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HYPERBOLIC GEOMETRY IN k-CONVEX REGIONS

DIEGO MEJIA AND DAVID MINDA

A simply connected region Ω in the complex plane $\mathbb{C}$ with smooth boundary $\partial\Omega$ is called k-convex ($k > 0$) if $k(z, \partial\Omega) \geq k$ for all $z \in \Omega$, where $k(z, \partial\Omega)$ denotes the euclidean curvature of $\partial\Omega$ at the point z. A different definition is used when $\partial\Omega$ is not smooth. We present a study of the hyperbolic geometry of k-convex regions. In particular, we obtain sharp lower bounds for the density λ_Ω of the hyperbolic metric and sharp information about the euclidean curvature and center of curvature for a hyperbolic geodesic in a k-convex region. We give applications of these geometric results to the family $K(k, \alpha)$ of all conformal mappings f of the unit disk $\mathbb{D}$ onto a k-convex region and normalized by $f(0) = 0$ and $f'(0) = \alpha > 0$. These include precise distortion and covering theorems (the Bloch-Landau constant and the Koebe set) for the family $K(k, \alpha)$.

1. Introduction. We study the hyperbolic geometry of certain types of convex regions called k-convex regions. It should be emphasized that our approach is geometric rather than analytic. Our work is a continuation of that of Minda ([**11**], [**12**], [**13**], [**14**]). The paper [**11**] deals with the hyperbolic geometry of euclidean convex regions, while [**12**] treats the hyperbolic geometry of spherically convex regions. A reflection principle for the hyperbolic metric was established in [**13**]; this reflection principle leads to a criterion for hyperbolic convexity that was employed in [**14**] to give a more penetrating analysis of certain aspects of the hyperbolic geometry of both euclidean and spherically convex regions.

Roughly speaking, a region Ω in the complex plane $\mathbb{C}$ is k-convex ($k > 0$), provided $k(z, \partial\Omega) \geq k$ for all $z \in \partial\Omega$. Here $k(z, \partial\Omega)$ denotes the euclidean curvature of $\partial\Omega$ at the point z. Of course, this only makes sense if $\partial\Omega$ is a closed Jordan curve of class C^2. This condition is actually sufficient for a region to be k-convex, but not necessary. Precisely, a region Ω is (euclidean) k-convex if $|a - b| < 2/k$ for any pair of distinct points $a, b \in \Omega$ and the intersection of the two closed disks of radii $1/k$ that have a and b on their boundary lies in Ω. For example, an open disk of radius $1/k$ is k-convex as is the intersection of finitely many such disks. Thus, for a region to be k-convex it must possess a certain degree of roundness.

In his Ph.D. dissertation Mejia **[10]** investigated the hyperbolic geometry of k-convex regions. Actually, he did even more; he also studied the hyperbolic geometry of k-convex regions Ω when $\Omega \subset \mathbb{P}$ ($\mathbb{P}$ denotes the Riemann sphere) is k-convex relative to spherical geometry or when $\Omega \subset \mathbb{D}$ ($\mathbb{D}$ is the open unit disk) is k-convex relative to hyperbolic geometry on $\mathbb{D}$. Independently, Flinn and Osgood **[3]** introduced the notion of hyperbolic k-convexity for a region $\Omega \subset \mathbb{D}$ and studied some of its properties in the special cases $k = 1, 2$. In subsequent work we will treat the hyperbolic geometry of both spherically k-convex and hyperbolically k-convex regions. Most of the results of this paper extend to spherically k-convex regions by using the same methods of proof, but for hyperbolically k-convex regions some new tools are needed, especially in case $0 < k < 2$.

Now, we outline the basic results of the paper. In §2 we present a discussion of a few elementary euclidean geometrical properties of k-convex regions that are crucial to our study of the hyperbolic geometry of such regions. In §§3 and 5 we give two different sharp lower bounds for the density $\lambda_\Omega(z)$ of the hyperbolic metric of a k-convex region Ω in terms of the geometrical quantity $\delta_\Omega(z)$, the distance from z to $\partial\Omega$. In §4 we show that one of these lower bounds, namely, $\lambda_\Omega(z) \geq 1/\delta_\Omega(z)[2 - k\delta_\Omega(z)] \geq 0$, actually characterizes k-convex regions. As consequences of these lower bounds, we obtain various results for the family $K(k, \alpha)$ of all holomorphic functions f such that f is univalent in $\mathbb{D}$, normalized by $f(0) = 0$ and $f'(0) = \alpha > 0$ and $f(\mathbb{D})$ is k-convex. In particular, we establish a sharp distortion theorem for $|f'(z)|$ and determine the Bloch-Landau constant for the family $K(k, \alpha)$. The reflection principle for the hyperbolic metric is used to establish a criterion for hyperbolic convexity in k-convex regions in §6; this result is best possible for an open disk of radius $1/k$. Several applications of this criterion are given in §§7 and 8. We obtain sharp information about the euclidean curvature and center of curvature of a hyperbolic geodesic in a k-convex region Ω. The center of curvature must always lie in $\mathbb{C}\backslash\overline{\Omega}$, and a sharp lower bound is given for the distance from it to $\overline{\Omega}$. This leads to both a sharp upper bound on the modulus of the second coefficient of a function in $K(k, \alpha)$, together with all extremal functions, and an analytic characterization of the family $K(k, \alpha)$. For $k = 0$ and $\alpha = 1$ all of these results for $K(k, \alpha)$ become well-known facts for the family K of normalized convex univalent functions.

The family $K(k, \alpha)$ is a special instance of the family $CV(R_1, R_2)$, $0 < R_1 \leq R_2$, of convex functions of bounded type that was introduced and studied by Goodman ([4], [5], [6]). Roughly speaking, a normalized conformal mapping of $\mathbb{D}$ onto a region Ω belongs to $CV(R_1, R_2)$ if $1/R_2 \leq k(z, \partial\Omega) \leq 1/R_1$ for all $z \in \partial\Omega$. Our family $K(k, 1)$, with $k \leq 1$, corresponds to Goodman's class $CV(1/k, \infty)$. Some of our results for the family $K(k, \alpha)$, such as the determination of the Koebe domain, follow from Goodman's work. On the other hand, we determine the Bloch-Landau constant, a problem not treated by Goodman, and find the sharp upper bound on the second coefficient. Goodman was the first to consider the problem of determining bounds on the second coefficient and he obtained a first approximation to the sharp bound. In any case, all of our results are obtained as corollaries of theorems dealing with the hyperbolic geometry of k-convex regions, an approach quite different from that of Goodman who used only elementary methods.

Finally, we recall a few fundamental facts about the hyperbolic metric that we shall use without further comment. For more details the reader should consult ([1], [11], [12], [13], [14]). The density $\lambda_{\mathbb{D}}$ of the hyperbolic metric on $\mathbb{D}$ is $\lambda_{\mathbb{D}}(z) = 1/(1 - |z|^2)$. If Ω is a region in $\mathbb{C}$ such that $\mathbb{C}\backslash\Omega$ contains at least two points, then there is a holomorphic universal covering projection f of $\mathbb{D}$ onto Ω. If Ω is simply connected, then f is a conformal mapping. The density λ_Ω of the hyperbolic metric on Ω is determined from $\lambda_\Omega(f(z))|f'(z)| = \lambda_{\mathbb{D}}(z)$. It is independent of the choice of the covering projection. In particular, if $f(0) = a$, then $\lambda_\Omega(a) = 1/|f'(0)|$. For example, if $D = \{z \colon |z-a| < r\}$, then $\lambda_D(z) = r/(r^2 - |z - a|^2)$. The hyperbolic metric has constant Gaussian curvature -4, that is,

$$\kappa(z, \lambda_\Omega) = \frac{\Delta \log \lambda_\Omega(z)}{\lambda_\Omega(z)^2} = -4.$$

The reader should note that some authors call $2\lambda_\Omega$ the density of the hyperbolic metric; note that $2\lambda_\Omega$ has Gaussian curvature -1. We shall require two basic properties of the hyperbolic metric. The first is the monotonicity property: If $\Omega \subset \Delta$, then $\lambda_\Omega(z) \leq \lambda_\Delta(z)$ for all $z \in \Omega$ and equality holds at a point if and only if $\Omega = \Delta$. The other is the principle of the hyperbolic metric: If f is holomorphic in $\mathbb{D}$ and $f(\mathbb{D}) \subset \Omega$, then $\lambda_\Omega(f(z))|f'(z)| \leq \lambda_{\mathbb{D}}(z)$ and equality holds at a point if and only if f is a covering projection onto Ω. When Ω is simply connected, equality holds if and only if f is a conformal mapping onto Ω.

2. Geometric properties of k-convex regions. In this section we introduce a measure of the "roundness" of a convex set called k-convexity.

We establish a few basic geometric properties of k-convex regions that will be needed in the remainder of the paper. Most of these results are analogs of known facts for convex regions.

Suppose that $k > 0$, $a, b \in \mathbb{C}$ and $|a - b| < 2/k$. Then there are two distinct closed disks $\overline{D}_1$ and $\overline{D}_2$ of radius $1/k$ such that $a, b \in \partial\overline{D}_j$ ($j = 1, 2$). Let $E_k[a, b] = \overline{D}_1 \cap \overline{D}_2$. Note that the boundary of $E_k[a, b]$ consists of two closed circular arcs Γ_1 and Γ_2, each of radius $1/k$ and angular length strictly less than π. These arcs have constant euclidean curvature k. We also let $E_0[a, b] = [a, b]$, the closed line segment joining a and b, and for $|a - b| = 2/k$, $E_k[a, b]$ is the closed disk with center $(a + b)/2$ and radius $1/k$. Then for $0 \leq k' < k \leq 2/|a - b|$ we have $E_{k'}[a, b] \subset E_k[a, b]$. Note that $E_k[a, b]$ is foliated by the collection of all arcs of constant absolute euclidean curvature k' that connect a and b and have angular length less than π, where k' varies over the interval $[0, k]$.

Definition. Suppose $k \in [0, \infty)$. A region $\Omega \subset \mathbb{C}$ is called k-convex provided $|a - b| < 2/k$ for any pair of points $a, b \in \Omega$ and $E_k[a, b] \subset \Omega$.

Observe that 0-convex is the same as convex. Henceforth, we shall employ the term "k-convex" only when $k > 0$. We will always use "convex" in place of "0-convex". Also, if $0 \leq k' \leq k$ and Ω is k-convex, then Ω is simultaneously k'-convex. In particular, a k-convex region is always convex and so simply connected. For $k > 0$ it is elementary to see that an open disk of radius $1/k$ is k-convex, but is not k'-convex for any $k' > k$. Also, if $\Omega_1, \ldots, \Omega_n$ are k-convex, then $\bigcap \Omega_j$ is k-convex. Finally, if $\Omega_1 \subset \Omega_2 \subset \cdots$ is an increasing sequence of k-convex regions, then $\bigcup \Omega_j$ is k-convex.

The next result gives an important sufficient condition for a region to be k-convex. Later (Corollary 2 to Theorem 8) we shall see that any k-convex region can be expressed as the increasing union of regions of the type indicated in the following proposition.

Proposition 1. *Suppose that Ω is a simply connected region in $\mathbb{C}$ bounded by a closed Jordan curve $\partial\Omega$ that is of class C^2. If $k(c, \partial\Omega) \geq k > 0$ for all $c \in \partial\Omega$, then Ω is k-convex.*

Proof. The hypotheses imply that Ω is convex [**16**, p. 46]. First, we show that if $\overline{D}$ is any closed disk that is contained in $\overline{\Omega}$, then the radius of $\overline{D}$ is at most $1/k$. Suppose $\overline{D} = \{z\colon |z - a| \leq r\}$, where $a \in \Omega$, and $\delta = \delta_\Omega(a)$. Then $r \leq \delta$ so it suffices to show $\delta \leq 1/k$.

Suppose that $c \in \partial\Omega \cap \{z\colon |z-a| = \delta\}$. Since the circle $|z-a| = \delta$ lies inside of or on $\partial\Omega$ and these curves are tangent at c, it follows that $k \le k(c, \partial\Omega) \le 1/\delta$ [7, pp. 28–30].

Next, we show that $E_k[a,b]$ is contained in $\overline{\Omega}$. Consider any pair of distinct points $a, b \in \Omega$. Since Ω is convex, $[a,b] \subset \Omega$. Because Ω is open, there exists $k' > 0$ such that $E_{k'}[a,b] \subset \Omega$. Let $K = \sup\{k'\colon E_{k'}[a,b] \subset \Omega\}$. Note that $E_K[a,b]$ is contained in the closure of Ω. Since $k' < 2/|a-b|$ for all admissible k', we must have $K \le 2/|a-b|$. If $K = 2/|a-b|$, then the closed disk with center $(a+b)/2$ and radius $1/K$ lies in $\overline{\Omega}$. From the first part of the proof we obtain $1/K \le 1/k$, or $k \le K$. Then $E_k[a,b] \subset E_K[a,b]$, so $E_k[a,b]$ is contained in $\overline{\Omega}$. The other possibility is that $K < 2/|a-b|$. If Γ_1 and Γ_2 are the two circular arcs of radius $1/K$ that bound $E_K[a,b]$, then at least one of them must meet $\partial\Omega$. Without loss of generality assume that Γ_1 meets $\partial\Omega$ at the point c. Since Γ_1 lies inside of or on $\partial\Omega$ near c, it follows that $k \le k(c, \partial\Omega) \le K$ [7, pp. 28–30]. Thus, $E_k[a,b] \subset E_K[a,b]$, so in this case $E_k[a,b]$ is again contained in $\overline{\Omega}$.

Finally, we show that $E_k[a,b] \subset \Omega$. Let L be the straight line through a and b. Select distinct points $a', b' \in (\Omega \cap L)\backslash[a,b]$ such that $[a,b] \subset [a',b']$. Then $E_k[a',b']$ is contained in $\overline{\Omega}$. Since $E_k[a,b] \subset \operatorname{int} E_k[a',b']$, the proof is complete.

It is not difficult to show that the converse also holds, that is, if Ω is k-convex, then $k(z, \partial\Omega) \ge k$ for all $z \in \partial\Omega$, provided $\partial\Omega$ is a C^2-closed Jordan curve.

Recall that if Ω is convex, then for any $a \in \Omega$ and $c \in \partial\Omega$ the half-open segment $[a,c) \subset \Omega$. The next result gives a refinement of this fact for k-convex regions.

PROPOSITION 2. *Suppose Ω is a k-convex region. Then for any $a \in \Omega$ and $c \in \partial\Omega$, $E_k[a,c]\backslash\{c\} \subset \Omega$.*

Proof. Since Ω is convex, the half-open segment $[a,c) \subset \Omega$. Note that $|a-c| < 2/k$ since $|a-c| = 2/k$ would imply that there exist $a' \in \Omega$ near a and $b' \in \Omega$ near c with $|a'-b'| \ge 2/k$ which violates the definition of k-convexity. We first show that $\operatorname{int} E_k[a,c] \subset \Omega$. Take $c_n \in [a,c)$ with $c_n \to c$. Now, $c_n \in \Omega$, so $E_k[a,c_n] \subset \Omega$ for all n. Consequently, $\operatorname{int} E_k[a,c] \subset \bigcup E_k[a,c_n] \subset \Omega$. Now, we establish the full result. Select $a' \in \Omega$ such that $a \in (a',c)$. The first part of the proof gives $\operatorname{int} E_k[a',c] \subset \Omega$. Because $E_k[a,c]\backslash\{c\} \subset \operatorname{int} E_k[a',c]$, the proof is complete.

COROLLARY. *Suppose Ω is a k-convex region. If $c, d \in \partial\Omega$, then* $\operatorname{int} E_k[c,d] \subset \Omega$.

Proof. Select $d_n \in \Omega$ with $d_n \to d$. Then$E_k[c, d_n]\backslash\{c\} \subset \Omega$ for all n. This yields $E_k[c,d] \subset \overline{\Omega}$, so $\operatorname{int} E_k[c,d] \subset \Omega$.

LEMMA 0. *Suppose D is an open disk of radius $1/k$, B is an open disk or half-plane such that $c \in \partial B \cap \partial D$ and B and D are externally tangent at c. If $|a-c| < 2/k$ and $a \notin \overline{D}$, then $E_k[a,c]\backslash\{c\} \cap B \neq \varnothing$.*

Proof. Let Γ_1 and Γ_2 be the two circular arcs of radius $1/k$ that bound $E_k[a,c]$, then, except for the point c, one of these two arcs lies entirely outside of $\overline{D}$. Suppose that Γ_1 is this arc. Since Γ_1 cannot be tangent to ∂D at c, it must intersect any disk that is externally tangent to D at c.

PROPOSITION 3. *Suppose Ω is a k-convex region. Assume $a \in \Omega$, $c \in \partial\Omega$ and $|a-c| = \delta_\Omega(a)$. If D is the open disk of radius $1/k$ that is tangent to the circle $|z-a| = \delta_\Omega(a)$ at c and that contains a in its interior, then $\Omega \subset D$.*

Proof. Let L be the straight line that is tangent to ∂D at c and let H be the open half-plane determined by L that does not contain a. Then Ω convex implies $\Omega \cap H = \varnothing$. If $\Omega\backslash D \neq \varnothing$, then there exists $a \in \Omega\backslash D$. Because Ω is open, we may even assume $a \in \Omega\backslash\overline{D}$. Note that $|a-c| < 2/k$ since Ω is k-convex. Now, Proposition 2 implies $E_k[a,c]\backslash\{c\} \subset \Omega$, while Lemma 0 gives $E_k[a,c]\backslash\{c\} \cap H \neq \varnothing$. These results contradict $\Omega \cap H = \varnothing$. Therefore, $\Omega \subset D$.

REMARK. If Ω is convex, $a \in \Omega$, $c \in \partial\Omega$, $|a-c| = \delta_\Omega(\delta)$, L is the line tangent to the circle $|z-a| = \delta_\Omega(a)$ and H' is the half-plane containing a, then $\Omega \subset H'$ and L is a support line for Ω at c. Proposition 3 gives an analog of this for k-convex regions: $\Omega \subset D$ and the circle ∂D of radius $1/k$ is a support line of constant euclidean curvature k for Ω at c.

PROPOSITION 4. *Suppose Ω is k-convex. Assume $a \in \mathbb{C}\backslash\Omega$, $c \in \partial\Omega$ and $|a-c| = \delta_\Omega(a)$. If D is the open disk of radius $1/k$ that is tangent to the circle $|z-a| = \delta_\Omega(a)$ at c and that does not meet the open segment (a,c), then $\Omega \subset D$.*

Proof. Let $B = \{z \colon |z-a| < \delta_\Omega(a)\} \subset \mathbb{C}\backslash\overline{\Omega}$. Then B and D are externally tangent at c and $B \cap \Omega = \varnothing$. If $\Omega\backslash D \neq \varnothing$, then there exists

$a \in \Omega \backslash \overline{D}$ and $|a-c| < 2/k$ since Ω is k-convex. Lemma 0 implies that $E_k[a,c] \backslash \{c\} \cap B \neq \varnothing$ and Proposition 2 gives $E_k[a,c] \backslash \{c\} \subset \Omega$. These facts contradict $B \cap \Omega = \varnothing$.

3. Lower bound for the density of the hyperbolic metric in a k-convex region. We obtain a sharp lower bound for the density of the hyperbolic metric in a k-convex region. This bound is used to obtain a precise distortion theorem and a covering theorem for the family of conformal mappings of the open unit disk $\mathbb{D}$ onto k-convex regions. These results are generally refinements of known facts for convex regions **[11]**.

THEOREM 1. *Suppose Ω is a k-convex region. Then for $z \in \Omega$*

$$\lambda_\Omega(z) \geq \frac{1}{\delta_\Omega(z)[2-k\delta_\Omega(z)]}$$

with equality at a point if and only if Ω is a disk of radius $1/k$.

Proof. First, assume that D is the open disk with center a and radius $1/k$. Then

$$\lambda_D(z) = \frac{1/k}{(1/k)^2 - |z-a|^2}.$$

For $z \in D, \delta_D(z) = (1/k) - |z-a|$ so that

$$\lambda_D(z) = \frac{1}{\delta_D(z)[1+k|z-a|]} = \frac{1}{\delta_D(z)[2-k\delta_D(z)]}.$$

Thus, equality holds for all points in the disk D.

Next, consider any k-convex region Ω. Fix $a \in \Omega$. Choose $c \in \partial\Omega$ with $|a-c| = \delta_\Omega(a)$. Let D be the disk of radius $1/k$ that is tangent to the circle $|z-a| = \delta_\Omega(a)$ at c and contains a in its interior. Proposition 3 gives $\Omega \subset D$. The monotonicity property of the hyperbolic metric yields $\lambda_\Omega(a) \geq \lambda_D(a)$ with equality if and only if $\Omega = D$. Since $\delta_\Omega(a) = \delta_D(a)$, the preceding inequality together with the work in the above paragraph completes the proof.

COROLLARY 1. *Suppose Ω is a k-convex region. If f is holomorphic in $\mathbb{D}$ and $f(\mathbb{D}) \subset \Omega$, then for $z \in \mathbb{D}$*

$$(1-|z|^2)|f'(z)| \leq \delta_\Omega(f(z))[2-k\delta_\Omega(f(z))].$$

Equality holds at a point if and only if Ω is a disk of radius $1/k$ and f is a conformal mapping of $\mathbb{D}$ onto Ω.

Proof. The principle of the hyperbolic metric gives

$$\lambda_\Omega(f(z))|f'(z)| \le \lambda_{\mathbb{D}}(z) = \frac{1}{1-|z|^2}$$

for $z \in \mathbb{D}$ with equality if and only if f is a conformal mapping of $\mathbb{D}$ onto Ω. The theorem yields

$$\frac{1}{\delta_\Omega(f(z))[2 - k\delta_\Omega(f(z))]} \le \lambda_\Omega(f(z))$$

with equality if and only if Ω is a disk of radius $1/k$. By combining the two preceding inequalities and the necessary and sufficient conditions for equality, we obtain the desired result.

DEFINITION. Let $K(k, \alpha)$ denote the family of all holomorphic functions f defined on $\mathbb{D}$ such that f is univalent, $f(0) = 0$, $f'(0) = \alpha > 0$ and $f(\mathbb{D})$ is a k-convex region.

The preceding corollary applied to $f \in K(k, \alpha)$ with $\Omega = f(\mathbb{D})$ and $z = 0$ yields $\alpha = |f'(0)| \le \delta_\Omega(0)[2 - k\delta_\Omega(0)] = h(\delta_\Omega(0))$, where $h(t) = t(2 - kt)$. Because h is increasing on $[0, 1/k]$ and $\delta_\Omega(0) \le 1/k$ since Ω is k-convex, we obtain $\alpha \le h(1/k) = 1/k$ whenever $f \in K(k, \alpha)$. Also, $\alpha = 1/k$ if and only if $f(z) = \alpha z$.

EXAMPLE. Set $f_k(z) = \alpha z/(1 - \sqrt{1 - \alpha k}z)$. Then $f_k \in K(k, \alpha)$ since $f(\mathbb{D})$ is a disk of radius $1/k$. In fact, $f_k(-1) = -\alpha/(1 + \sqrt{1 - \alpha k})$ and $f_k(1) = \alpha/(1 - \sqrt{1 - \alpha k})$, so $f_k(\mathbb{D})$ is the disk with center $\sqrt{1 - \alpha k}/k$ and radius $1/k$. Also, the largest disk contained in $f_k(\mathbb{D})$ and centered at the origin has radius $\alpha/(1 + \sqrt{1 - \alpha k})$. This example was considered by Goodman [**4**].

As an easy consequence, we obtain the Koebe domain for the family $K(k, \alpha)$. This result was first obtained by Goodman [**4**, Theorem 3].

COROLLARY 2. *Suppose $f \in K(k, \alpha)$. Then either the closure of the disk $\{w : |w| < \alpha/(1 + \sqrt{1 - \alpha k})\}$ is contained in $f(\mathbb{D})$ or else $f(z) = e^{-i\theta} f_k(e^{i\theta} z)$ for some $\theta \in \mathbb{R}$.*

Proof. Set $\Omega = f(\mathbb{D})$. Then the preceding corollary with $z = 0$ produces $\alpha = |f'(0)| \le \delta_\Omega(0)[2 - k\delta_\Omega(0)]$. This inequality implies that $\delta_\Omega(0) \ge \alpha/(1 + \sqrt{1 - \alpha k})$. Therefore, either $\delta_\Omega(0) > \alpha/(1 + \sqrt{1 - \alpha k})$ or else f is a conformal mapping of $\mathbb{D}$ onto a disk of radius $1/k$ that contains the origin and is tangent to $\{w : |w| = \alpha/(1 + \sqrt{1 - \alpha k})\}$. In the second it is elementary to verify that f must have the form prescribed in the statement of the corollary.

4. A characterization of convex and k-convex regions. We show that the inequality for the density of the hyperbolic metric in Theorem 1 actually characterizes k-convex regions. But first we establish the analogous result for convex regions; we will need this in our proof of the characterization of k-convex regions. The characterization of convex regions was demonstrated by Hilditch **[8]** but not published. Here we present a simple proof that is based on a result of Keogh **[9]**.

THEOREM 2. *Suppose Ω is a hyperbolic region in $\mathbb{C}$. Then Ω is convex if and only if $\lambda_\Omega(z) \geq 1/2\delta_\Omega(z)$ for $z \in \Omega$.*

Proof. Minda **[11]** showed the necessity. Now, we establish the sufficiency. Let $f(z) = a_0 + a_1 z + \cdots$ be holomorphic in $\mathbb{D}$ with $f(\mathbb{D}) \subset \Omega$. Set $\sigma_1(z) = a_0 + (a_1/2)z$. The principle of the hyperbolic metric gives $\lambda_\Omega(f(0))|f'(0)| \leq \lambda_{\mathbb{D}}(0) = 1$, or $|a_1| \leq 1/\lambda_\Omega(a_0)$. Then for $z \in \mathbb{D}$ we have

$$|\sigma_1(z) - a_0| = \frac{|a_1 z|}{2} < \frac{|a_1|}{2} \leq \frac{1}{2\lambda_\Omega(a_0)} \leq \delta_\Omega(a_0).$$

Thus, $\sigma_1(\mathbb{D}) \subset \{w\colon |w - a_0| < \delta_\Omega(a_0)\} \subset \Omega$. It follows **[9**, Theorem 1] that Ω is convex.

THEOREM 3. *Suppose Ω is a hyperbolic region in $\mathbb{C}$. Then Ω is k-convex if and only if $\lambda_\Omega(z) \geq 1/\delta_\Omega(z)[2 - k\delta_\Omega(z)] \geq 0$ for $z \in \Omega$.*

Proof. We need only establish the sufficiency. The proof is given in a series of steps.

First, we show that $\delta_\Omega(a) \leq 1/k$ for $a \in \Omega$ with equality only if Ω is a disk of radius $1/k$ and center a. Fix $a \in \Omega$ and set $\delta = \delta_\Omega(a)$, $D = \{z\colon |z - a| < \delta\}$. Then $D \subset \Omega$ so the monotonicity property of the hyperbolic metric in conjunction with the hypothesis gives

$$0 \leq \frac{1}{\delta[2 - k\delta]} \leq \lambda_\Omega(a) \leq \lambda_D(a) = \frac{1}{\delta}.$$

Hence, $2 - k\delta \geq 1$, or $\delta \leq 1/k$. If equality holds, then $\lambda_\Omega(a) = \lambda_D(a)$ which implies $\Omega = D$. The inequality $\delta_\Omega(z) \leq 1/k$ for $z \in \Omega$ implies that $\lambda_\Omega(z) \geq 1/2\delta_\Omega(z)$. Therefore, Ω is convex by Theorem 2.

Second, we prove that if there exist $a', b' \in \Omega$ with $|a' - b'| \geq 2/k$, then there exist $a, b \in \Omega$ with $|a - b| < 2/k$ and $E_k[a, b] \not\subset \Omega$. Let m be the midpoint of the straight line segment $[a', b']$; $m \in \Omega$ because Ω is convex. Since $a', b' \in \Omega$ and $|a' - b'| \geq 2/k$, Ω cannot be a disk of radius $1/k$. Therefore, the preceding paragraph implies $\delta_\Omega(m) < 1/k$.

Select $c \in \partial\Omega$ with $|m-c| = \delta_\Omega(m)$. Then $c \in \{z\colon \delta_\Omega(m) \le |z-m| < 1/k\}$. By selecting a, b on $[a', b'] \cap \{z\colon |z-m| < 1/k\}$ but close to the two points in which $[a', b']$ meets the circle $|z-m| = 1/k$, we can insure that $|a-b| < 2/k$ and $c \in E_k[a,b]$ so that $E_k[a,b] \not\subset \Omega$.

Finally, we show that Ω must be k-convex. Suppose Ω were not k-convex. Then the definition of k-convexity implies that either there exist points $a, b \in \Omega$ with $|a-b| \ge 2/k$ or points $a, b \in \Omega$ with $|a-b| < 2/k$ but $E_k[a,b] \not\subset \Omega$. The foregoing paragraph shows that the first alternative actually implies the second, so we need only show that this second possibility cannot occur. Since Ω is open, we can even assume that $a, b \in \Omega$, $|a-b| < 2/k$ but $\operatorname{int} E_k[a,b] \not\subset \Omega$. Because Ω is convex and open, there exists $k' > 0$ with $E_{k'}[a,b] \subset \Omega$. Let $K = \sup\{k'\colon E_{k'}[a,b] \subset \Omega\}$. Clearly, $K \le k$ and $E_K[a,b]$ is contained in the closure of Ω. But $\operatorname{int} E_k[a,b] \not\subset \Omega$, so we must actually have $K < k$. Let Γ_1 and Γ_2 be the two closed circular arcs of radius $1/K$ that bound $E_K[a,b]$. One of these arcs must meet $\partial\Omega$, say there is a point $c \in \Gamma_1 \cap \partial\Omega$. Now select α and β on $\Gamma_1 \backslash \{c\}$ so that they are symmetric about the normal line to Γ_1 at c. By performing a euclidean motion, if necessary, we may assume that $c = iM$, $M > 0$, $\alpha = -R$ and $\beta = R > 0$. Note that both λ_Ω and δ_Ω are invariant under euclidean motions. Then $E_K[-R, R]$ is contained in the closure of Ω and $iM \in \partial\Omega$. For $\varphi = \arctan(M/R)$ the function

$$f(z) = R\tanh[(2\varphi/\pi)\operatorname{Log}(1+z)/(1-z)]$$

is a conformal mapping of $\mathbb{D}$ onto $\operatorname{int} E_K[-R, R]$. The density of the hyperbolic metric for $E = \operatorname{int} E_K[-R, R]$ is [**12**, p. 133]

$$\lambda_E(z) = \frac{\pi R}{4\varphi|R^2 - z^2|\cos\left[\frac{\pi}{4\varphi}\arctan\frac{2R\operatorname{Im}(z)}{R^2 - |z|^2}\right]}.$$

Let $z = i(M-\delta)$, $0 < \delta \le M$. Obviously, $\delta_\Omega(z) = \delta$. Set $g(\delta) = \delta(2-k\delta)\lambda_E(i(M-\delta))$. Elementary computations show that $g(0) = 1$ and

$$g'(0) = \frac{M}{R^2+M^2} - \frac{k}{2}.$$

But

$$R^2 + M^2 = \frac{2M - M^2K}{K} + M^2$$

so that $g'(0) = (K-k)/2 < 0$. Hence, $g(\delta) = 1 + (K-k)\delta/2 + O(\delta^2) < 1$ for δ small and positive. The monotonicity property of the hyperbolic metric gives

$$\delta(2-k\delta)\lambda_\Omega(i(M-\delta)) \le \delta(2-k\delta)\lambda_E(i(M-\delta)) = g(\delta) < 1$$

for δ small and positive. But this contradicts our hypothesis that $\lambda_\Omega(z) \geq 1/\delta_\Omega(z)[2-k\delta_\Omega(z)]$ for $z \in \Omega$. Consequently, Ω must actually be k-convex.

5. The Bloch-Landau constant for the family $K(k, \alpha)$. We establish a sharp constant lower bound for the density of the hyperbolic metric of a k-convex region in terms of the least upper bound for δ_Ω. This lower bound leads to an easy determination of the Bloch-Landau constant for the family $K(k, \alpha)$.

We start by introducing a region that plays the central role in this section. Fix $M \in (0, 1/k]$. Set $R = \sqrt{M(2-kM)/k}$; note that $R = 1/k$ when $M = 1/k$. The number R is determined so that the circle through $-R, iM$ and R has radius $1/k$. Let $E = E(M)$ denote the k-convex region $\operatorname{int} E_k[-R, R]$. Note that E contains the disk $\{z\colon |z| < M\}$, but no larger disk, and is contained in the disk $D = \{z\colon |z| < R\}$. Also, observe that $E(1/k) = D$. If 2φ is the acute angle that each of the boundary arcs of E makes with the real axis, then $\varphi = \arctan(M/R)$. Essentially this region E was employed in **[12]** except that in this reference the region E was normalized by $R = 1$. We shall use the results of **[12]** but for arbitrary R rather than just $R = 1$; the extension of the results of **[12]** to arbitrary R is elementary. The region E has the feature that circular arcs in E through $\pm R$ have constant hyperbolic distance from the hyperbolic geodesic $(-R, R)$ and from each other. For $z \in E$ let $\gamma_D(z)$ denote the hyperbolic distance, relative to D, from z to ∂E. Then the quotient λ_E/λ_D can be expressed in terms of γ_D; we restate the result here for the general R **[12**, pp. 133–135**]**.

LEMMA 1. *For $z \in E$*

$$\frac{\lambda_E(z)}{\lambda_D(z)} = \frac{\pi \cos[2 \arctan \tanh(\gamma_D(0) - \gamma_D(z))]}{4\varphi \cos\left[\frac{\pi}{2\varphi} \arctan \tanh(\gamma_D(0) - \gamma_D(z))\right]} \geq \frac{\pi}{4\varphi},$$

and equality holds if and only if $z \in (-R, R)$.

Next, we extend this inequality from E to certain "triangular" k-convex regions that contain $\{z\colon |z| < M\}$. Let $\mathscr{T} = \mathscr{T}(M)$ denote the family of k-convex regions that contain $\{z\colon |z| < M\}$ and are bounded by three distinct circular arcs each of radius $1/k$ and having the property that the full circles are tangent to $|z| = M$ and contain $\{z\colon |z| < M\}$ in their interior. Observe that if $\Delta \in \mathscr{T}$, then each of the three circular arcs bounding Δ meets the boundary of the disk D

in diametrically opposite points when extended. The following result is essentially Theorem 2 of **[12]**.

LEMMA 2. *If $\Delta \in \mathscr{T}$, then for $z \in \Delta$*

$$\lambda_\Delta(z) > \frac{\pi}{4\varphi}\lambda_D(z) \geq \frac{\pi}{4\varphi R}.$$

Now, we can establish the desired lower bound.

THEOREM 4. *Let Ω be a k-convex region and $M = \sup\{\delta_\Omega(z) \colon z \in \Omega\}$. Then for $z \in \Omega$*

$$\lambda_\Omega(z) \geq \frac{\pi}{4\varphi R} = \frac{\pi}{4}\sqrt{\frac{k}{M(2-kM)}}\frac{1}{\arctan\sqrt{kM/(2-kM)}}.$$

Equality holds at a point $a \in \Omega$ if and only if there is a euclidean motion T of $\mathbb{C}$ such that $\Omega = T(E)$ and $a = T(0)$.

Proof. The proof is an adaptation of that given for the analogous result for spherically convex regions [**12**, Theorem 3]. Since Ω is bounded, we actually have $M = \max\{\delta_\Omega(z) \colon z \in \Omega\}$. Also, $M \in (0, 1/k]$ since Ω is k-convex. Take $a \in \Omega$ with $\delta_\Omega(a) = M$.

First, suppose that $M = 1/k$. Then Ω is a disk of radius $1/k$ with center a (see the proof of Theorem 1), so $\Omega = T(E)$, where $T(z) = z + a$. Thus,

$$\lambda_\Omega(z) = \frac{1/k}{(1/k)^2 - |z-a|^2} \geq k = \lambda_\Omega(a),$$

with equality if and only if $z = a$. This establishes the theorem in the special case that $M = 1/k$.

Now, assume that $M \in (0, 1/k)$. Because both λ_Ω and δ_Ω are invariant under euclidean motions, we may assume that $a = 0$ without loss of generality. Let $I = \{z \colon |z| = M \text{ and } z \in \partial\Omega\}$; I is nonempty and closed. A result of Blaschke **[2]** implies that I cannot be contained in a closed subarc of the circle $|z| = M$ with angular length strictly less than π. We consider two cases.

The first is when I is contained in a closed subarc of angular length exactly π. Then I contains a pair of diametrically opposite points of the circle $|z| = M$. By rotating Ω about the origin, if necessary, we may assume that $\pm iM \in I$. Then Proposition 3 permits us to conclude that $\Omega \subset E$. The monotonicity property of the hyperbolic metric in conjunction with Lemma 1 implies that

$$\lambda_\Omega(z) \geq \lambda_E(z) \geq \frac{\pi}{4\varphi}\lambda_D(z) \geq \frac{\pi}{4\varphi}\lambda_D(0) = \frac{\pi}{4\varphi R}$$

with strict inequality between the extremes unless $z = 0$ and $\Omega = E$.

The second case occurs when I is not contained in any closed subarc of angular length π. Then there exist three points c_1, c_2 and c_3 in I that partition the circle $|z| = M$ into three subarcs each having angular length strictly less than π. Let D_j ($j = 1, 2, 3$) be the open disk of radius $1/k$ that is tangent to the circle $|z| = M$ at c_j and contains 0 in its interior. Proposition 3 implies that $\Omega \subset D_j$, so $\Omega \subset \bigcap D_j = \Delta$ and $\Delta \in \mathcal{T}$. Thus, Lemma 2 together with the monotonicity property of the hyperbolic metric gives

$$\lambda_\Omega(z) \geq \lambda_\Delta(z) > \frac{\pi}{4\varphi R}.$$

This completes the proof.

The function

$$g_k(t) = \sqrt{\frac{t(2-kt)}{k}}\arctan\sqrt{\frac{kt}{2-kt}}$$

is strictly increasing on $[0, 1/k]$ with maximum value $g_k(1/k) = \pi/4k$. Hence, for $\alpha \in [0, 1/k]$ the equation $g_k(t) = \alpha\pi/4$ has a unique solution $M(\alpha) \in [0, 1/k]$. More precisely, we should write $M_k(\alpha)$ in place of $M(\alpha)$, but we suppress the dependence on k. Note that $M(1/k) = 1/k$.

COROLLARY (*Bloch-Landau constant for $K(k,\alpha)$*). *Let $f \in K(k,\alpha)$. Then either $f(\mathbb{D})$ contains an open disk of radius strictly larger than $M(\alpha)$, or else $f(z) = e^{-i\theta}F(e^{i\theta}z)$ for some $\theta \in \mathbb{R}$, where*

$$F(z) = \sqrt{\frac{M(\alpha)(2-kM(\alpha))}{k}}\tanh\left(\frac{\alpha}{2}\sqrt{\frac{k}{M(\alpha)(2-kM(\alpha))}}\operatorname{Log}\frac{1+z}{1-z}\right)$$

belongs to $K(k,\alpha)$ and maps $\mathbb{D}$ conformally onto $E(M(\alpha))$.

Proof. Set $\Omega = f(\mathbb{D})$ and $M = \max\{\delta_\Omega(f(z)) \colon z \in \mathbb{D}\}$. If $M > M(\alpha)$, then we are done. Suppose that $M \leq M(\alpha)$. Then$g_k(M) \leq g_k(M(\alpha)) = \alpha\pi/4$. Now, $\lambda_\Omega(0) = 1/f'(0) = 1/\alpha$, so the theorem with

$z = f(0) = 0$ gives $1/\alpha \geq \pi/(4g_k(M))$, or $g_k(M) \geq \alpha\pi/4$. Hence, $g_k(M) = \alpha\pi/4$, which gives $M = M(\alpha)$. Thus, equality holds in the theorem at the origin, so Ω is just a rotation of $E(M(\alpha))$. Because $F \in K(k, \alpha)$ and maps $\mathbb{D}$ conformally onto $E(M(\alpha))$, it follows that $f(z) = e^{-i\theta}F(e^{i\theta}z)$ for some $\theta \in \mathbb{R}$.

Note that $g_k(t) \to t$ as $k \to 0$, so for a fixed value of α, $M(\alpha) \to \alpha\pi/4$ as $k \to 0$. Thus, for $\alpha = 1$ and $k = 0$ we obtain $M(1) = \pi/4$ which is the Bloch-Landau constant for the family K of normalized convex univalent functions. This result is due to Szegö **[17]**; also, see **[11]**.

6. Hyperbolic convexity in k-convex regions. Minda **[13]** proved that if $\Omega \neq \mathbb{C}$ is convex and $a \in \overline{\Omega}$, then $\Omega \cap \{z: |z-a| < r\}$ is hyperbolically convex as a subset of Ω for any $r > 0$. Also, if a is in the exterior of Ω, then this set need not be hyperbolically convex; this is readily seen to be true when Ω is a half-plane. Of course, this result also holds for k-convex regions. In this section we improve this result for k-convex regions. We show that a can actually lie in the exterior of Ω, provided there is a restriction on r.

EXAMPLE. Let $D = \{z: |z - b| < 1/k\}$. Suppose $a \in \mathbb{C}\backslash\overline{D}$, $\delta = \delta_D(a), r > 0$ and $\delta = kr^2/(1 + \sqrt{1 + k^2r^2})$. Let $\Gamma' = \partial D$ and $\Gamma = \{z: |z - a| = r\}$. Then $|a - b|^2 = (1/k + \delta)^2 = (1/k)^2 + r^2$, so the circles Γ and Γ' are orthogonal. Thus, $\Gamma \cap D$ is a hyperbolic geodesic in D and $D \cap \{z: |z-a| < r\}$ is a hyperbolic half-lane which is trivially hyperbolically convex as a subset of D. If $\delta > ks^2/(1 + \sqrt{1 + k^2s^2})$, then it is easy to see geometrically that $D \cap \{z: |z - a| < s\}$ is not hyperbolically convex in D.

THEOREM 5. *Suppose Ω is a k-convex region. Let $a \in \mathbb{C}\backslash\overline{\Omega}$, $\delta = \delta_\Omega(a)$, $r > 0, R = \{z: |z - a| < r\}$, $\Gamma = \partial R$ and j denote reflection in the circle Γ. If $\delta \leq kr^2/(1 + \sqrt{1 + k^2r^2})$, then $j(\Omega\backslash R) \subset \Omega$. In particular, $\Omega \cap \{z: |z - a| < r\}$ is hyperbolically convex as a subset of Ω.*

Proof. Select $c \in \partial\Omega$ with $|a - c| = \delta$. Let D be the open disk of radius $1/k$ that is tangent to the circle $|z - a| = \delta$ at c and does not meet the open segment (a, c). Suppose b is the center of D and $\Gamma' = \partial D$. Proposition 4 implies that $\Omega \subset D$.

First, we consider the case in which $\delta = kr^2/(1+\sqrt{1+k^2r^2})$. As in the preceding example, the circles Γ and Γ' are orthogonal. Consider $z \in \Omega\backslash R$ and let γ be the circular arc through c, z and $j(c)$. Note that $j(c) \in \Gamma'$ since Γ' perpendicular to Γ implies that $j(\Gamma') = \Gamma'$. In fact, c and $j(c)$ are diametrically opposite on the circle Γ'. Let d be the point in which γ meets Γ. Then it is clear that $j(\gamma) = \gamma$ since j fixes d and interchanges the points c and $j(c)$. The point d divides γ into two subarcs γ_1 and γ_2, with $\gamma_1 \subset \overline{R}$, $\gamma_2 \subset \mathbb{C}\backslash\overline{R}$ and $j(\gamma_2) = \gamma_1$. If s is the radius of γ, then $s > 1/k$ since γ is inside of D and passes through diametrically opposite points of $\Gamma' = \partial D$. Let $\gamma' \supset \gamma_1$ be the subarc of γ from z to c. Proposition 2 implies that $\gamma'\backslash\{c\} \subset \Omega$. Then $j(z) \in j(\gamma' \cap \gamma_2) \subset \gamma_1\backslash\{c\}$, so $j(z) \in \Omega$. This proves $j(\Omega\backslash R) \subset \Omega$. This inclusion implies that $\Omega \cap R$ is hyperbolically convex in Ω ([**13**, Theorem 6], [**14**, Proposition 1]).

The remaining case $\delta < kr^2/(1+\sqrt{1+k^2r^2})$ can be reduced to the preceding case as follows. Select $k_0 > 0$ so that

$$\delta = k_0r^2/(1+\sqrt{1+k_0^2r^2}).$$

Then $k > k_0$, so Ω is also k_0 convex. The first part of the proof applied with k_0 in place of k allows us to conclude that $j(\Omega\backslash R) \subset \Omega$ and that $\Omega \cap R$ is hyperbolically convex.

The example prior to the theorem shows that the inequality between δ and r in the theorem cannot be improved.

7. Applications to euclidean curvature. In [**14**, Theorem 2] Minda obtained precise information about the location of the center of the euclidean circle of curvature for a hyperbolic geodesic in a convex region. In particular, the center of curvature always lies in the complement of the region. Now, we determine a sharp relationship between the euclidean curvature and center of curvature for a hyperbolic geodesic in a k-convex region.

THEOREM 6. *Suppose that Ω is a k-convex region, γ is a hyperbolic geodesic in Ω, $z_0 \in \gamma$ and the euclidean curvature $k(z_0,\gamma)$ of γ at z_0 is nonvanishing. Let a denote the euclidean center of curvature and $r(z_0,\gamma) = 1/|k(z_0,\gamma)|$ the radius of curvature for γ at z_0. Then $a \in \mathbb{C}\backslash\overline{\Omega}$ and*

$$\delta_\Omega(a) \geq \frac{kr(z_0,\gamma)^2}{1+\sqrt{1+k^2r(z_0,\gamma)^2}}.$$

Equality holds if and only if Ω is an open disk of radius $1/k$ and γ is a circular arc orthogonal to $\partial\Omega$.

Proof. We start by showing that equality holds when Ω is a disk of radius $1/k$. In this situation the hyperbolic geodesics are the circular arcs and straight line segments that are orthogonal to the boundary. Suppose γ is a circular arc that is perpendicular to $\partial\Omega$. Let a be the center and r the radius of γ and $\delta = \delta_\Omega(a)$. The fact that γ is orthogonal to $\partial\Omega$ gives $(1/k)^2+r^2 = (\delta+1/k)^2$, or $\delta^2+(2/k)\delta-r^2 = 0$. The positive root of this quadratic equation is $\delta = kr^2/(1+\sqrt{1+k^2r^2})$. Thus, equality does hold for a disk of radius $1/k$.

Now, we establish the inequality in the general case of an arbitrary k-convex region Ω. There is no harm in assuming that γ is oriented so that $k(z_0,\gamma)$ is positive. From [**14**, Theorem 2] we know that $a \in \mathbb{C}\backslash\Omega$. Actually, we will show that if Γ_0 is any positively oriented circle through z_0 with radius r_0, center $a_0 \in \mathbb{C}\backslash\Omega$ and the same unit tangent as γ at z_0, then Γ_0 is not the circle of curvature for γ at z_0 provided $\delta_\Omega(a_0) < kr_0^2/(1+\sqrt{1+k^2r_0^2})$. Fix such a circle Γ_0. Because the preceding inequality is strict, we can choose a strictly larger circle Γ through z_0 with radius $r > r_0$, center $a \in \mathbb{C}\backslash\Omega$ and the same unit tangent as γ at z_0 such that $\delta_\Omega(a) < kr^2/(1+\sqrt{1+k^2r^2})$. Then Theorem 5 implies that $j(\Omega\backslash R) \subset \Omega$, where $R = \{z\colon |z-a| < r\}$ and j denotes reflection in Γ, so [**14**, Theorem 1] implies $k(z_0,\gamma) \le k(z_0,\Gamma)$. Since $k(z_0,\Gamma) < k(z_0,\Gamma_0)$, it follows that Γ_0 cannot be the circle of curvature for γ at z_0.

Finally, we determine the form of Ω when equality holds. Suppose that Γ is the circle of curvature for γ at z_0, has center a, radius r and $\delta_\Omega(a) = kr^2/(1+\sqrt{1+k^2r^2})$. Theorem 5 again implies that $j(\Omega\backslash R) \subset \Omega$, so that $k(z_0,\gamma) \le k(z_0,\Gamma)$ with equality if and only if Ω is symmetric about Γ and $\gamma \subset \Gamma$ [**14**, Theorem 1]. But $k(z_0,\gamma) = k(z_0,\Gamma)$ since Γ is the circle of curvature for γ at z_0, so Ω must be symmetric about Γ and $\gamma \subset \Gamma$. Select $c \in \partial\Omega$ with $|a-c| = \delta_\Omega(a)$. If D is the disk of radius $1/k$ that is tangent to the circle $|z-a| = \delta_\Omega(a)$ at c and does not meet the segment (a,c), then $\Omega \subset D$ by Proposition 4. The fact that equality holds implies that Γ is orthogonal to ∂D as in the first part of the proof. Because Ω is symmetric about Γ, $j(c) \in \partial\Omega$. Note that $j(c)$ is diametrically opposite c on ∂D, so $|c-j(c)| = 2/k$. But then the corollary to Proposition 2 implies that $D = \operatorname{int} E_k[c, j(c)] \subset \Omega$. Hence, $\Omega = D$, so Ω is a disk of radius $1/k$ when equality holds.

The inequality in Theorem 6 is equivalent to each of the following inequalities:

$$\delta_\Omega(a) \geq \frac{k}{|k(z_0,\gamma)|[\sqrt{k^2+k(z_0,\gamma)^2}+|k(z_0,\gamma)|]},$$

$$|k(z_0,\gamma)| \leq \sqrt{\frac{k}{k\delta_\Omega^2(a)+2\delta_\Omega(a)}}.$$

COROLLARY 1. *Suppose Ω is a k-convex region, γ is a hyperbolic geodesic in Ω and $z_0 \in \gamma$. Then*

$$|k(z_0,\gamma)| \leq \frac{2(1-k\delta_\Omega(z_0))}{\delta_\Omega(z_0)[2-k\delta_\Omega(z_0)]}.$$

Equality implies that Ω is a disk of radius $1/k$ and γ is a circular arc orthogonal to $\partial\Omega$.

Proof. We may assume that $k(z_0,\gamma) \neq 0$. Let a be the center and r the radius of the circle of curvature for γ at z_0. Set $\delta = \delta_\Omega(a)$. The segment $[a, z_0]$ meets $\partial\Omega$ in some point c, so $\delta_\Omega(z_0) \leq |z_0 - c| = |a - z_0| - |a - c| = r - \delta$. The theorem gives

$$r - \delta_\Omega(z_0) \geq \delta \geq \frac{kr^2}{1+\sqrt{1+k^2r^2}}.$$

We solve this inequality for $1/r$ by means of elementary manipulations and obtain

$$|k(z_0,\gamma)| = \frac{1}{r} \leq \frac{2(1-k\delta_\Omega(z_0))}{\delta_\Omega(z_0)[2-k\delta_\Omega(z_0)]}.$$

COROLLARY 2. *Let Ω be a k-convex region. Then for $z \in \Omega$*

(i) $$|\nabla \log\lambda_\Omega(z)| \leq \frac{2(1-k\delta_\Omega(z))}{\delta_\Omega(z)[2-k\delta_\Omega(z)]}$$

and

(ii) $$|\nabla \log\lambda_\Omega(z)| \leq 2\sqrt{\lambda_\Omega(z)[\lambda_\Omega(z)-k]}$$

with equality if and only if Ω is a disk of radius $1/k$.

Proof. First, we show that equality holds for the disk D with center 0 and radius $1/k$. Then $\lambda_D(z) = k/(1-k^2|z|^2)$, so that $\nabla\log\lambda_D(z) = 2k^2 z/(1-k^2|z|^2)$. Thus, $\delta_D = (1/k) - |z|$ gives

$$|\nabla\log\lambda_D(z)| = \frac{2|z|}{[(1/k)-|z|][(1/k)+|z|]} = \frac{2(1-k\delta_D(z))}{\delta_D(z)[2-k\delta_D(z)]}.$$

Similarly, it is elementary to establish equality in (ii) for D.

Next, we establish (i). Fix $z_0 \in \Omega$ and set $\nu = \nabla \log \lambda_\Omega(z_0)$. There is nothing to prove if $\nu = 0$, so we may assume $\nu \neq 0$. Set $n = \nu/|\nu|$ and let γ be a hyperbolic geodesic through z_0 with normal n at z_0. From [**15**, Formula 19] and Corollary 1 we have

$$|\nabla \log \lambda_\Omega(z_0)| = \frac{\partial \log \lambda_\Omega(z_0)}{\partial n} = k(z_0, \gamma) \le \frac{2(1 - k\delta_\Omega(z_0))}{\delta_\Omega(z_0)[2 - k\delta_\Omega(z_0)]}$$

and equality implies Ω is a disk of radius $1/k$.

Finally, we establish (ii). From Theorem 3 in conjunction with the fact that $\delta_\Omega(z) < 1/k$, we obtain

$$\frac{1 - \sqrt{1 - (k/\lambda_\Omega(z))}}{k} \le \delta_\Omega(z) \le \frac{1}{k}$$

with equality in the left-hand inequality if and only if Ω is a disk of radius $1/k$. Since $h(t) = 2(1 - kt)/t(2 - kt)$ is decreasing on $[0, 1/k]$, we obtain

$$h(\delta_\Omega(z)) \le h\left(\frac{1 - \sqrt{1 - (k/\lambda_\Omega(z))}}{k}\right) = 2\sqrt{\lambda_\Omega(z)(\lambda_\Omega(z) - k)}.$$

Thus, inequality (ii) follows from inequality (i).

8. Applications to k-convex mappings. We now establish some additional results for the family $K(k, \alpha)$. In particular, we obtain a sharp estimate for the second coefficient of a function in $K(k, \alpha)$ and an analytic characterization of the class $K(k, \alpha)$.

THEOREM 7. *Suppose Ω is a k-convex region and $f \colon \mathbb{D} \to \Omega$ is a conformal mapping. Then*

$$\frac{|f''(0)|}{|f'(0)|^2} \le \frac{2[1 - k\delta_\Omega(f(0))]}{\delta_\Omega(f(0))[2 - k\delta_\Omega(f(0))]} \le 2\sqrt{\lambda_\Omega(f(0))[\lambda_\Omega(f(0)) - k]}$$

and equality holds if and only if Ω is a disk of radius $1/k$.

Proof. From $\lambda_\Omega(f(z))|f'(z)| = \lambda_\mathbb{D}(z) = 1/(1 - |z|^2)$, we obtain

$$\frac{\partial}{\partial w}[\log \lambda_\Omega(f(z))]|f'(z)| + \frac{f''(z)}{2f'(z)} = \frac{\bar{z}}{1 - |z|^2}.$$

For $z = 0$ this gives

$$\frac{1}{2}|\nabla \log \lambda_\Omega(f(0))| = \left|\frac{\partial}{\partial w} \log \lambda_\Omega(f(0))\right| = \frac{|f''(0)|}{2|f'(0)|^2}.$$

The desired result now follows from Corollary 2 to Theorem 6.

COROLLARY. *If $f \in K(k,\alpha)$, then $|f''(0)| \le 2\alpha\sqrt{1-\alpha k}$. Equality holds if and only if $f(z) = e^{-i\theta} f_k(e^{i\theta} z)$ for some $\theta \in \mathbb{R}$.*

Proof. Set $\Omega = f(\mathbb{D})$. Then $\lambda_\Omega(0) = \lambda_\Omega(f(0)) = 1/f'(0) = 1/\alpha$. The theorem gives

$$|f''(0)| \le 2|f'(0)|^2\sqrt{\lambda_\Omega(0)[\lambda_\Omega(0) - k]} = 2\alpha\sqrt{1-\alpha k}$$

with equality if and only if Ω is a disk of radius $1/k$. The only functions which belong to $K(k,\alpha)$ and map onto a disk of radius $1/k$ have the form specified in the corollary.

It is well known that if K is the class of normalized convex univalent functions defined in $\mathbb{D}$, then a normalized holomorphic function belongs to K if and only if for $z \in \mathbb{D}$, $1+\mathrm{Re}(zf''(z)/f'(z)) > 0$. Moreover, whenever this inequality is true, then the stronger inequality

$$\left|\frac{f''(z)}{f'(z)} - \frac{2\bar{z}}{1-|z|^2}\right| \le \frac{2}{1-|z|^2}$$

also holds [1, p. 5]. We shall obtain analogs of these results for the family $K(k,\alpha)$.

THEOREM 8. *If $f \in K(k,\alpha)$, then $z \in \mathbb{D}$*

$$\left|\frac{f''(z)}{f'(z)} - \frac{2\bar{z}}{1-|z|^2}\right| \le \frac{2}{1-|z|^2}\sqrt{1-(1-|z|^2)|f'(z)|k}.$$

Equality holds at a point if and only if $f(z) = e^{-i\theta} f_k(e^{i\theta} z)$ for some $\theta \in \mathbb{R}$.

Proof. For $z = 0$ the inequality in the theorem becomes $|f''(0)| \le 2\alpha\sqrt{1-\alpha k}$. Thus, for $z = 0$ the theorem reduces to the corollary of Theorem 7. Next, we show that the general case can be reduced to this special situation. Consider any $f \in K(k,\alpha)$. Fix $a \in \mathbb{D}$ and define $g(z) = f((z+a)/(1+\bar{a}z)) - f(a)$. Then $g(0) = 0$ and $g(\mathbb{D})$ is a k-convex region. Also,

$$g'(0) = (1-|a|^2)f'(a)$$

and

$$g''(0) = (1-|a|^2)[f''(a)(1-|a|^2) - 2\bar{a}f'(a)].$$

Let $\varphi = -\operatorname{Arg} f'(a)$. Then $G = e^{i\varphi} g \in K(k, (1-|a|^2)|f'(a)|)$. The corollary of Theorem 7 applied to G gives

$$|G''(0)/G'(0)| \leq 2\sqrt{1 - kG'(0)}.$$

The formulas for $g'(0)$ and $g''(0)$ reveal that this inequality is equivalent to that stated in the theorem. All that remains is to determine when equality holds. If $f(z) = e^{-i\theta} f_k(e^{i\theta} z)$ for some $\theta \in \mathbb{R}$, then equality holds for $z = re^{-i\theta}$, $0 \leq r < 1$. On the other hand, if equality holds at some point of $\mathbb{D}$, then the proof shows that G, and hence f, maps $\mathbb{D}$ onto a disk of radius $1/k$. Therefore, f must have the form $e^{-i\theta} f_k(e^{i\theta} z)$ for some $\theta \in \mathbb{R}$.

COROLLARY 1. *Let f be holomorphic and univalent in $\mathbb{D}$ and normalized by $f(0) = 0$, $f'(0) = \alpha > 0$. Then $f \in K(k, \alpha)$ if and only if*

$$1 + \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) \geq k|zf'(z)|$$

for $z \in \mathbb{D}$.

Proof. Suppose $f \in K(k, \alpha)$. Then the inequality in Theorem 8 holds. If we multiply this inequality by $|z|$, square the resulting inequality and then simplify, we obtain

$$\frac{4|z|^4}{(1-|z|^2)^2} + \left|\frac{zf''(z)}{f'(z)}\right|^2 \leq \frac{4|z|^2}{1-|z|^2}\left[1 + \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) - k|f'(z)|\right].$$

This implies the desired result.

Conversely, assume that the inequality in the corollary holds. Consider the path $\gamma\colon z = z(t) = re^{it}$, $t \in [0, 2\pi]$. The euclidean curvature of the path $f \circ \gamma$ at the point $f(z)$, $z \in \gamma$, is given by

$$k(f(z), f \circ \gamma) = \frac{1 + \operatorname{Re}(zf''(z)/f'(z))}{|zf'(z)|}.$$

Consequently, $k(f(z), f \circ \gamma) \geq k$ for all $z \in \gamma$. Proposition 1 implies that $f(\{z : |z| < r\})$ is a k-convex region. Because $f(\mathbb{D})$ is an increasing union of k-convex regions, it is also k-convex.

COROLLARY 2. *Suppose f is holomorphic and univalent in $\mathbb{D}$. Then $f(\mathbb{D})$ is k-convex if and only if f maps each subdisk of $\mathbb{D}$ onto a k-convex region.*

Proof. First, suppose that f maps each subdisk of $\mathbb{D}$ onto a k-convex region. Then $f(\{z : |z| < r\})$ is a k-convex region for $0 < r < 1$, so

$f(\mathbb{D})$ is also k-convex. Conversely, assume that $f(\mathbb{D})$ is k-convex. There is no harm in supposing that $f(0) = 0$ and $\alpha = f'(0) > 0$, since k-convexity is invariant under translations and rotations. We begin by showing that $f(\{z: |z| < r\})$ is a k-convex region for $0 < r < 1$. Corollary 1 implies that if $\gamma: z = z(t) = re^{it}, t \in [0, 2\pi]$, then $k(f(z), f \circ \gamma) \geq k$ for all $z \in \gamma$, so Proposition 1 shows that $f(\{z: |z| < r\})$ is a k-convex region. Now, if Δ is a subdisk of $\mathbb{D}$ with $\overline{\Delta} \subset \mathbb{D}$, then there is a conformal automorphism T of $\mathbb{D}$ such that $T(\Delta)$ is a disk centered at the origin. Now $g = f \circ T^{-1}$ maps $\mathbb{D}$ onto a k-convex region, so the first case shows that $f(\Delta) = g(T(\Delta))$ is k-convex. Finally, if Δ is a subdisk $\mathbb{D}$ of that is tangent to the unit circle, then there is an increasing sequence $\{\Delta_n\}$ of subdisks of $\mathbb{D}$ with $\overline{\Delta}_n \subset \Delta \subset \mathbb{D}$ for all n and $\Delta = \bigcup \Delta_n$. Since $f(\Delta_n)$ is k-convex, it follows that $f(\Delta) = \bigcup f(\Delta_n)$ is also k-convex.

Added in proof. Professor Wolfram Koepf has pointed out that some of the results of this paper as well as similar results for negtive curvature are contained in the paper, E. Peschl, *Über die Krümmung von Niveaukurven bei der konformen Abbildung einfachzusammenhängender Gebiete auf das Innere eines Kreises*, Math. Ann., **106** (1932), 574–594.

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UNIVERSIDAD NACIONAL
APARTADO AÉREO 568
MEDELLIN, COLOMBIA

AND

UNIVERSITY OF CINCINNATI
CINCINNATI, OH 45221–0025