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KAPLANSKY'S THEOREM AND BANACH PI-ALGEBRAS

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By the theorem of Kaplansky a bounded operator in a Banach space is algebraic if and only if it is locally algebraic. We prove a generalization of this theorem. As a corollary we obtain the analogous result for finite (or countable) families of operators. Further we prove that a Banach algebra is PI (i.e. it satisfies a polynomial identity) if and only if it is locally PI.

Let T be a bounded operator on a Banach space X. The classical theorem of Kaplansky [**5**] states that T is algebraic (i.e. $p(T) = 0$ for some polynomial $p \neq 0$) if and only if it is locally algebraic (i.e. for every $x \in X$ there exists a non-zero polynomial p_x such that $p_x(T)x = 0$). In this paper we prove (Theorem 1) a generalized version of this theorem. As its corollaries it is possible to obtain the original theorem of Kaplansky, the theorem of Sinclair [**9**] and also new analogical results for finite or countable families of operators.

In the second part of the paper we deal with Banach PI-algebras (i.e. Banach algebras satisfying a polynomial identity). PI-rings and PI-algebras were studied intensely from the algebraic point of view, see e.g. [**4**], [**8**]. On the other hand Banach PI-algebras are much less known even though they form a very interesting class of Banach algebras. They are a natural generalization of commutative Banach algebras and it is possible to develop the complete analogy of the Gelfand theory, see [**6**].

In this paper we prove a theorem of Kaplansky's type for Banach PI-algebras. This result is closely related to earlier results of Grabiner [**2**] and Dixon [**1**].

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Let n be a positive integer. We denote by $\mathscr{P}^{(n)}$ the set of all complex polynomials in n non-commutative indeterminates i.e. the free algebra over $\mathbb{C}$ with n generators and with the unit element. Similarly we denote by $\mathscr{P}^{(\infty)} = \bigcup_{n=1}^{\infty} \mathscr{P}^{(n)}$ the set of all complex polynomials with countably many indeterminates.

Let X and Y be Banach spaces. Then $B(X,Y)$ denotes the set of all bounded operators from X to Y; we write shortly $B(X)$ instead of $B(X,X)$.

Let X be a Banach space, $1 \le n < \infty$ and let $T_1,\dots,T_n \in B(X)$. We say that the n-tuple $(T_1,\dots,T_n)$ is algebraic if $p(T_1,\dots,T_n) = 0$ for some $p \in \mathscr{P}^{(n)}$, $p \neq 0$. We say that $(T_1,\dots,T_n)$ is locally algebraic if, for every $x \in X$, there exists a non-zero polynomial $p_x \in \mathscr{P}^{(n)}$ such that $p_x(T_1,\dots,T_n)x = 0$.

These definitions can be used also for an infinite sequence $\{T_i\}_{i=1}^\infty$ of bounded operators on X (for $p \in \mathscr{P}^{(n)} \subset \mathscr{P}^{(\infty)}$ we have $p(T_1,T_2,\dots) = p(T_1,\dots,T_n)$). Equivalently, the sequence $\{T_i\}_{i=1}^\infty$ is locally algebraic if, for every $x \in X$ there exist n and $0 \neq p \in \mathscr{P}^{(n)}$ such that $p(T_1,\dots,T_n)x = 0$.

We start with the following generalization of Kaplansky's theorem.

THEOREM 1. *Let M be a linear space of countable (infinite) dimension, let Y,Z be Banach spaces and let $R\colon M \to B(Y,Z)$ be a linear mapping with the property that for every $y \in Y$ there exists $m \in M$, $m \neq 0$ such that $R(m)y = 0$. Then there exists $m \in M$, $m \neq 0$ such that $R(m)$ is a finite-dimensional operator.*

Proof. Let $e_1,e_2,\dots$ be a basis in M. Put $M_0 = \{0\}$ and denote by M_k $(k = 1,2,\dots)$ the linear subspace of M spanned by the vectors $e_1,\dots,e_k$.

Let F be a finite-dimensional subspace of Z. For $j = 1,2,\dots$ denote by $Y_{F,j}$ the set of all $y \in Y$ for which there exists $m \in M_j$, $m \neq 0$, such that $R(m)y \in F$ and $R(m')y \notin F$ for every $m' \in M_{j-1}$, $m' \neq 0$. By the assumption $\bigcup_{j=1}^\infty Y_{F,j} = Y$ so there exists $k = k(F)$ such that $Y_{F,k}$ is of the second category and $Y_{F,l}$ is of the first category for every $l < k$. Fix a finite-dimensional subspace $F \subset Z$ with the property that

$$k = k(F) = \min_{\substack{G\subset Z\\ \dim G<\infty}} k(G).$$

We have $Y_{F,k} = \bigcup_{s=1}^\infty Y_{F,k}^{(s)}$ where

$$Y_{F,k}^{(s)} = \left\{ y \in Y_{F,k},\ \text{there exists } m = e_k + \sum_{i=1}^{k-1} \alpha_i e_i \in M_k \right.$$
$$\left. \text{such that } \sum_{i=1}^{k-1} |\alpha_i| \le s \text{ and } R(m)y \in F \right\}.$$

We prove that $Y_{F,k}^{(s)}$ is a closed set for every s. Let $y_j \in Y_{F,k}^{(s)}$ ($j = 1, 2, \dots$), $y_j \to y$. Then there exist elements $m_j \in M_k$, $m_j = e_k + \sum_{i=1}^{k-1} \alpha_{ji} e_i$ such that $\sum_{i=1}^{k-1} |\alpha_{ji}| \leq s$ and $R(m_j)y_j \in F$. Using the compactness argument it is possible to find a subsequence $\{y_{j_r}\}_{r=1}^{\infty}$ and a vector $m \in M_k$ such that $m_{j_r} \to m$ coordinate-wise and $R(m_{j_r}) \to R(m)$ in the norm topology. It is easy to show that

$$R(m)y = \lim_{r\to\infty} R(m_{j_r})y_{j_r} \in F;$$

hence $y \in Y_{F,k}^{(s)}$ and $Y_{F,k}^{(s)}$ is closed. Therefore there exists $w \in Y$, $r > 0$ and a positive integer s such that

$$\{y \in Y, \|y - w\| < r\} \subset Y_{F,k}^{(s)} \subset Y_{F,k}.$$

Let $a = e_k + \sum_{i=1}^{k-1} \alpha_i e_i$ be the element of M_k satisfying

$$R(a)w \in F. \tag{1}$$

Denote by $F' = F \vee \bigvee_{i=1}^{k} \{R(e_i)w\}$. Clearly $\dim F' \leq \dim F + k < \infty$. Put $V = Y_{F,k} - \bigcup_{l<k} Y_{F,l}$. It follows from the choice of the subspace F that V is of the second category. Let $v \in V$. Then $v \in Y_{F,k}$ and

$$R(b)v \in F \tag{2}$$

for some $b \in M_k$, $b = e_k + \sum_{i=1}^{k-1} \beta_i e_i$.

Further $w + \lambda v \in Y_{F,k}$ for some complex number $\lambda \neq 0$, i.e. there exists $c = e_k + \sum_{i=1}^{k-1} \gamma_i e_i \in M_k$ such that

$$R(c)(w + \lambda v) = R(c)w + \lambda R(c)v \in F. \tag{3}$$

This implies $R(c)v \in F'$ and together with (2) $R(c - b)v \in F'$ where $c - b = \sum_{i=1}^{k-1} (\gamma_i - \beta_i) e_i \in M_{k-1}$. Since $v \notin \bigcup_{l<k} Y_{F',l}$, we conclude $c - b = 0$, $c = b$.

By (2), (3) and (1) we have $R(c)v = R(b)v \in F$, $R(c)w \in F$ and $R(c - a)w \in F$, where $c - a \in M_{k-1}$. Since $w \notin \bigcup_{l<k} Y_{F,l}$ we conclude again that $c = a$, i.e. $R(a)v \in F$ for every $v \in V$. Thus $R(a)^{-1}F \supset V$ and $R(a)^{-1}F$ is a linear subspace of the second category in Y, therefore $R(a)^{-1}F = Y$, $R(a)Y \subset F$ and $R(a)$ is a finite dimensional operator.

REMARK. One is tempted to expect in Theorem 1 that there exists $m \in M$, $m \neq 0$, such that $R(m) = 0$. However, the following example shows that this is not true in general. Let $Y = Z$ be a separable

Hilbert space with an orthonormal basis $\{h_i\}_{i=1}^{\infty}$. Define operators $R(m)$, $m \in M$, by

$$R(e_1)h_1 = h_1, \quad R(e_1)h_j = 0 \qquad (j \geq 2),$$
$$R(e_2)h_1 = 0, \quad R(e_2)h_2 = h_1, \quad R(e_2)h_j = 0 \qquad (j \geq 3),$$
$$R(e_i)h_j = \delta_{ij}h_j \qquad (i \geq 3; \delta_{ij} \text{ means the Kronecker's symbol}).$$

It is easy to show that the conditions of Theorem 1 are satisfied and $R(m) \neq 0$ $(m \neq 0)$.

THEOREM 2. *Let X be a Banach space, $1 \leq n \leq \infty$. Let $T = \{T_i\}_{i=1}^{n}$ be a (finite or infinite) sequence of bounded operators on X. Then T is algebraic if and only if it is locally algebraic.*

Proof. Suppose T is locally algebraic. We prove that it is algebraic (the converse implication is trivial). Put $M = \mathscr{P}^{(n)}$, $Y = Z = X$. For $p \in \mathscr{P}^{(n)}$ put $R(p) = p(T)$. By Theorem 1 there exist a polynomial $p \in \mathscr{P}^{(n)}$, $p \neq 0$, such that $\dim p(T)X < \infty$. Hence $(q \circ p)(T) = 0$ where $q \in \mathscr{P}^{(1)}$ is the characteristic polynomial of the finite-dimensional operator $p(T)|_{p(T)X}$.

In **[9]**, the following generalization of the Kaplansky's theorem was proved: Let $T \in B(X)$ be a non-algebraic operator. Then there exists a sequence $x_1, x_2, \ldots$ of elements of X such that $\sum_{i=1}^{k} p_i(T)x_i \neq 0$ for every $k \geq 0$ and for every polynomial $p_1, \ldots, p_k \in \mathscr{P}^{(1)}$ not all of which are equal to 0.

This result can be extended to the case of more than one operator.

THEOREM 3. *Let X be a Banach space, $1 \leq n \leq \infty$. Let $T = \{T_i\}_{i=1}^{\infty}$ be a (finite or infinite) sequence of bounded operators on X which is not algebraic. Then there exist vectors $x_1, x_2, \ldots \in X$ such that $\sum_{i=1}^{k} p_i(T)x_i \neq 0$ for every k and for every polynomial $p_1, \ldots, p_k \in \mathscr{P}^{(n)}$ not all of which are equal to* 0.

Proof. Suppose on the contrary that for every sequence $x_1, x_2, \ldots$ of elements of X there exist k and polynomials $p_1, \ldots, p_k \in \mathscr{P}^{(n)}$, $(p_1, \ldots, p_k) \neq (0, \ldots, 0)$ such that $\sum_{i=1}^{k} p_i(T)x_i = 0$.

Let M be the linear space of all sequences $\{p_i\}_{i=1}^{\infty}$ of polynomials $p_i \in \mathscr{P}^{(n)}$ only a finite number of which are non-zero. Put $Z = X$ and

$$Y = \{\{x_i\}_{i=1}^{\infty}, x_i \in X \ (i = 1, 2, \ldots), \ \sup\{\|x_i\|, i = 1, 2, \ldots\} < \infty\}.$$

Then Y with the norm $\|\{x_i\}_{i=1}^{\infty}\| \sup\{\|x_i\|, i = 1, 2, \dots\}$ is a Banach space. For $p = \{p_i\}_{i=1}^{\infty} \in M$ and $y = \{x_i\}_{i=1}^{\infty} \in Y$ put $R(p)y = \sum_{i=1}^{\infty} p_i(T)x_i$ (in fact the sum is finite). By Theorem 1 there exist a finite-dimensional subspace $F \subset X$, a positive integer k and polynomials $p_1, \dots, p_k \in \mathscr{P}^{(n)}$, $(p_1, \dots, p_k) \neq (0, \dots, 0)$, such that

$$\sum_{i=1}^{k} p_i(T)x_i \in F \quad \text{for every } x_1, \dots, x_k \in X.$$

Choose $j \in \{1, \dots, k\}$ such that $p_j \neq 0$. Let $x \in X$ be arbitrary. If we put $x_j = x$, $x_i = 0$ $(i \neq j)$ then we get $p_j(T)x \in F$ for every $x \in X$, i.e. $p_j(T)$ is a finite-dimensional operator. The rest is the same as in the proof of Theorem 2.

REMARK. Theorem 1 unifies some of the results of Kaplansky's type (cf. problem of Halmos [**3**]). On the other hand there are some results of this type which do not fit into this frame (see e.g. [**10**] where bounded analytic functions are used instead of polynomials or "approximative" results of Kaplansky's type [**7**], [**11**]). Another example will be the result for Banach PI-algebras which we prove in the following section.

Let A be a Banach algebra with the unit (we shall always assume that a Banach algebra has a unit element although this assumption is not essential). We say that A is PI if there exist a positive integer n and a non-zero polynomial $p \in \mathscr{P}^{(n)}$ such that $p(a_1, \dots, a_n) = 0$ for every $a_1, \dots, a_n \in A$. We say that A is locally PI if for every sequence $\{a_i\}_{i=1}^{\infty}$ of elements of A there exist n and a non-zero polynomial $p \in \mathscr{P}^{(n)}$ such that $p(a_1, \dots, a_n) = 0$ (both n and p depend on the sequence $\{a_i\}_{i=1}^{\infty}$).

THEOREM 4. *Let A be a Banach algebra with the unit. Then A is PI if and only if A is locally PI.*

Proof. The implication PI $\Rightarrow$ locally PI is trivial. Suppose that A is locally PI. Denote by $\tilde{A}$

$$\tilde{A} = \{\{a_i\}_{i=1}^{\infty},\ a_i \in A,\ i = 1, 2, \dots, \sup\{\|a_i\|, i = 1, 2, \dots\} < \infty\}.$$

Then A with the norm $\|\{a_i\}_{i=1}^{\infty}\| = \sup\{\|a_i\|, i = 1, 2, \dots\}$ is a Banach space. Further $\tilde{A} = \bigcup_{n=1}^{\infty} \tilde{A}_n$ where

$$\tilde{A}_n = \{\{a_i\}_{i=1}^{\infty} \in \tilde{A},\ \text{there exists } p \in \mathscr{P}^{(n)},\ \deg p \leq n,\ n^{-1} \leq |p| \leq n,\ p(a_1, \dots, a_n) = 0\}$$

(we denote by $\deg p$ the degree of a polynomial p and $|p|$ denotes the sum of moduli of coefficients of p).

Since $\tilde{A}_n$ is a closed subset for every n, Baire's theorem implies that there exist a positive integer n, $\tilde{y} \in \tilde{A}$ and $r > 0$ such that

$$\{\tilde{a} \in \tilde{A}, \|\tilde{a} - \tilde{y}\| < r\} \subset \tilde{A}_n.$$

Let $\tilde{z} = \{z_i\}_{i=1}^{\infty} \in \tilde{A}_n$. Then $p(z_1, \dots, z_n) = 0$ for some $p \in \mathscr{P}^{(n)}$, $p \neq 0$, $\deg p \leq n$, i.e. the set

$$C = \{z_{i_1}, \dots, z_{i_k},\ 0 \leq k \leq n,\ i_1, \dots, i_k \in \{1, \dots, n\}\}$$

is linearly dependent and $\sum_{c \in C} \alpha_c c = 0$ where α_c denotes the coefficient of p standing at the term c. Therefore $\sum_{c \in C} \alpha_c (c z_{n+1} - z_{n+1} c) = 0$. Let $C = \{c_1, \dots, c_s\}$. Denote by

$$e_s(x_1, \dots, x_s) = \sum_{\sigma \in S_s} (-1)^{\operatorname{sign} \sigma} x_{\sigma(1)} \cdots x_\sigma(s)$$

the standard polynomial (the sum is taken over all permutations of the set $\{1, \dots, s\}$). Clearly,

$$e_s(c_1 z_{n+1} - z_{n+1} c_1, \dots, c_s z_{n+1} - z_{n_1} c_s) = 0,$$

i.e. there exists a non-zero polynomial $p_n \in \mathscr{P}^{(n+1)}$ such that $p_n(z_1, \dots, z_{n+1}) = 0$ for every sequence $\{z_i\}_{i=1}^{\infty} \in \tilde{A}_n$. Let $\tilde{a} = \{a_i\}_{i=1}^{\infty} \in \tilde{A}$ be arbitrary. Then $\tilde{y} + \lambda \tilde{a} \in \tilde{A}_n$ for all complex λ, $|\lambda| \|\tilde{a}\| < r$, i.e.

$$p_n(y_1 + \lambda a_1, \dots, y_{n+1} + \lambda a_{n+1}) = 0.$$

We can write

$$\begin{aligned} &p_n(y_1 + \lambda a_1, \dots, y_{n+1} + \lambda a_{n+1}) \\ &\quad = p_n(y_1, \dots, y_{n+1}) + \lambda q^{(1)}(y_1, \dots, y_{n+1}, a_1, \dots, a_{n+1}) \\ &\qquad + \cdots + \lambda^{\deg p_n - 1} q^{(\deg p_n - 1)}(y_1, \dots, y_{n+1}, a_1, \dots, a_{n+1}) \\ &\qquad + \lambda^{\deg p_n} p_n(a_1, \dots, a_{n+1}). \end{aligned}$$

Since this expression is equal to 0 for all λ such that $|\lambda| \|\tilde{a}\| < r$, we conclude that $p_n(a_1, \dots, a_{n+1}) = 0$ for every $(n+1)$-tuple $a_1, \dots, a_{n+1}$ of elements of A. Thus A is a PI-algebra.

REMARK. In [2], S. Grabiner proved that a nil Banach algebra (i.e. consisting of nilpotent elements) is nilpotent (i.e. $A^n = 0$ for some n). The previous theorem is closely related to this result.

An algebra A is called algebraic if every element $a \in A$ is algebraic, i.e. $p(a) = 0$ for some non-zero polynomial $p \in \mathscr{P}^{(1)}$. An algebra is called locally finite if every finite subset of A generates a finite-dimensional subalgebra.

Clearly, a locally finite algebra is algebraic.

As an easy corollary of the previous theorem we can obtain the following result of Dixon **[1]** that the converse implication is true for Banach algebras.

COROLLARY 5. *Let A be a Banach algebra with the unit. Then A is algebraic if and only if A is locally finite.*

Proof. If A is algebraic then A is locally PI and thus PI by Theorem 4. An algebraic PI-algebra is locally finite (see **[4]**, X/12, Theorem 1).

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