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COMPLEMENTATION OF CERTAIN SUBSPACES OF $L_\infty(G)$ OF A LOCALLY COMPACT GROUP

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Let G be a locally compact group, $\mathrm{WAP}(G)$ be the space of continuous weakly almost periodic functions on G and $C_0(G)$ the space of continuous functions on G vanishing at infinity. We prove in this paper, among other things, that if G is infinite and X is any subspace of $\mathrm{WAP}(G)$ (or $\mathrm{CB}(G)$, the space of bounded continuous functions in case G is nondiscrete) containing $C_0(G)$, then X is uncomplemented in $L_\infty(G)$. If G is non-compact, then $\mathrm{WAP}(G)$ is uncomplemented in $\mathrm{LUC}(G)$. Furthermore, $\mathrm{AP}(G)$, the space of continuous almost periodic functions on G, is complemented in $\mathrm{LUC}(G)$ if and only if G/N is compact, where N is the intersection of the kernels of all finite-dimensional continuous unitary representations of G. We also prove that if A is any left translation invariant C^*-subalgebra of $C_0(G)$, then A is the range of a continuous projection commuting with left translations.

1. Introduction and some preliminaries. Let G be a locally compact group and $\mathrm{CB}(G)$ be the space of bounded continuous complex-valued functions on G with supremum norm. Let $\mathrm{LUC}(G)$ denote the space of bounded left uniformly continuous complex-valued functions on G, i.e. all $f \in \mathrm{CB}(G)$ such that the map $g \to l_g f$ from G into $\mathrm{CB}(G)$ is continuous when $\mathrm{CB}(G)$ has the norm topology where $l_g f(x) = f(gx)$, $x \in G$. Let $\mathrm{WAP}(G)$ (respectively $\mathrm{AP}(G)$) denote the space of continuous weakly almost periodic (respectively almost periodic) functions on G i.e. all $f \in \mathrm{CB}(G)$ such that $\{l_a f; a \in G\}$ is relatively compact in the weak (resp. norm) topology of $\mathrm{CB}(G)$. Let $L_\infty(G)$ denote the Banach space of essentially bounded complex-valued functions on G with the essential supremum norm $\|\cdot\|_\infty$ as defined in [**12**, p. 141]. Then $\mathrm{CB}(G)$, $\mathrm{LUC}(G)$, $\mathrm{WAP}(G)$ and $\mathrm{AP}(G)$ are translation invariant subalgebras of $L_\infty(G)$ with $\mathrm{AP}(G) \subseteq \mathrm{WAP}(G) \subseteq \mathrm{LUC}(G) \subseteq \mathrm{CB}(G)$. Furthermore, $C_0(G) \cap \mathrm{AP}(G) = \{0\}$ unless G is compact, where $C_0(G)$ is the closed subalgebra of $\mathrm{CB}(G)$ consisting of all $f \in \mathrm{CB}(G)$ vanishing at infinity. Recall that an application of the Ryll-Nardzewski fixed point theorem ([**21**]) shows that $\mathrm{WAP}(G)$ has a unique invariant mean m_G i.e. m_G is a positive linear functional on $\mathrm{WAP}(G)$ of norm one and $m_G(l_a f) = m_G(r_a f) = m_G(f)$ for all $f \in \mathrm{WAP}(G)$, where

$r_a f(x) = f(xa)$, $x \in G$. Let $W_0(G) = \{f \in \mathrm{WAP}(G);\ m_G(|f|) = 0\}$. Then $\mathrm{WAP}(G) = \mathrm{AP}(G) \oplus W_0(G)$ (see **[6]** or **[2]**). i.e. $\mathrm{AP}(G)$ is always complemented in $\mathrm{WAP}(G)$.

B. B. Wells proved in **[26]** that $\mathrm{AP}(\mathbf{R})$ and $\mathrm{WAP}(\mathbf{R})$ are uncomplemented in $\mathrm{LUC}(\mathbf{R})$, where $\mathbf{R}$ denotes the additive group of the reals. It was also shown by I. Glicksberg **[9]** that if G is a compact group, A is a closed translation invariant subalgebra of $C(G)$ (continuous complex-valued functions on G) and A is not self-adjoint, then A is uncomplemented in $C(G)$. More recently, Y. Takahashi **[23]** proves that a weak*-closed non-self-adjoint translation invariant subalgebra of $L_\infty(G)$ is uncomplemented in $L_\infty(G)$ (see **[14]** for proof of Lemma 4 in **[23]**). Furthermore, [**24**, Theorem 1] if G is an infinite maximally almost periodic group, then $\mathrm{WAP}(G)$ and $\mathrm{AP}(G)$ are uncomplemented in $L_\infty(G)$. Also, as shown by Lau in **[13]**, if G is an amenable locally compact group, then any weak*-closed self-adjoint left translation invariant subalgebra of $L_\infty(G)$ is the range of a continuous projection commuting with left translations.

In this paper, we prove among other things, (Corollary 3) that if G is an infinite locally compact group and X is any closed subspace of $\mathrm{WAP}(G)$ containing $C_0(G)$, then X is uncomplemented in $L_\infty(G)$. If G is non-discrete and X is any closed subspace of $\mathrm{CB}(G)$ containing $C_0(G)$, then X is not complemented in $L_\infty(G)$ (Theorem 4). Furthermore, (Theorem 6), if G is a locally compact non-compact group, then $\mathrm{WAP}(G)$ is *not* complemented in $\mathrm{LUC}(G)$. We prove that (Theorem 7) if H is a closed subgroup of a locally compact group G, then $\mathrm{CB}(G/H)$ (when identified as a closed subspace of $\mathrm{CB}(G)$) is always complemented in $\mathrm{CB}(G)$. This result is used to show that (Theorem 8) $\mathrm{AP}(G)$ is complemented in $\mathrm{LUC}(G)$ if and only if G/N is compact where N is the intersection of the kernels of all finite dimensional continuous unitary representations of G. In particular, if G is maximally almost periodic, then $\mathrm{AP}(G)$ is complemented in $\mathrm{LUC}(G)$ if and only if G is compact. However (Theorem 11), if A is a left translation invariant C^*-subalgebra of $C_0(G)$, then there exists a continuous projection P from $C_0(G)$ onto A and P commutes with left translations.

2. Uncomplemented subspaces of $L_\infty(G)$. In this section we show that if G is an infinite locally compact group, then any subspace X of $\mathrm{WAP}(G)$ containing $C_0(G)$ is uncomplemented in $L_\infty(G)$. We first establish the following lemma which follows directly from the corollary in Losert and Rindler [**16**, p. 74] when G contains a countable dense subset.

LEMMA 1. *Let G be an infinite σ-compact locally compact group. Then there exists a sequence $\{\mu_n\}$ of probability measures on G such that for each $f \in \mathrm{WAP}(G)$*

$$\lim_{n\to\infty} \int r_y f \, d\mu_n = m_G(f)$$

and the convergence is uniform with respect to y, $y \in G$.

Proof. We may assume that G is nondiscrete (otherwise, G is countable, and the lemma follows directly from Losert and Rindler [**16**, p. 74]).

Let K be a compact *normal* subgroup such that G/K is metrizable separable (see Remark 14(b)). For each $x \in G$, $f \in \mathrm{WAP}(G)$, let f^K be a function on G defined by

$$f^K(x) = m_K(f_x), \quad x \in G,$$

where $f_x(k) = f(xk)$.

Then f^K is constant on each coset of K, $f^K \in \mathrm{WAP}(G/K)$ and $m_G(f) = m_{G/K}(f^K)$ (see Chou [**4**, Lemma 2.3]). By the corollary in [**16**, p. 74], there exists a sequence $\{\overline{x}_n\}$ in G/K such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} r_{\overline{y}}(f^K)(\overline{x}_n) = m_G(f)$$

holds uniformly in $\overline{y} \in G/K$.

For each n, let $\theta_n = (1/N)\sum_{n=1}^{N} \delta_{\overline{x}}$, $\overline{x} \in G/K$, where $\delta_{\overline{x}}(f) = f(\overline{x})$. Let μ_n denote the probability measure on G defined by the functional $\tilde{\theta}_n$ on $C_0(G)$, where $\tilde{\theta}_n(f) = \theta_n(f^K)$, $f \in C_0(G)$. If $f \in \mathrm{WAP}(G)$, $y \in G$, then

$$m_G(f) = m_{G/K}(f^K) = \lim_n \theta_n(r_{\overline{y}} f^K) = \lim_n \theta_n((r_y f)^K) = \int r_y f \, d\mu_n$$

and the convergence is uniform in y. □

THEOREM 2. *Let G be a locally compact group. The following are equivalent*:

(a) *G is finite.*

(b) *There exists a continuous linear operator S from $L_\infty(G)$ into $\mathrm{WAP}(G)$ such that $S(f) = f$ for all $f \in C_0(G)$.*

Proof. (a) implies (b) is clear.

(b) implies (a). Let G_0 be an infinite open and closed subgroup of G which is σ-compact. For $f \in L_\infty(G)$, define $(\pi f)(x) = f(x)$ for $x \in G_0$ (restriction to G_0). Then π is a norm decreasing linear map from $L_\infty(G)$ onto $L_\infty(G_0)$.

Given $h \in L_\infty(G_0)$, write $h' \in L_\infty(G)$, where $h'(x) = h(x)$ if $x \in G_0$ and $h'(x) = 0$ if $x \notin G_0$. Define $S'(g) = \pi S(h')$. Then S' is a bounded linear map from $L_\infty(G_0)$ into $L_\infty(G_0)$. Also if $x \in G_0$, then $l_x S'(h) = \pi(l_x S(h'))$. In particular, the range of S' is contained in $\mathrm{WAP}(G_0)$. Furthermore, if $h \in C_0(G_0)$, then $h' \in C_0(G)$, and $S'(h) = \pi(Sh') = \pi(h') = h$.

Let $\{\mu_n\}$ be a sequence of probability measures on G_0 satisfying the conclusion of Lemma 1. Let $\tilde{\mu}_n(f) = \int S'(f)\,d\mu_n$, $f \in L_\infty(G_0)$. Then for each $f \in L_\infty(G_0)$,

$$\lim_n \tilde{\mu}_n(f) = \lim_n \int S'(f)\,d\mu_n = m_{G_0}(S'(f)).$$

Let $\tilde{m}_{G_0}(f) = m_{G_0}(S'(f))$, $f \in L_\infty(G)$. Since $f \in L_\infty(G_0)$ is an abelian W^*-algebra, its spectrum Ω is Stonean (see [**22**, p. 46] or [**25**, p. 109]). Since $C(\Omega)$ and $L_\infty(G_0)$ are isometrically isomorphic via the Gelfand transform, it follows from Theorem 9 [**121**, p. 168] that weak* convergence of a sequence in $L_\infty(G_0)^*$ implies weak convergence. Consequently $\tilde{m}_{G_0}$ is the weak limit of the sequence $\tilde{\mu}_n$. Let K be the convex hull of $\{\tilde{\mu}_n; n = 1, 2, \dots\}$ in the Banach space $L_\infty(G_0)^*$; then there exists a sequence ψ_n in K such that $\|\psi_n - \tilde{m}_{G_0}\| \to 0$. For $\psi \in L_\infty(G_0)^*$, let ψ' denote the restriction of ψ to $C_0(G_0)$. Since S' is the identity on $C_0(G_0)$, it follows that for $\psi \in L_\infty(G_0)^*$, $f \in C_0(G_0)$, we have $\tilde{\psi}(f) = \psi(S'(f)) = \psi(f)$ i.e. $\tilde{\psi}' = \psi'$. In particular if G_0 is non-compact, then $\tilde{m}'_{G_0} = 0$. Now for each n, there exists a continuous function f on G with compact support, $0 \le f \le 1$, $f(x) = 1$, if $x \in \operatorname{supp}\mu_i$, $i = 1, \dots, n$. Since $\tilde{\mu}'_i = \mu'_i$ (as shown above), it follows (by linearity) that if $\varphi = \sum_{i=1}^n \lambda_i \tilde{\mu}'_i$, $\lambda_i \ge 0$, $\sum_{i=1}^n \lambda_i = 1$, then $\varphi(f) = 1$. Hence $\|\varphi\| = 1$. Consequently, each φ in $K' = \{\psi'; \psi \in K\}$ has norm one. But this is impossible. Hence G_0 is again finite. This implies that G is discrete (otherwise take $G_0 = \bigcup_{n=1}^\infty U^n$ where U is a compact symmetric neighbourhood of the identity) and then that G is finite.

If G_0 is compact and infinite (hence not discrete), we may assume that the measures μ_n are singular with respect to the Haar measure m_{G_0}. Then for each n, there exists $f \in C_0(G_0)$ with $0 \le f \le 1$, $\int f(x)\,d\mu_i(x) = 0$ for $i = 1, \dots, n$ and $\int f(x)\,dm_{G_0}(x) > m_{G_0}(G_0)/2$.

It follows that $\|\varphi - m'_{G_0}\| > m_{G_0}(G_0)/2$ for each $\varphi \in K$, which is impossible. So G_0 must again be finite. □

The following is a generalization of Theorem 1 (i) ↔ (ii) in **[24]**:

COROLLARY 3. *Let G be a locally compact group. The following are equivalent*:

(a) *G is finite.*

(b) *There exists a closed subspace X of* WAP(G), *$X \supseteq C_0(G)$ and X is complemented in $L_\infty(G)$.*

When G is non-discrete, we have a much stronger result:

THEOREM 4. *Let G be a locally compact group. The following are equivalent:*

(a) *G is discrete.*

(b) *There exists a closed subspace X of* CB(G), *$X \supseteq C_0(G)$, and X is complemented in $L_\infty(G)$.*

Proof. (a) implies (b) is clear.

(b) implies (a). If G is not discrete, let U be a compact symmetric neighbourhood of the identity of G and $G_0 = \bigcup_{n=1}^{\infty} U^n$. Then G_0 is an infinite open and closed compactly generated subgroup of G. Let K be a compact normal subgroup of G_0 such that G_0/K is metrizable and not discrete (see **[12**, p. 71]). Then G_0/K is open in G/K. In particular, $H = G/K$ is also metrizable. By Corollary 3, G is non-compact. Since H is locally compact and not discrete, there exists an infinite compact subset L of H. By the Borsuk-Dugundji Theorem [7, Theorem 5.1], there exists a continuous linear extension operator S_0: CB$(L) \to$ CB(H). Let f be a continuous real-valued function on H with compact support satisfying $f(x) = 1$ for all $x \in L$ and let $\pi: G \to H$ be the canonical mapping. Then $S(g) = [f \cdot S_0(g)] \circ \pi$ defines a continuous linear mapping from CB(L) into $C_0(G)$. Let λ be the normalized Haar measure of K. If $g \in$ CB(G), let $R(g)$ denote the restriction of g^K to L, where $g^K(x) = m_K(f_x)$, $x \in G$. Observe that $R \circ S$ is the identity on CB(L); hence $S \circ R$: $X \to X$ is a continuous projection on $Y = \operatorname{Im} S$, i.e., Y is a complemented subspace of X. Now if X is complemented in $L_\infty(G)$, then the same is true for Y. Since L is infinite and metrizable, CB(L) is infinite dimensional and separable. Hence Y (being isomorphic to CB(L)) is also infinite dimensional and separable. However, as in the proof of Theorem 1, $L_\infty(G)$, being an abelian von Neumann algebra, is isometrically isomorphic to $C(\Omega)$

of a Stonean space Ω. This is impossible by Corollary 2 in [**11**, p. 169]. □

3. Uncomplemented subspaces in LUC(G). B. B. Wells proved in [**26**] that if $G = \mathbf{R}$, then the space WAP($\mathbf{R}$) is not complemented in LUC($\mathbf{R}$) using Phillips' lemma [**21**] (or [**25**, p. 117]). We now show that this result also holds for all locally compact non-compact groups.

LEMMA 5. *Let G be a non-compact group, $\{F_n; n = 1, 2, \dots\}$ be a family of compact subsets of G and U be a compact neighbourhood of the identity e of G. There exists a sequence $\{y_n\}$ in G and a sequence g_n of continuous functions on G with compact support, $0 \le g_n \le 1$ such that*

(a) *$\{UF_ny_n\}$ is pairwise disjoint,*

(b) *$g_n(x) = 1$ for each $x \in F_ny_n$ and $g_n(x) = 0$ for each $x \notin UF_ny$.*

(c) *For any subset E of $\mathbf{N} = \{1, 2, \dots\}$, the function $g_E(x) = \sum\{g_n(x); n \in E\}$ is left uniformly continuous.*

Proof. By induction, we can construct a sequence $\{y_n\}$ in G such that $\{UF_ny_n\}$ is pairwise disjoint. Let V be a compact symmetric neighbourhood of e such that $V^3 \subseteq U$. By Urysohn's Lemma, there exists a continuous function $f\colon G \to [0,1]$ such that $f(e) = 1$ and $f(G \sim V) = \{0\}$. Define a pseudometric d on G by

$$d(x,y) = \|l_xf - l_yf\|, \quad x, y \in G.$$

Also for each $n = 1, 2, \dots,$ define

$$g_n(x) = 1 - d(x, F_ny_n).$$

Clearly, each g_n is continuous, $0 \le g_n \le 1$ and $g_n(x) = 1$ for all $x \in F_ny_n$. Furthermore, if $g_n(x) > 0$, then $x \in V^2F_ny_n$. (Indeed, in this case, $d(x,y) < 1$ for some $y \in F_ny_n$, and hence $Vx \cap Vy \neq \emptyset$. For otherwise $(l_xf)(x^{-1}) = 1$ and $(l_yf)(x^{-1}) = 0$ and $d(x,y) = 1$ i.e. (b) holds.)

Finally, since $\{UF_ny_n\}$ is pairwise disjoint, the function g_E, $E \subseteq \mathbf{N}$ is well defined. To see that g_E is left uniformly continuous, let $x \in V$, $t \in G$ be such that $|g_E(xt) - g_E(t)| > 0$. If $g_E(xt) \neq 0$, then $xt \in V^2F_ny_n$ for some unique n, $n \in E$, and this gives $t \in V^3F_ny_n$. Similarly, if $g_E(t) \neq 0$, then both xt and t are in UF_ny_n for some unique n, $n \in E$. Thus

$$\begin{aligned} |g_E(xt) - g_E(t)| &= |g_n(xt) - g_n(t)| = |d(xt, F_ny_n) - d(t, F_ny_n)| \\ &\le d(xt, t) = \|l_xf - f\|. \end{aligned}$$

Consequently $\|l_x g_E - g_E\| \leq \|l_x f - f\|$. Hence $g_E \in \mathrm{LUC}(G)$ since $f \in \mathrm{LUC}(G)$. □

THEOREM 6. *Let G be a non-compact group. Then* WAP(G) *is not complemented in* LUC(G).

Proof. We first assume that G is σ-compact. Let $\{\mu_n\}$ be the sequence of probability measures on G constructed in Lemma 1. Let $F_n = \operatorname{supp}\mu_n$. Let $\{y_n\}$ be a sequence of elements in G and $0 \leq g_n \leq 1$ be a sequence of continuous functions of G satisfying the conditions in Lemma 5. Define for each $f \in \mathrm{WAP}(G)$

$$\psi_n(f) = m_G(f) - \int r_{y_n} f \, d\mu_n.$$

Then, by Lemma 1, $\lim_{n\to\infty} \psi_n(f) = 0$ for each $f \in \mathrm{WAP}(G)$. Assume that P is a continuous projection of LUC(G) onto WAP(G) and define for each subset $E \subset \mathbf{N}$

$$\nu_n(E) = \psi_n(P(g_E)).$$

Then ν_n is a finitely additive function on the algebra of subsets of $\mathbf{N}$ and

$$\lim_n \nu_n(E) = 0 \quad \text{for all } E \subseteq \mathbf{N}.$$

But if $n \in \mathbf{N}$, $g_n \in \mathrm{WAP}(G)$ and hence

$$\nu_n(\{n\}) = \psi_n(Pg_n) = \psi_n(g_n) = \int r_{y_n} g_n \, d\mu_n = 1$$

since $0 \leq r_{y_n} g_n \leq 1$, and $r_{y_n} g_n(x) = 1$ for each $x \in F_n = \operatorname{supp}\mu_n$. This contradicts Phillips' Lemma [**20**].

If G is not σ-compact, let H be an open σ-compact but non-compact subgroup of G. For each $f \in \mathrm{LUC}(H)$, let f' be the continuous function on G which agrees with f on H and is zero outside H. Then $f' \in \mathrm{LUC}(G)$. Also, if $f \in \mathrm{WAP}(H)$, then $f' \in \mathrm{WAP}(G)$ (see Chou [**3**, Lemma 2.4] or Milnes [**17**, Theorem 2]).

Assume once more that P is a continuous projection of LUC(G) onto WAP(G). Define for each $f \in \mathrm{LUC}(H)$

$$Qf = P(f')|_H.$$

Since $h|_H \in \mathrm{WAP}(H)$ for each $h \in \mathrm{WAP}(G)$, it follows that Q is a continuous projection of LUC(H) onto WAP(H). By the first part, this is impossible. □

B. B. Wells [**26**, Theorem 3.2] also proved that if $G = \mathbf{R}$, then AP(G), the space of almost periodic functions on G, is uncomplemented in LUC(G). Of course, if AP(G) is finite dimensional (e.g. $G = \mathrm{SL}(2, \mathbf{R})$), then AP($G$) is complemented in LUC(G). It also follows from Takahashi [**24**, Theorem 2] that if G is a discrete group, then AP(G) is complemented in $l_\infty(G)$ if and only if AP(G) is finite dimensional. We shall prove an extension of these results. First we establish the following theorem that we need:

THEOREM 7. *Let G be a locally compact group, H a closed subgroup of G. Then there exists a contractive linear projection P from* CB(G) *onto* CB(G/H). *In particular,* CB(G/H) *is complemented in* CB(G).

Proof. Let $\pi: G \to G/H$ be the canonical mapping. We consider CB(G/H) as a subspace of CB(G) by identifying $f \in \mathrm{CB}(G/H)$ and $f \circ \pi \in \mathrm{CB}(G)$. First we show that it is sufficient to prove the theorem for almost connected groups. Indeed, assume that G_1 is an open, almost connected subgroup of G. Then for $x \in G$, we have $\pi(G_1 x) = G_1 x H/H$ and this is homeomorphic to $G_1/(G_1 \cap xHx^{-1})$. Now let R be a set of representatives for the $G_1 - H$-double cosets in G and assume that for each $x \in R$, we have a linear contractive projection $P_x\colon \mathrm{CB}(G_1) \to \mathrm{CB}(G_1/G_1 \cap xHx^{-1})$ (i.e. $P_x(f \circ \pi_x) = f$ for $f \in \mathrm{CB}(G_1/(G_1 \cap xHx^{-1}))$, if again $\pi_x\colon G_1 \to G_1/(G_1 \cap xHx^{-1})$ denotes the canonical mapping). P_x gives rise to a continuous projection $P'_x\colon \mathrm{CB}(G_1 x) \to \mathrm{CB}(\pi(G_1 x))$: for $f \in \mathrm{CB}(G_1 x)$, $y \in G_1 x$, we put

$$P'_x(f)(\pi(y)) = P_x(r_x f)(yx^{-1}(G_1 \cap xHx^{-1})).$$

If $f \in \mathrm{CB}(\pi(G_1 x))$, then $r_x(f \circ \pi)$ is right-$G_1 \cap xHx^{-1}$ periodic (i.e. $r_k(r_x(f \circ \pi)) = r_x(f \circ \pi)$ for all $k \in G_1 \cap xHx^{-1}$). Hence $P'_x(f \circ \pi) = f$. Observe also that $G/H = \bigcup\{\pi(G_1 x); x \in R\}$. For $y \in G_1 x$, $f \in \mathrm{CB}(G)$, put

$$P(f)(yH) = P_x(f|_{G_1 x})(yH).$$

Then P is a contractive linear projection onto CB(G/H).

If G is almost connected, let K be a compact normal subgroup of G such that G/K is a Lie group. By convolution with the normalized Haar measure of $K \cap H$, we get a contractive linear projection from CB(G) to CB($G/(K \cap H)$) (compare with proof of Lemma 1). Hence, it is sufficient to construct a contractive linear projective from CB($G/(K \cap H)$) to CB(G/H).

Let $\pi_K\colon G \to G/K$ be the canonical mapping, similarly π_H and $\pi_{K\cap H}$ are defined. Let $v_1,\dots,v_n$ be a basis for the Lie algebra of G/K such that $v_{k+1},\dots,v_n$ span the Lie algebra of $\pi_K(H) = HK/K$ for some k. Let $\dot{x}_i(t)$ $(1 \leq i \leq n)$ be the corresponding one parameter subgroups of G/K. By **[19]**, 4.15, Theorem 1, there are continuous one-parameter subgroups $x_i(t)$ in G $(1 \leq i \leq n)$ such that $\pi_K(x_i(t)) = \dot{x}_i(t)$. For $k < i \leq n$, we can even accomplish that $x_i(t) \in H$. There exists $\varepsilon > 0$ such that $(t_1,\dots,t_n) \to \dot{x}_1(t_1)\cdots\dot{x}_n(t_n)$ is a homeomorphism of the cube C

$$\{(t_1,\dots,t_n) \in \mathbf{R}^n\colon |t_1| \leq \varepsilon \text{ for } i = 1,\dots,n\}$$

onto a neighbourhood V of $\dot{e}$ $(= K)$ in G/K and $V \cap (HK)/K$ corresponds to $\{(t_1,\dots,t_n) \in C\colon t_1 = \cdots = t_k = 0\}$. Put

$$M_1 = \{x_1(t_1)\cdots x_k(t_k)\colon |t_i| \leq \varepsilon \text{ for } i = 1,\dots,k\}$$

and

$$M_2 = \{x_{k+1}(t_{k+1})\cdots x_n(t_n)\colon |t_i| \leq \varepsilon \text{ for } i = k+1,\dots,n\}.$$

(If $n = 0$, i.e. K is open in G, we put $M_1 = M_2 = \{e\}$, $V = \{\dot{e}\}$. Similarly if $k = 0$ or $k = n$.) Then $(x,y) \to xy$ maps $M_1 \times M_2$ homeomorphically to M_1M_2, the restriction of π_K to M_1M_2 is a homeomorphism onto V and the restriction of π_{HK} to M_1 is a homeomorphism onto $\pi_{HK}(V)$. Put $W = \pi_K^{-1}(V)$, $U = \pi_H(W)$. Then

$$W = \{abc\colon a \in M_1,\ b \in K,\ c \in M_2\}$$

and the elements a, b, c are uniquely determined by $x = abc$. Assume that $x, x' \in W$ are decomposed as above: $x = abc$, $x' = a'b'c'$, and that $\pi_H(x) = \pi_H(x')$. Then $\pi_{HK}(x) = \pi_{HK}(x')$ and, since $\pi_{HK}(x) = \pi_{HK}(a)$, $\pi_{HK}(x') = \pi_{HK}(a')$, it follows that $a = a'$. Hence $\pi_H(bc) = \pi_H(b'c')$ and this gives $\pi_{H\cap K}(b) = \pi_{H\cap K}(b')$ (recall that $M_2 \subseteq H$). Given $\pi_H(x) \in U$ with $x = abc \in W$, we put $\psi(\pi_H(x)) = \pi_{K\cap H}(ab)$. It follows from the above argument that $\psi\colon U \to G/K \cap H$ is well defined. Also ψ is continuous. This follows easily from the compactness of M_1, M_2 and K and from the fact that a, b, c depend continuously on $x = abc$. Furthermore, $\psi \circ \pi_H = \pi_{K\cap H}$ on M_1K and the canonical mapping $\pi_{H,K\cap H}\colon G/K \cap H \to G/H$ maps $\psi(\pi_H(ab)) = \pi_{K\cap H}(ab)$ to $\pi_H(ab)$. Since $\pi_H(M_1K) = U$, we conclude that $\pi_{H,K\cap H} \circ \psi$ is the identity on U. The covering $\{xU; x \in G\}$ of G/H has a locally finite refinement. Let $\{\varphi_x\colon x \in G\}$ be a partition of unity, subordinate to this covering, i.e. $\varphi_x \in C_0(G/H)$, $0 \leq \varphi_x \leq 1$,

$\operatorname{supp}\varphi_x \subseteq xU$ for each $x \in G$ and $\sum_{x\in G}\varphi_x(y) = 1$ for all $y \in G/H$, where the sum is finite on each compact subset of G/H.

For $f \in \mathrm{CB}(G/(K\cap H))$ define

$$Pf = \sum_{x\in G}\varphi_x \cdot l_{x^{-1}}((l_x f)\circ\psi).$$

(The sum is actually finite on each compact subset of G/H.) Then it is easy to see that P is a contractive linear projection from $\mathrm{CB}(G/(K\cap H))$ to $\mathrm{CB}(G/H)$. □

If G is a locally compact group, the *von Neumann-kernel* is defined as the intersection of the kernels of all finite-dimensional (continuous, unitary) representations of G. It coincides with the kernel of the canonical mapping of G into its *Bohr compactification* bG. The quotient group G/N is maximally almost periodic (for short: $G/N \in \mathrm{MAP}$).

THEOREM 8. *Let G be a locally compact group. The following statements are equivalent*:

(a) $\mathrm{AP}(G)$ *is complemented in* $\mathrm{LUC}(G)$.

(b) *G/N is compact, where N denotes the von Neumann kernel of G.*

(c) *The canonical mapping of G into its Bohr compactification bG is surjective.*

Proof. The equivalence of (b) and (c) is almost immediate.

If (b) holds, then (a) follows from Theorem 7, since $\mathrm{AP}(G) = \mathrm{AP}(G/N) = \mathrm{CB}(G/N)$ (we get a contractive linear projection even from $\mathrm{CB}(G)$ to $\mathrm{AP}(G)$).

For the proof of (a) $\to$ (b) assume that $\mathrm{AP}(G)$ is complemented in $\mathrm{LUC}(G)$. We start with three observations:

If G_1 is a subgroup of G with finite index, and $f \in \mathrm{AP}(G_1)$ is extended to G by putting $f(x) = 0$ for $x \notin G_1$, then $f \in \mathrm{AP}(G)$. In this way, $\mathrm{AP}(G_1)$ becomes a subspace of $\mathrm{AP}(G)$ and it follows now as in the proof Theorem 2 that $\mathrm{AP}(G_1)$ is complemented in $\mathrm{LUC}(G_1) \subseteq \mathrm{LUC}(G)$.

For the second observation assume that $G = H + K$ is the direct sum of closed subgroups H and K. Let $\pi\colon G \to H$ be the corresponding projection. If P: $\mathrm{LUC}(G) \to \mathrm{AP}(G)$ is a projection, then $Qf = P[(f\circ\pi)]|_H$ (where $f \in \mathrm{LUC}(H)$) defines a projection from $\mathrm{LUC}(H)$ to $\mathrm{AP}(H)$.

For the third observation, assume that G_1 is an open subgroup of G that is also closed for the Bohr topology, i.e. the topology induced by bG (in particular $N \subseteq G_1$). We claim that (under the assumption that $\mathrm{AP}(G)$ is complemented in $\mathrm{LUC}(G)$) G_1 has finite index in G. Let L be the closure of the image of G_1 in bG. Then the isomorphism between $\mathrm{AP}(G)$ and $\mathrm{CB}(bG)$ maps $\mathrm{AP}(G) \cap \mathrm{CB}(G_1\backslash G)$ onto $\mathrm{CB}(L\backslash bG)$ (where $G_1\backslash G$ resp. $L\backslash bG$ denote the spaces of *right* cosets). As in the proof of Theorem 7, $\mathrm{CB}(L\backslash bG)$ is complemented in $\mathrm{CB}(bG) = \mathrm{AP}(G)$. It follows that $\mathrm{CB}(L\backslash bG)$ is complemented in $\mathrm{LUC}(G)$. Since $\mathrm{AP}(G) \cap \mathrm{CB}(G_1\backslash G) \subseteq \mathrm{CB}(G_1\backslash G) \subseteq \mathrm{LUC}(G)$ and $G_1\backslash G$ is discrete (hence $\mathrm{CB}(G_1\backslash G) = l^\infty(G_1\backslash G)$), there exists a bounded linear projection from $l^\infty(G_1\backslash G)$ to $\mathrm{CB}(L\backslash bG)$ and also to $\mathrm{CB}((KL)\backslash bG)$ if K is any compact normal subgroup of bG. If $(KL)\backslash bG$ is metrizable, it follows from Corollary 2, p. 169 of **[11]** that $\mathrm{CB}((KL)\backslash bG)$ can be complemented in $l^\infty(G_1\backslash G)$ only if it is reflexive, hence, only if $(KL)\backslash bG$ is finite. Now if $L\backslash bG$ would happen to be infinite, there would exist $f \in \mathrm{CB}(L\backslash bG) \subseteq \mathrm{CB}(bG)$ such that $f(L\backslash bG)$ is infinite. Then, by the Kakutani-Kodaira theorem, there would exist a closed normal subgroup K of G such that bG/K is metrizable and f is K-periodic i.e. $f \in \mathrm{CB}(bG/K)$. This would imply that $f \in \mathrm{CB}((KL)\backslash bG)$. But by the argument above, this is impossible. This shows that $L\backslash bG$ is finite, and since G_1 is the preimage of L in G, it follows that $G_1\backslash G$ is finite too.

To prove (b), we can assume that $G \in \mathrm{MAP}$ (otherwise replace G by G/N and observe that $\mathrm{AP}(G) = \mathrm{AP}(G/N) \subseteq \mathrm{LUC}(G/N) \subseteq \mathrm{LUC}(G)$). We want to show that G is compact.

Let H be an open, almost connected subgroup of G. Then $H \in \mathrm{MAP}$; hence by Theorem 2.9 of **[10]**, it has an open subgroup of finite index which is a direct sum $V + L$ of a compact group L and a vector group V (i.e. $V \simeq \mathbf{R}^n$ for some $n \geq 0$). Replacing H by this open subgroup, we may assume that $H = V + L$.

Let V_1 be the closure of V in G with respect to the Bohr topology. Then (by continuity) L centralizes V_1; hence V_1L is an open subgroup of G which is closed for the relative topology of bG. From the third observation above, it follows that V_1L has finite index in G and, by the first observation above, we can assume that $G = V_1L$ (The Bohr topology induces on a subgroup of finite index again the Bohr topology). This implies that L is normal in G.

Let $\pi\colon G \to G/L$ be the canonical projection. Since L is compact, $\pi(V)$ is closed in G/L and, since $\pi(V_1) = G/L$, it follows that G/L

is abelian. Assume that $\pi(V) \neq G/L$. Take $\dot{x} \notin \pi(V)$. Then there exists a continuous character $\chi \in (G/L)^\wedge$ such that $\chi(\dot{x}) \neq 1$ and $\chi(\pi(V)) = \{1\}$. Then $\chi \circ \pi \in \mathrm{AP}(G)$ and if $x \in V_1$ satisfies $\pi(x) = \dot{x}$, then $\chi(\pi(x)) \neq 1$. But this would imply that x does not belong to the closure of V with respect to the Bohr topology, which is a contradiction. Thus $\pi(V) = G/L$ and hence $G = V \oplus L$. If it would happen that $n > 0$, then we could write G as a direct sum of two groups, one of them being isomorphic to $\mathbf{R}$. By the second observation above, this would imply that $\mathrm{AP}(\mathbf{R})$ is complemented in $\mathrm{LUC}(\mathbf{R})$, contradicting Theorem 3.2 of Wells [**26**]. Hence $n = 0$, i.e. $G = L$ is compact. □

COROLLARY 9. *If G is a locally compact, maximally almost periodic group, then* $\mathrm{AP}(G)$ *is complemented in* $\mathrm{LUC}(G)$ *if and only if G is compact.*

REMARK. In general, the conditions of Theorem 8 do not imply that N is minimally almost periodic group (i.e. that $\mathrm{AP}(N)$ contains only the constant functions). Take e.g. $G = \mathbf{C} \times_\sigma T$ (semidirect product), where $T = \mathbf{R}/\mathbf{Z}$ and the multiplication is defined by $(z,s)(w,t) = (z + e^{2\pi i s}w, s + t)$. Then $N = \mathbf{C}$ and $G/N \simeq T$ is compact (see also Theorem 2.3 in [**18**]).

4. Subspaces of $\mathrm{WAP}(G)$**.** Let G be a locally compact group. For each $m, n \in \mathrm{WAP}(G)^*$, define a multiplication

$$\langle m \odot n, f \rangle = \langle m, n_l(f) \rangle, \qquad f \in \mathrm{WAP}(G),$$

where $n_l(f)(g) = \langle n, l_g f \rangle$, $g \in G$. Then $n_l(f) \in \mathrm{WAP}(G)$ (see [**2**, p. 36]) and, as readily checked, $\mathrm{WAP}(G)^*$ with $\odot$ is a Banach algebra. Furthermore, for each $g \in G$, let δ_g denote the point evaluation at g. Then the map $g \to \delta_g$ is a natural embedding of G into $\mathrm{WAP}(G)^*$.

Let X be a Banach space and $\mathscr{B}(X)$ be the space of bounded linear operators from X into X. Let $\{U_g; g \in G\}$ be continuous representation of G on X i.e. for each $g \in G$, $U_g \in \mathscr{B}(X)$, $U_{g_1}U_{g_2} = U_{g_1 g_2}$, $g_1, g_2 \in G$, and for each $x \in X$, the map $g \to U_g(x)$ from G into X is continuous. We say that $\{U_g; g \in G\}$ is weakly almost periodic if for each $x \in X$, $\{U_g x, g \in G\}$ is a relatively weakly compact subset of X.

LEMMA 10. *Let G be a locally compact group and $\{U_g; g \in G\}$ be a weakly almost periodic continuous representation of G. Then there* *exists a representation* $\{U(m); m \in \mathrm{WAP}(G)^*\} \subseteq \mathscr{B}(X)$ *of the Banach algebra* $\mathrm{WAP}(G)^*$ *on* X *such that:*

(i) $\|U(m)\| \leq K\|m\|$ *for each* $m \in \mathrm{WAP}(G)^*$ *and some fixed* $K > 0$.

(ii) $U(\delta_g) = U_g$ *for each* $g \in G$.

(iii) $P = U(m_G)$ *is a projection of* X *onto the closed subspace* $F_X = \{x \in X; U_g x = x \text{ for all } g \in G\}$.

(iv) P *commutes with any continuous linear operator* T *from* X *into* X *which commutes with* $\{U_g, g \in G\}$.

Proof. Since $\{U_g; g \in G\}$ is weakly almost periodic, it follows from the principle of uniform boundedness that there exists $K > 0$ such that $\|U_g\| \leq K$ for all $g \in G$. For each $x \in X$, $\varphi \in X^*$, define $h_{x,\varphi}(g) = \langle U_g x, \varphi\rangle$, $g \in G$. Then, it is well known [2, p. 36] that $h_{x,\varphi} \in \mathrm{WAP}(G)$. Given $m \in \mathrm{WAP}(G)^*$, let $\langle U(m)x, \varphi\rangle = \langle m, h_{x,\varphi}\rangle$. Then, it is readily checked that $U(m)$ is a continuous linear operator on X, and $\|U(m)\| \leq K\|m\|$. Furthermore $U(m \odot n) = U(m) \circ U(n)$, $m, n \in \mathrm{WAP}(G)^*$, and $U(\delta_g) = U_g$ for each $g \in G$.

Now if $x \in X$, $g \in G$, then

$$\begin{aligned} U_g P(x) &= U(\delta_g) \circ U(m_G)(x) = U(\delta_g \odot m_G)(x) \\ &= U(m_G)(x) = P(x) \end{aligned}$$

i.e. $P(x) \in F_X$. Also if $x \in F_X$, $\varphi \in X^*$

$$\langle P(x), \varphi\rangle = \langle m_G, h_{x,\varphi}\rangle = \langle x, \varphi\rangle.$$

Hence P is a projection from X onto F_X.

Finally if $T \in \mathscr{B}(X)$ and $TU_g = U_g T$, let $m_\alpha = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha \delta_{g_i}^\alpha$ denote a convex combination of point evaluations such that m_α converges to m_G in the weak*-topology of $\mathrm{WAP}(G)^*$, then for each $x \in X$, and $\varphi \in X^*$, $\langle U(m_\alpha)x, \varphi\rangle \to \langle U(m_G)x, \varphi\rangle$, i.e. $U(m_\alpha)$ converges to $U(m_G)$ in the weak operator topology of $\mathscr{B}(X)$. Replacing by a different net if necessary, we may assume that $U(m_\alpha)$ even converges to $U(m_G)$ in the strong operator topology of (X). Hence for each $x \in X$,

$$T \circ P(x) = \lim_\alpha TU(m_\alpha)(x) = \lim_\alpha U(m_\alpha)T(x) = PT(x). \qquad \square$$

THEOREM 11. *Let* G *be a locally compact group and* X *be a closed translation invariant subspace of* $\mathrm{WAP}(G)$. *Let* N *be a closed subgroup of* G *and*

$$A = \{f \in X; r_g f = f \text{ for all } g \in N\}.$$

There exists a projection P from X onto A and P commutes with any continuous linear operator from X into X which commutes with right translations. In particular, P commutes with any left translations.

Proof. This follows directly from Lemma 10 with the observation that left translation always commutes with right translation. □

Parts of the following Lemma were proved in [**5**, Theorem 5.1] for G abelian.

LEMMA 12. *Let G be a locally compact group. Then A is a non-zero left translation invariant C^*-subalgebra of $C_0(G)$ if and only if there exists a unique compact subgroup N_A of G such that*

$$A = \{f \in C_0(G); r_g f = f \text{ for all } g \in N_A\}.$$

Furthermore, A is translation invariant if and only if N_A is normal.

Proof. Let N be a compact subgroup of G, it is easy to see that

$$A = \{f \in C_0(G); r_g f = f \text{ for each } g \in N\}$$

is a left translation invariant C^*-subalgebra of $C_0(G)$. Also, since $C_0(G/N) \simeq A$ (using the identification $f \leftrightarrow f \circ \pi$, where π is the canonical mapping of G onto G/N), $A \neq \{0\}$.

Conversely, if A is a left translation invariant C^*-algebra of $C_0(A)$ let

$$N = N_A = \{g \in G; r_g f = f \text{ for all } f \in A\}.$$

Then N is a closed subgroup of G. Also, if $f \in A$, and $f \neq 0$, let $g_0 \in G$ such that $f(g_0) = \lambda \neq 0$. Then for each $g \in N$, $f(g_0 g) = f(g_0) = \lambda$. Consequently N is compact.

Let $B = \{f \in C_0(G); r_g f = f \text{ for each } g \in N\}$. Clearly $B \supseteq A$. To prove equality, we observe that each $f \in B$ may be regarded as a function $\overline{f}$ in $C_0(G/N)$. Let $\mathscr{A} = \{\overline{f}; f \in A\}$ and $\mathscr{B} = \{\overline{f}; f \in B\}$. Clearly $\mathscr{B} \supseteq \mathscr{A}$. However as in the proof of Theorem 5.1 in [**5**], an application of the Stone-Weierstrass theorem shows that $\mathscr{A} = \mathscr{B}$.

Suppose N_0 is another compact subgroup of G such that $A = \{f \in C_0(G); r_g f = f \text{ for each } g \in N_0\}$ then $N_0 \subseteq N$. If $a \in N$, $a \notin N_0$, there exists $h \in C_{00}(G/N_0)$ such that $h(aN_0) \neq h(N_0)$. Let $f \in C_{00}(G)$ such that

$$\tilde{f}(x) = \int_{N_0} f(x\xi)\, d\xi = h(x).$$

Then $\tilde{f} \in A$ and $r_a\tilde{f} \neq \tilde{f}$, which is impossible. Hence $N_0 = N$.

Finally if A is translation invariant, $g \in G$, $a \in N$, then

$$r_{g^{-1}ag}(f) = r_{g^{-1}}r_a(r_g f) = r_{g^{-1}}r_g f = f$$

since $r_g f \in A$. Hence N is normal. Conversely, if N is normal, $f \in A$ and $g \in G$, then for each $a \in N$, $r_a(r_g f) = r_{ag}f = r_{gb}f = r_g f$ where $b = g^{-1}ag \in N$. In particular, $r_g f \in A$. □

The following is an analogue of Theorem 3.3 in **[13]**:

THEOREM 13. *Let G be any locally compact group and A be a left translation invariant C^*-subalgebra of $C_0(G)$. Then there exists a continuous projection P from $C_0(G)$ onto A and P commutes with any continuous linear operator from $C_0(G)$ into $C_0(G)$ which commutes with right translations. In particular, P commutes with any left translations.*

REMARK 14. (a) Let $N = N_A$, then the projection P in Theorem 13 corresponds to the mapping $T_N(f)(x) = \int_N f(x\xi)\,d\xi$, $x \in G$, which maps $C_0(G)$ onto $C_0(G/N)$ **[8**, p. 261] and $C_0(G/N) \simeq A$.

(b) Lemma 12 can be applied to obtain a well-known result of Kakutani-Kodaira: If G is a σ-compact group, there exists a compact normal subgroup N of G such that G/N is metrizable. Let $f \in C_0(G)$, $f \neq 0$. Since G is σ-compact, the translation invariant C^*-subalgebra A of $C_0(G)$ generated by f is separable. Let $N = N_A$. Then $C_0(G/N) \simeq A$ is also separable. In particular, G/N is metrizable.

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