $W = (1/\psi'(0)) f \circ \psi$ satisfies the hypotheses of f in Lemma 3. Since the desired result is intrinsic to $f(D)$ and not to W, our claim follows at once.

Hence, suppose that f is exactly as in Lemma 3. Also, suppose that $h'(z) \neq 0$ for all $z \in J$; otherwise we take a subinterval of J with this condition. Let ζ_1, ζ_2 be a pair of distinct points of J, and I the subinterval of J with endpoints ζ_1 and ζ_2. Also let l_1, l_2 be the radii of D ending at ζ_1, ζ_2 respectively, S the closed sector bounded

by $I \cup l_1 \cup l_2$, $\Omega = f(S)$, $L_1 = f(l_1)$, and $L_2 = f(l_2)$. Then it follows directly that Ω is the simply connected subdomain of $f(D)$ bounded only by $L_1 \cup L_2$.

First, we observe that the inner angle of Ω at w_0 exists and has size

$$\left| \arg \frac{f_r(\zeta_1)}{f_r(\zeta_2)} \right| = \left| \arg \frac{\zeta_1 h'(\zeta_1) + \overline{\zeta_1 g'(\zeta_1)}}{\zeta_2 h'(\zeta_2) + \overline{\zeta_2 g'(\zeta_2)}} \right|$$

$$= \left| \arg \frac{\zeta_1 h'(\zeta_1)}{\zeta_2 h'(\zeta_2)} \right|$$

because of Lemma 3.c.

Now suppose that $f(D)$ does not satisfy the wedge condition at w_0. Then it is immediate in view of the above that

$$\left| \arg \frac{f_r(\zeta_1)}{f_r(\zeta_2)} \right| = 0$$

for all $\zeta_1, \zeta_2 \in J$; or equivalently,

$$zh'(z) = |zh'(z)|e^{i\eta}, \qquad z \in \text{Int}(J),$$

where η is a constant. Then by Lemma 3 we also have

$$zg'(z) = |zh'(z)|e^{-i\eta}, \qquad z \in \text{Int}(J),$$

Hence

$$\frac{h'}{g'}(z) = e^{2i\eta}, \qquad z \in \text{Int}(J),$$

and h'/g' is a constant function in D by the identity theorem, and $|h'| = |g'|$ in D. This contradicts that $f \in S_H$. Therefore, our assumption is false and $f(D)$ satisfies the wedge condition at w_0.

REMARK. The conclusion of theorem is best possible. In other words, P may not be a prime-end of the first kind, that is the impression of P, $I(P)$, is non-degenerate, as shown by the following example.

EXAMPLE 1. Let

$$K_0(z) = \text{Re} \frac{z + \frac{1}{3}z^3}{(1-z)^3} + i \, \text{Im} \frac{z}{(1-z)^2}.$$

It was shown in [3] that $K_0 \in S_H$, K_0 maps D to the complement of the ray on the negative real-axis from $-1/6$ to ∞ and $K_0(\zeta) = -1/6$ for all $\zeta \in \partial D \backslash \{1\}$. Let α be the circular arc in D with endpoints 1, i and perpendicular to ∂D (see Figure 1). Let $\beta = K_0(\alpha)$, G the lens domain between ∂D and α and $H = K_0(G)$. Then H is the domain in the upper half-plane bounded by β and $[-\infty, -1/6]$. For $n = 1, 2, \ldots$
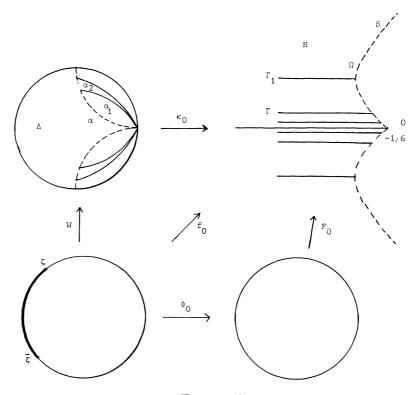
FIGURE 1

we let Γ_n to be the horizontal slit in $\overline{H}$ ending at β and satisfying $\operatorname{Im} w = \frac{1}{n}$ for $w \in \Gamma_n$ (see Figure 1). Then the arcs $K_0^{-1}(\Gamma_n) = \alpha_n$ lie in G and converge to ∂D as shown. Let $\overline{A}$ denote the reflection of A about the real axis. Now we let

$$W: D \to D \setminus \bigcup_{j=1}^{\infty} (\alpha_n \cup \overline{\alpha}_n) = \Delta$$

$$F_0: D \to K_0(D) \setminus \bigcup_{n=1}^{\infty} (\Gamma_n \cup \overline{\Gamma}_n) = \Omega$$

be the two conformal maps satisfying $W(0) = F_0(0) = 0$ and $W'(0)$, $F'(0) > 0$ and define

$$f_0 = K_0 \circ W,$$
$$\phi_0 = F_0^{-1} \circ f_0.$$

Observe that Δ has two accessible prime-ends P_1, P_2 of non-degenerate impressions with accessible points i, $-i$ and Ω has one accessible prime end P with impression $(-\infty, -1/6]$, accessibility point $-1/6$

and which satisfy the wedge condition. Note that there exist symmetric points $\zeta, \overline{\zeta} \in \partial D$ ($\zeta \neq -1, 1$) corresponding, under W, to P_1 and P_2 respectively, and that -1 corresponds, under F_0, to P. Then f_0 extends continuously to the circular arc ending at ζ and $\overline{\zeta}$ and containing -1, $f_0 \equiv -1/6$ there, and $f_0(D)$ satisfies the wedge condition at P.

We finally prove two corollaries which extend a result for normal functions [8, p. 267].

COROLLARY 2. *Let $f = u + v \in S_H$ satisfy $u, v \in h^p$, $p > 1$, and α: $z(t)$, $0 \leq t \leq 1$, be a Jordan arc in D with $|z(t)| \to 1$ ($t \to 1$). If $f(z(t)) \to w_0(t \to 1)$ and*

$$\lim_{t \to 1^-} \arg(f(z(t)) - w_0) \tag{10}$$

exists, then $z(t) \to \zeta(t \to 1)$ for some $\zeta \in \partial D$ so that α is an asymptotic path.

For the definition of an *asymptotic path* see [8, p. 267].

Proof. Suppose that the set of accumulation points of α on ∂D form a nondegenerate arc I. Then $f(\zeta) = w_0$ for every $\zeta \in I$ where $f(\zeta)$ exists. Because radial limits exist a.e. and $u, v \in h^p$, $p > 1$, f extends continuously to a constant function in $\text{Int}(I)$. Using this, we conclude from the proof of Theorem 3 that for distinct $\zeta_1, \zeta_2 \in \text{Int}(I)$ the images l_1, l_2, under f, of the radii ending at ζ_1, ζ_2, respectively, are Jordan half-intervals in $f(D)$, both ending at w_0 and make, in $f(D)$, an angle of positive size at w_0. This contradicts (10), since each of the arcs l_1 and l_2 meets $f(\alpha)$ in a sequence converging to w_0, and the proof is complete.

REMARK. Condition (10) of Corollary 2 is essential. For example:

EXAMPLE 2. Let f be the function defined by

$$f(e^{i\theta}) = \begin{cases} 1, & 0 \leq \theta < 2\pi/3, \\ i, & 2\pi/3 \leq \theta < 4\pi/3, \\ -i, & 4\pi/3 \leq \theta < 2\pi, \end{cases}$$

and

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \left(\text{Re} \frac{e^{i\theta} + z}{e^{i\theta} - z} \right) f(e^{i\theta}) \, d\theta.$$

Then f is a univalent harmonic map and $f(D)$ is a triangle (see [3]). Let γ be a curve in D converging to a non-degenerate arc subset to the arc $\{e^{i\theta}: 0 \leq \theta \leq \pi/2\}$. Then $f(\gamma)$ converges to 1.

COROLLARY 3. *Let f be as in Corollary 2, w_0 an accessible point of $f(D)$, and α_j: $z_j(t)$, $0 \le t < 1$ $(j = 1, 2)$ be Jordan arcs in D with $|z_j(t)| \to 1$ $(t \to 1)$. If $f(z_j(t)) \to w_0$ $(t \to 1)$,*

$$(11) \qquad \lim_{t \to 1^-} \arg(f(z_j(t)) - w_0) = \Theta \qquad (j = 1, 2)$$

and the zero cusp at w_0 with sides $f(\alpha_j)$ lies in $f(D)$, then $z_j(t) \to \zeta$ $(t \to 1; j = 1, 2)$ for some $\zeta \in \partial D$.

Proof. It follows at once from Corollary 2 that each α_j is an asymptotic path with $z_j(t) \to \zeta_j$ $(t \to 1)$. If $\zeta_1 \ne \zeta_2$, then there exists a Jordan arc Γ: $z(t)$, $0 \le t < 1$, in D such that for some positive sequence $t_n \to 1^-$ Γ meets α_j at $z_j(t_n)$ $(j = 1, 2)$ for all n, $f(z(t)) \to w_0$ $(t \to 1)$, and

$$\lim_{t \to 1} \arg(f(z(t)) - w_0) = \Theta.$$

By Corollary 2 Γ is an asymptotic path, a contradiction. This completes the proof.

REMARK. Condition (11) is essential. For example, let f be as in Example 2 and α_1, α_2 be two disjoint curves terminating at $e^{i\pi/4}$ and i. Then $f(\alpha_1)$, $f(\alpha_2)$ terminate at 1 with a positive angle between them.

4. Correspondence under f between D and the prime-ends of $f(D)$. Throughout this section, we let

$$(12) \qquad E = \{\zeta \in \partial D \colon C(\phi, \zeta) \text{ is not a singleton}\}, \quad \text{and}$$
$$G = \{\lambda \in \partial D \colon C(\phi^{-1}, \lambda) \text{ is not a singleton}\}$$

where ϕ is as given by (8) and $C(\phi, \zeta)$ is the cluster set of ϕ at ζ. For the definition of a cluster set, see [1, p. 66] or [8, p. 276]. In addition, let $C_R(\phi, \zeta)$ and $C_L(\phi, \zeta)$ denote the right and left cluster sets of ϕ at ζ [1, p. 83].

We next state Theorem 4, whose proof is a consequence of Theorem 3 and the forthcoming Lemma 5.

THEOREM 4. *Let $\lambda \in G$. Then the prime-end P of $f(D)$ corresponding to λ, under F, is accessible and satisfies the wedge condition at P.*

The next lemma is purely topological and is true for all automorphisms of D with radial limits almost everywhere.

LEMMA 4. *Let* ζ, $\zeta' \in \partial D$.

(a) *If* $C(\phi, \zeta) = \partial D$, *then* ϕ *extends continuously to a constant* λ, $|\lambda| = 1$, *on* $\partial D \backslash \zeta$, *and* ζ *is unique*.

(b) *If* $C(\phi, \zeta)$, $C(\phi, \zeta')$ *are not* ∂D, *then* $\operatorname{Int} C(\phi, \zeta) \cap C(\phi, \zeta') = C(\phi, \zeta) \cap \operatorname{Int} C(\phi, \zeta') = \phi$.

(c) *If* $C(\phi, \zeta) \cap C(\phi, \zeta') = \lambda$, $|\lambda| = 1$, *then* ϕ *extends continuously to an open circular arc between* ζ *and* ζ', *and* $\phi \equiv \lambda$ *there*.

Proof. (a) Let γ be a union of two radii of D so that ζ and ζ' are on different sides of γ and ϕ has radial limits along these radii. Since $C(\phi, \zeta) = \partial D$, $\phi(\gamma)$ is a loop that meets ∂D at a unique point λ. We infer at once that $C(\phi, \zeta') = \lambda$ for all $\zeta' \neq \zeta$, and ζ is unique. This proves (a).

(b) Choose $\zeta_1, \zeta_2 \in \partial D$, different from ζ, ζ', so that the radial limits $\hat{\phi}(\zeta_1)$, $\hat{\phi}(\zeta_2)$ exist and the crosscup γ, made up of the radii terminating at ζ_1 and ζ_2, separates ζ and ζ' in D. Then the crosscut, $\phi(\gamma)$ of D, separates D into two simply connected domains such that one contains $C(\phi, \zeta)$ on its boundary and the other $C(\phi, \zeta')$. This proves (b).

(c) If $C(\phi, \zeta) = \partial D$ for some $\alpha \in \partial D$, then we are done. Suppose otherwise and let I, I' be the subarcs of ∂D determined by ζ, ζ'. If for every $\zeta_1 \in \operatorname{Int}(I)$, $\phi(\alpha)$ exists and $\phi(\zeta_1) = \lambda$, then (c) follows, otherwise choose $\zeta_1 \in \operatorname{Int}(I)$, $\zeta_2 \in \operatorname{Int}(I')$ such that $\phi(\zeta_1)$, $\phi(\zeta_2)$ exists and $\phi(\zeta_1) \neq \lambda$. Then an argument similar to that of (b) yields (c) and the proof is complete.

REMARK. (1) K_0 indicates that (a) is possible.

(2) It follows from Lemma 4, that the set E (see [12]) is at most countable.

LEMMA 5. *Let* $\zeta, \zeta' \in \partial D$.

(a) *If* $C(\phi, \zeta) = \partial D$, *then* f *extends continuously to* $\partial D \backslash \zeta$ *and* f *is constant there*.

(b) *If* $C(\phi, \zeta) \cap C(\phi, \zeta') = \lambda$, $|\lambda| = 1$, *then* f *extends continuously to an open circular arc between* ζ *and* ζ', *and* $f \equiv F(\lambda)$ *there*.

Proof. (a) By Lemma 4 ϕ extends continuously to some constant, λ, $|\lambda| = 1$ on $\partial D \backslash \zeta$. Let $\zeta_1, \zeta_2 \in \partial D \backslash \zeta$ be chosen such that $f(\zeta_1)$, $f(\zeta_2)$ exist. If γ_1, γ_2 are the images, under ϕ, of the radii ending at ζ_1, ζ_2, respectively, then γ_1, γ_2 are asymptotic paths of F both ending at λ. Since F is univalent, F is normal (see [8, pp. 262–263]), and

$F(\zeta_1) = F(\zeta_2)$ (see [**8**, p. 267]). Therefore, $f(\zeta_1) = f(\zeta_2)$. Since ζ_1, ζ_2 are arbitrary, the desired result follows.

The truth of (b) follows in the same manner. This completes the proof.

As an immediate consequence of Theorem 4 we have

COROLLARY 4. *Suppose that $f(D)$ does not satisfy the wedge condition at the prime-end P corresponding to $\lambda \in \partial D$; then there exists a unique $\zeta \in \partial D$ such that $\lambda \in C(\phi, \zeta)$.*

In what follows, we assume that $f = u + iv$ satisfies $u, v \in h^p, p > 1$.

Let $[a, b]$, $a \neq b \in \mathbb{C} \cup \{\infty\}$, denote the line segment from a to b. We call $[a, b]$ a *side* of a domain H if no point of the interior, (a, b), of $[a, b]$ is a point of accumulation of $\partial H \backslash [a, b]$ and H lies on the left of $[a, b]$. For $\lambda_1, \lambda_2 \in \partial D$, let $\widehat{\lambda_1\lambda_2}$ be the circular arc described by going positively from λ_1 to λ_2.

Suppose that f is defined on an open subarc $\{e^{i\theta} : \theta_0 - \delta < \theta < \theta_0 + \delta\}$ of ∂D. We say that f is *constant on the right (left)* of $e^{i\theta_0}$ if $f(e^{i\theta})$ is constant for all θ, $\theta_0 - \delta < \theta < \theta_0$ ($\theta_0 < \theta < \theta_0 + \delta$).

The following theorem describes the cluster set of f at a point in E.

THEOREM 5. *Let $f = u + iv$, $u, v \in h^p$, $p > 1$, $\zeta_0 = e^{i\theta_0} \in E$ and $C(\phi, \zeta_0) = \widehat{\lambda_1\lambda_2}$. Then*

$$C(f, \zeta_0) = I_1 \cup [a, b] \cup I_2,$$

where $[a, b]$ is a side of $\partial f(D)$, and $I_1 = a$ ($I_2 = b$) if f is constant on the right (left) of ζ_0, otherwise $I_1 = I(P_1)$ ($I_2 = I(P_2)$), where P_1 are P_2 are the prime ends of $f(D)$ corresponding to λ_1 and λ_2, respectively. In this case, P_1 (P_2) is accessible at a (b), and $I(P_1)$ ($I(P_2)$) has at most one continuum of subsidiary points given by $C_R(F, \lambda_1)$ ($C_L(F, \lambda_2)$).

Proof. We show first that ϕ^{-1} has radial limits a.e. on $C(\phi, \zeta_0)$. Choose $\lambda_0 \in \text{Int } C(\phi, \zeta_0)$ so that $F(\lambda_0)$ exists, and let γ_0 be the radius ending at λ_0, and $\alpha_0 = \phi^{-1}(\gamma_0)$. We claim that α_0 is a Jordan half-interval ending at ζ_0. Let J denote the interval of accumulation points of α_0 on ∂D. Since $u, v \in h^p; p > 1$, J is a proper subinterval of ∂D; otherwise f is identically $F(\lambda_0)$, a contradiction. Pick $\zeta \in (\text{Int}(J)) \backslash \zeta_0$, and $\zeta' \in J$ so that $\phi(\zeta')$ exists and $\phi(\zeta') \neq \lambda_0$, and let α, α' be the radii ending at ζ, ζ', respectively. Then $\phi(\alpha) \cup \phi(\alpha')$ is a crosscut of

D dividing D into two simply connected domains the boundary of each of which contains a subarc of $C(\phi, \zeta_0)$ not shared by the other, a contradiction.

Now we choose $\lambda_1', \lambda_2' \in C(\phi, \zeta_0)$ so that $\widehat{\lambda_1' \lambda_2'}$ lies in $\widehat{\lambda_1 \lambda_2}$, and $F(\lambda_1')$, $F(\lambda_2')$ exist. For $j = 1, 2$, let γ_j be the radius ending at λ_j', $\beta_j = F(\gamma_j)$ and $\alpha_j = \phi^{-1}(\gamma_j)$. Then $\alpha_1 \cup \alpha_2$ is a loop which, except for ζ_0, lies in D, and bounds a simply connected subdomain, G, of D. If $W: D \to G$ is a conformal map satisfying $W(1) = \zeta_0$, and $H = f \circ W$, then H is a harmonic univalent map which equals $F(\lambda_1')$ on the right of 1 and $F(\lambda_2')$ on the left, and $\operatorname{Re} H$, $\operatorname{Im} H \in h^p$, $p > 1$, by Littlewood's subordination theorem [4, pp. 10–11]. Call Ω the domain bounded by $\beta_1 \cup \beta_2 \cup [F(\lambda_1'), F(\lambda_2')]$, and S the sector bounded by $\gamma_1 \cup \gamma_2 \cup \widehat{\lambda_1' \lambda_2'}$. Then $C(H, 1) = [F(\lambda_1'), F(\lambda_2')]$ by a theorem of Schwarz [9, p. 131], the cluster set of f at ζ within G is $[F(\lambda_1'), F(\lambda_2')]$, and F extends to a homeomorphism between $\overline{S}$ and $\overline{\Omega}$ mapping $\widehat{\lambda_1' \lambda_2'}$ to $[F(\lambda_1'), F(\lambda_2')]$. Letting $\lambda_1' \to \lambda_1$ and $\lambda_2' \to \lambda_2$, then $F(\lambda_1') \to a$ and $F(\lambda_2') \to b$, where $a, b \in \mathbb{C} \cup \{\infty\}$, and $C(f, \zeta_0)$ contains $[a, b]$ which obviously is a side of $f(D)$.

Now suppose that f is constant on the left (right) of ζ_0, then by replacing α_1 (α_2) by some circular arc $\{e^{i\theta}: \theta_0 - \delta \le \theta < \theta_0\}$ ($\{e^{i\theta}: \theta_0 < \theta \le \theta_0 + \delta\}$) and carrying the same argument above, we infer that $I_1 = a$ ($I_2 = b$).

Suppose to the contrary that f is not constant either on the right or on the left of ζ_0, then because $u, v \in h^p$, $p > 1$, there exist $\theta_n \theta_n', \theta_n \to \theta_0^-$, $\theta_n' \to \theta_0^+$ and $r_n \to 1^-$ such that if, for $n = 1, 2, \ldots$,

$$Q_n = (e^{i\theta_n}, r_n e^{i\theta_n}] \cup \{r_n e^{it}: \theta_n \le t \le \theta_n'\} \cup [r_n e^{i\theta_n'}, e^{i\theta_n'}),$$

then $\phi(Q_n)$ is a crosscut of D (see [8, p. 267]) which meets $\widehat{\lambda_1 \lambda_2}$ nowhere, and $\phi(Q_n) \to \widehat{\lambda_1 \lambda_2}$ ($n \to \infty$). Hence $I_1 = I(P_1)$ and $I_2 = I(P_2)$. It is also immediate that P_1 and P_2 are accessible at a and b, respectively, since F extends conformally to $\operatorname{Int}(\widehat{\lambda_1 \lambda_2})$ and maps $\operatorname{Int}(\widehat{\lambda_1 \lambda_2})$ to (a, b). Therefore the radial limits $F(\lambda_1)$, $F(\lambda_2)$ exist and they are a, b, respectively. Consequently, by Lindelöf's theorem [1, p. 72] $C_L(F, \lambda_1) = a$ and $C_R(F, \lambda_2) = b$, which forces each of P_1 and P_2 to have at most one continuum of subsidiary points given by $C_R(F, \lambda_1)$ and $C_L(F, \lambda_2)$, respectively, (see [1, pp. 188–189]). This completes the proof.
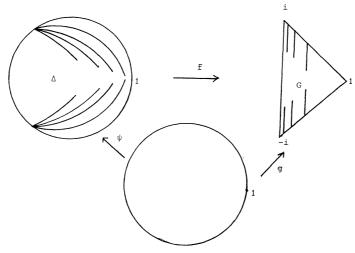
FIGURE 2

EXAMPLE 3. Let f be the function defined in Example 2. For $n = 2, 3, \ldots$, let Γ_n: $w(t) = \frac{1}{n} + it$ $(\frac{1}{2}(1 - \frac{1}{n}) \leq t \leq 1 - \frac{1}{n})$ and $\overline{\Gamma}_n$ be the reflection of Γ_n about the real axis. Then $G = f(D)\backslash(\bigcup_{n=2}^{\infty}(\Gamma_n \cup \overline{\Gamma}_n))$ is the simply connected domain shown in Figure 2. Let $\gamma_n = f^{-1}(\Gamma_n)$ and $\overline{\gamma}_n$ be the reflection of γ_n about the real axis, for all n. Because of the symmetry $\overline{f(z)} = f(\overline{z})$, it follows that $\overline{\gamma}_n = f^{-1}(\overline{\Gamma}_n)$. It is immediate that each of the arcs γ_n and $\overline{\gamma}_n$ is an asymptotic path, under f, in D meeting ∂D at $e^{i2\pi/3}$ and $e^{i4\pi/3}$, respectively, such that the sequences (γ_n) and $(\overline{\gamma}_n)$ converge uniformly to the circular arcs $\{e^{i\theta} : 0 \leq \theta \leq 2\pi/3\}$ and $\{e^{i\theta} : 4\pi/3 \leq \theta \leq 2\pi\}$, respectively. If

$$\Delta = D\backslash \left(\bigcup_{n=2}^{\infty} (\gamma_n \cup \overline{\gamma}_n) \right),$$

then Δ is a simply connected domain symmetric about the real axis and has a prime end, P, accessible at 1, with impression the arc $\{e^{i\theta} : -2\pi/3 \leq \theta \leq 2\pi/3\}$, see Figure 2. So if $\psi : D \to \Delta$ is the Riemann mapping satisfying $\psi(0) = 0$ and $\psi'(0) > 0$, then 1 corresponds, in the Carathéodory sense, to P under ψ. Furthermore, it can be easily verified that the harmonic univalent map $g = f \circ \psi$ has a nonempty set E such that $1 \in E$ and $C(g, 1)$ satisfies the conclusion of Theorem 5.

The next two corollaries follow from Theorem 5.

COROLLARY 5. *Under the assumptions of Theorem 5, if $f(D)$ has no sides, then $E = \varnothing$.*

COROLLARY 6. *Under the assumptions of Theorem 5, if P is a prime end of $f(D)$ which is either non-accessible, or accessible but has two subsidairy continua without the wedge condition, then there exists a unique $\zeta \in \partial D$, with $\zeta \in \partial D\backslash E$ such that $C(\phi, \zeta) = \lambda$, where λ is the corresponding point to P under F.*

In the final part of this section, we give conclusive statements of correspondence between various points of ∂D and the prime-ends of $f(D)$, where f is assumed univalent and in h^p, $p > 1$. First we need the following definition.

DEFINITION 3. Let A and B be subsets of ∂D (B corresponds to a set of prime ends of $f(D)$ under F). Then we say
   (a) *A corresponds to B* if $C(\phi, \zeta) \subset B$ and $C(\phi, \zeta') \subset \partial D\backslash B$ for all $\zeta \in A$ and $\zeta' \in \partial D\backslash A$.
   (b) *B corresponds to A* if $C(\phi^{-1}, \zeta) \subset A$ and $C(\phi^{-1}, \zeta') \subset \partial D\backslash A$ for all $\zeta \in B$ and $\zeta' \in \partial D\backslash B$.
   (c) *A and B correspond to each other* if (a) and (b).
   (d) There is *one to one correspondence* between A and B if every point of A corresponds to a single point of B and conversely.
   In accordance with this definition, if $\mathscr{A}$ is the set of all $\zeta \in \partial D\backslash E$ where f is constant neither on the left nor the right of ζ, and $\mathscr{B} = \{\hat{\phi}(\zeta): \zeta \in A\}$, then Theorem 3 and an argument of the proof of Theorem 5 yield the following:

THEOREM 6. *There is a one-to-one correspondence between $\mathscr{A}$ and $\mathscr{B}$.*

In view of this result, special attention need be given to the case when f is constant on some non-degenerate open subinterval of ∂D. Let $J = \widehat{\zeta_1 \zeta_2}$ be such a subinterval. Since Schwarz's Theorem [9, p. 131] offers an exact description of f at ζ_1 (ζ_2) if f is also constant on the right of ζ_1 (left of ζ_2), we restrict ourselves to the complementary case when f is constant neither on the right of ζ_1 nor the left of ζ_2.

THEOREM 7. *Suppose that $f(\zeta) \equiv w$ for all $\zeta \in \text{Int} \widehat{\zeta_1 \zeta_2}$ and f is constant neither on the right of ζ_1 nor the left of ζ_2. Then the following is true:*
   (a) *If $\zeta_1, \zeta_2 \in \partial D\backslash E$, then there is a prime end, λ, of $f(D)$ satisfying the wedge condition at w_0 such that $\widehat{\zeta_1 \zeta_2}$ and λ correspond to each other with $C(\phi, \zeta_1) = C_R(F, \lambda)$ and $C(\phi, \zeta_2) = C_L(F, \lambda)$ (see Figure 3.a).*
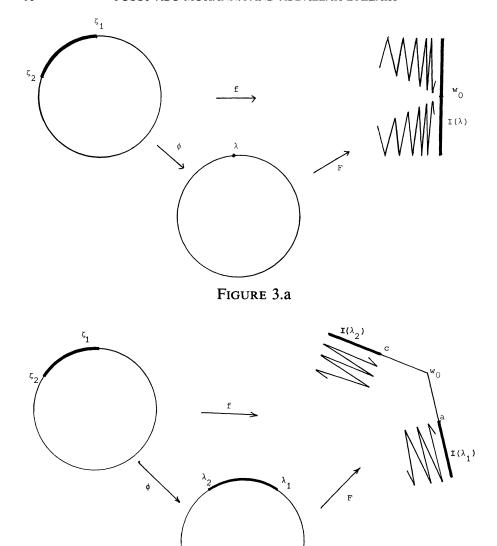
FIGURE 3.a

FIGURE 3.b

(b) *If* $\zeta_1, \zeta_2 \in E$, *then there is a non-degenerate interval* $\widehat{\lambda_1 \lambda_2}$ *of prime ends of* $f(D)$ *such that* $\widehat{\zeta_1 \zeta_2}$ *and* $\widehat{\lambda_1 \lambda_2}$ *correspond to each other. Both of* λ_1 *and* λ_2 *are (prime ends) accessible, say at* a *and* c, *respectively, and the impressions of all the prime ends in* $\widehat{\lambda_1 \lambda_2}$ *is* $I(\lambda_1) \cup [a, w_0] \cup [w_0, c] \cup I(\lambda_2)$. *(See Figure 3.b)*
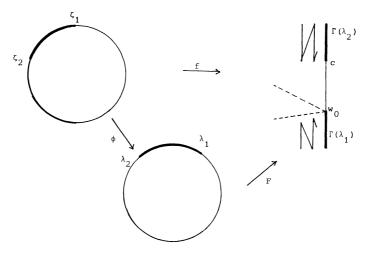
FIGURE 3.c

(c) *If $\zeta_1 \in \partial D \backslash E$ and $\zeta_2 \in E$, then there is a non-degenerate interval $\widehat{\lambda_1 \lambda_2}$ of prime ends of $f(D)$ such that $\widehat{\zeta_1 \zeta_2}$ and $\widehat{\lambda_1 \lambda_2}$ correspond to each other. Both of λ_1 and λ_2 are (prime ends) accessible, say at w_0 and c, respectively, and the former satisfy the wedge condition. Furthermore, the impressions on all the prime ends is $I(\lambda_1) \cup [w_0, c] \cup I(\lambda_2)$ (see Figure 3.c).*

The proof of this theorem follows immediately from Theorems 3, 4 and 5 and the fact that $f \in h^p$, $p > 1$.

REMARK. Wherever and throughout the paper, the assumption $f \in h^p$, $p > 1$ or equivalently $u, v \in h^p$, $p > 1$ can be replaced by the property that sets of positive measure on ∂D are sets of uniqueness for f or u, v.

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KUWAIT UNIVERSITY
P.O. BOX 5969
13060-SAFAT-KUWAIT