PACIFIC JOURNAL OF MATHEMATICS
Vol. 141, No. 2, 1990

# SHEAVES AND FUNCTIONAL CALCULUS

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**Let $A$ be a commutative Banach algebra with identity over the complex field, $\mathbb{C}$. Let $a_1, \dots, a_n$ be elements of $A$, and $\operatorname{sp}(a)$ their joint spectrum. In this paper we seek to characterize the functional calculus**

$$T_a\colon \mathscr{O}(\operatorname{sp}(a), A) \to A$$

**as part of a cohomology sequence of certain sheaves, and the algebra $A$ as the algebra of sections**

$$H^0(\operatorname{sp}(a), \mathscr{A}) = A$$

**of a sheaf $\mathscr{A}$, which is related to the Putinar structural sheaf. This is obtained under certain conditions on $a_1, \dots, a_n$. The problem is related also to the unique extension property and to the local analytic spectrum $\sigma(a, x)$ of $x$ with respect to $a$.**

**Section 2 is devoted to attacking this problem. In §1, some preliminary results are obtained. We also prove that if $\sigma(a, x)$ is empty, then $x$ is nilpotent.**

**1.** Let us start by briefly recalling (some details may be found in [2], [5]) the construction of a holomorphic functional calculus morphism

$$T_a\colon \mathscr{O}(\operatorname{sp}(a), A) \to A.$$

Let $U$ be an open neighborhood of $\operatorname{sp}(a)$, and $u_1, \dots, u_n$, $\psi$ infinitely differentiable $A$-valued functions defined on $U$ and verifying:

(i) $\sum_{i=1}^n u_i(z)(z_i - a_i) + \psi(z) = 1$, for all $z$ in $U$.
(ii) $\psi$ has compact support contained in $U$.
(iii) $\psi = 1$ in some neighborhood of $\operatorname{sp}(a)$.

Then

$$T_a^U(f) = f(a) = n!(2\pi i)^{-n} \int_U f\, du_1\, dz_1 \cdots du_n\, dz_n$$

defines a continuous $A$-linear morphism from $\mathscr{O}(U, A)$ to $A$. The compatibility of these morphisms as $U$ varies over open neighborhoods of $\operatorname{sp}(a)$ produces $T_a$. We have the following theorem, where $U$ denotes a neighborhood of $\operatorname{sp}(a)$.

THEOREM 1.1. *Let $f \in \mathscr{O}(U,A)$, and suppose there are $g_1,\dots,g_n$ in $C^\infty(U,A)$ such that*

$$f(z) = \sum_{i=1}^{n} g_i(z)(z_i - a_i) \quad \text{for all } z \text{ in } U.$$

*Then there exists an $A$-valued differential $2n$-form $\alpha$ over $U$, verifying:*

(i) $(z_i - a_i)\alpha = 0$ *for* $i = 1,\dots,n$; *and* $f\alpha = 0$.

(ii) *For every $h$ in $\mathscr{O}(U,A)$.*

$$n!(2\pi i)^{-n} \int_U h\alpha = f(a)^n h(a).$$

*Proof*. Let $u_1,\dots,u_n$, $\psi$ be as above, and let $f_k = u_k f$, $q_k = g_k\psi$, and $r_{jk} = g_k u_j - u_k g_j$. Then

$$\begin{aligned} g_k - f_k &= g_k \left( \sum_{j=1}^{n} u_j(z_j - a_j) + \psi \right) - u_k \sum_{j=1}^{n} g_j(z_j - a_j) \\ &= \sum_{j=1}^{n} r_{jk}(z_j - a_j) + q_k. \end{aligned}$$

Also

$$\sum_{j=1}^{n} f_j(z_j - a_j) = (1 - \psi)f = f - \psi f$$

and therefore, differentiating and multiplying by $dz_1 \cdots dz_n$,

$$\sum_{j=1}^{n} (z_j - a_j)\, df_j\, dz_1 \cdots dz_n = -d(\psi f)\, dz_1 \cdots dz_n.$$

Since $\operatorname{supp}(\psi f)$ and $\operatorname{supp}(q_k)$ are compact sets contained in $U$, we may proceed as in **[5]** (III, 4.9), and obtain an $n-1$ differential form $\tau$ with $\operatorname{supp}(\tau)$ contained in $U$ and such that

$$\begin{aligned} (1)\quad d\tau\, dz_1 \cdots dz_n &= dg_1\, dz_1 \cdots dg_n\, dz_n - df_1\, dz_1 \cdots df_n\, dz_n \\ &= dg_1\, dz_1 \cdots dg_n\, dz_n - f^n\, du_1\, dz_1 \cdots du_n\, dz_n \end{aligned}$$

Now set

$$\alpha = dg_1\, dz_1 \cdots dg_n\, dz_n \quad \text{and} \tag{2}$$

$$\omega = du_1\, dz_1 \cdots du_n\, dz_n. \tag{3}$$

Differentiating $f$ we obtain

$$df = \sum_{j=1}^{n} g_j\, dz_j + \sum_{j=1}^{n} (z_j - a_j)\, dg_j$$

and multiplying by $dg_1\, dz_1 \cdots \widehat{dg_k}\, dz_k \cdots dg_n\, dz_n$,

$$0 = (z_k - a_k)\alpha.$$

Multiplying by $g_k$ and adding gives $f\alpha = 0$ and so, (i) is proved.

Now let $h$ be an element of $\mathscr{O}(U, A)$. By (1), (2), and (3) we have

$$h\alpha - hf^n\omega = h\, d\tau\, dz_1 \cdots dz_n = d(h\tau)\, dz_1 \cdots dz_n.$$

Hence, by construction of the functional calculus,

$$n!(2\pi i)^{-n} \int_U h - h(a) f(a)^n = n!(2\pi i)^{-n} \int_U d(h\tau)\, dz_1 \cdots dz_n$$

but this is zero by Stokes' theorem, for $\operatorname{supp}(h\tau)$ is contained in $U$.

COROLLARY 1.2. *Under the hypothesis of the theorem,* $f(a)^{n+1} = 0$.

*Proof*. Simply put $h = f$.

COROLLARY 1.3. *Let $U$ be an open neighborhood of* $\operatorname{sp}(a)$, *and* $f \in \mathscr{O}(U, A)$. *Suppose that for every $z^0$ in $U$, there are $f_1, \ldots, f_n$ infinitely differentiable functions near $z^0$ such that*

$$f(z) = \sum_{i=1}^{n} f_i(z)(z_i - a_i) \quad \textit{for } z \textit{ near } z^0.$$

*Then* $f(a)^{n+1} = 0$.

*Proof*. A partition of unity will put us in a situation where the theorem is applicable.

Now suppose $x$ is an element of $A$ and consider $\sigma(a, x)$, the local analytic spectrum of $x$ with respect to $a$ (**[1]**, **[4]**). Putting $f = x$, we obtain that if $\sigma(a, x)$ is empty, then $x^{n+1} = 0$. The conclusion $x = 0$ is known only under additional hypotheses **[4]**.

**2.** Let $\mathscr{O}^A$ be the sheaf of germs of holomorphic $A$-valued functions over $\mathbb{C}^n$. If $a = (a_1, \ldots, a_n) \in A^n$, the morphism $\lambda_a \colon (\mathscr{O}^A)^n \to \mathscr{O}^A$ defined by

$$\lambda_a(f_1, \ldots, f_n) = \sum_{i=1}^{n} (z_i - a_i) f_i$$

induces an exact sequence of sheaves

$$0 \to \mathcal{N}_a \to (\mathcal{O}^A)^n \to \mathcal{O}^A \to \mathcal{A} \to 0.$$

Here the stalk of $\mathcal{N}_a$ over $z^0$, $\mathcal{N}_{a_{z^0}}$, consists of germs of $n$-tuples $(g_1, \dots, g_n)$ of functions analytic near $z^0$ and verifying $\sum(z_i - a_i)g_i = 0$ in some neighborhood of $z^0$. $\mathcal{A}_{z^0} = \mathcal{O}^A_{z^0}/\mathcal{J}_{z^0}$, where $\mathcal{J} \subset \mathcal{O}^A$ is the sheaf of ideals generated by $(z_i - a_i)$ for $i = 1, \dots, n$. Note that if $z^0$ is not in $\operatorname{sp}(a)$, $\mathcal{J}_{z^0} = \mathcal{O}^A_{z^0}$, and therefore $\mathcal{A}_{z^0} = 0$.

Clearly, if $U$ is an open, holomorphically convex subset of $C^n$, then

$$\begin{aligned} H^0(U, \mathcal{N}_a) &= \mathcal{N}_a(U) \\ &= \left\{ (f_1, \dots, f_n) \in \mathcal{O}(U, A)^n \colon \sum_{i=1}^{n} (z_i - a_i) f_i = 0 \right\}. \end{aligned}$$

On the other hand, if $I(U)$ denotes the ideal generated by $(z_i - a_i)$, $i = 1, \dots, n$ in $\mathcal{O}(U, A)$, then $I(U) \subset H^0(U, \mathcal{J})$, but the equality does not, in general, hold.

Suppose $U$ contains the joint spectrum of $a_1, \dots, a_n$. Since $T_a^U(z_i - a_i) = 0$ for $i = 1, \dots, n$; we have the inclusion $I(U) \subset \operatorname{Ker} T_a^U$. The following proposition shows that the ideals are the same.

**Proposition** 2.1. *Let $U$ be a holomorphically convex open neighborhood of* $\operatorname{sp}(a)$. *Then* $\operatorname{Ker} T_a^U = I(U)$.

*Proof*. The ideal of $\mathcal{O}(U \times U, \mathbb{C})$ generated by the functions

$$(z, w) \mapsto z_i - w_i, \qquad i = 1, \dots, n,$$

is the ideal of functions analytic on $U \times U$ and zero on the diagonal $\Delta \subset U \times U$, for both ideals are closed, and they coincide locally.

Since $\mathcal{O}(U \times U, A) = \mathcal{O}(U \times U, \mathbb{C}) \otimes_\varepsilon A$, it follows from [3] that all $g \colon U \times U \to A$ null over $\Delta$ belong to the ideal generated by $(z_i - w_i)$, for $i = 1, \dots, n$.

Therefore, it $f \in \mathcal{O}(U, A)$, there are analytic $g_k \colon U \times U \to A$, such that

$$f(z) - f(w) = \sum_{k=1}^{n} g_k(z, w)(z_k - w_k).$$

Applying the functional calculus morphism in the $w$-variable,

$$f(z) - f(a) = \sum_{k=1}^{n} g_k(z, a)(z_k - a_k).$$

Hence, if $f(a) = 0$, $f \in I(U)$.

Now we can relate this fact with the homological approach of Putinar ([7]; see also [6]); to do this we consider the presheaf $\mathscr{P}$ over $\mathbb{C}^n$ defined by

$$\mathscr{P}(U) = \mathscr{O}(U)\hat{\otimes}_{\mathscr{O}(\mathbb{C}^n)}A \qquad (U \text{ open}, U \in \mathbb{C}^n),$$

and let $\mathscr{F}$ be the sheaf defined by $\mathscr{P}$. The standard definitions ([6]) give the identification

$$(*) \qquad \mathscr{P}(U) = \mathscr{O}^A(U)/I(U).$$

Hence looking at the germs we have

LEMMA 2.2. *The sheaf* $\mathscr{A}$ *is the sheaf* $\mathscr{F}$ *defined by the presheaf* $\mathscr{P} = \mathscr{O}\hat{\otimes}_{\mathscr{O}(C^n)}A$.

We also have the following fact:

PROPOSITION 2.3. *Let* $U$ *be a holomorphically convex open neighborhood of* $\operatorname{sp}(a)$. *Then*

(i) *The functional calculus induces a topological isomorphism*

$$\mathscr{P}(U) \approx A$$

(ii) *The kernel of the canonical map*

$$\mathscr{P}(U) \to \mathscr{A}(U)$$

*consists of nilpotent elements.*

*Proof*. The first assertion follows easily from Proposition 2.1 and $(*)$ above, since $I(U)$ is closed in $\mathscr{O}(U, A)$. For the second, let $f \in \mathscr{O}^A(U)$ and assume that the image of $f$ is zero in $\mathscr{A}(U)$; this means that the class of the germ of $f$ in $\mathscr{A}_z$ is zero for every $z \in U$.

Then for every $z^0 \in U$ we have an $n$-tuple $(g_{g_1}^{z^0}, \dots, g_n^{z^0})$ of functions analytic near $z^0$ such that $f = \sum (z_i - a_i) g_i^{z^0}$ in some neighborhood of $z^0$.

Using a partition of unity we are in the situation of Corollary 1.3; hence $f(a)^{n+1} = 0$. But this implies $f^{n+1} \in I(U)$ and this means that $f^{n+1} = 0$ in $\mathscr{P}(U)$.

We shall now study, for a neighborhood $U$ of $\operatorname{sp}(a)$, the cohomology sequence resulting from the exact sequence of sheaves

$$0 \to \mathscr{I} \to \mathscr{O}^A \to \mathscr{A} \to 0.$$

Note that when $U$ is holomorphically convex, $H^p(U,\mathscr{O}^A) = 0$, for all $p > 0$, due to [3] and the well-known case $A = \mathbb{C}$. We have then the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & I(U) & \to & \mathscr{O}(U,A) & \xrightarrow{T_a^A} & A & \to & 0 \\ & & \downarrow & & \| & & \downarrow L & & \\ 0 & \to & H^0(H,\mathscr{J}) & \to & \mathscr{O}(U,A) & \to & H^0(U,\mathscr{A}) & \to & H^1(U,\mathscr{J}) \to 0 \end{array}$$

The ideal $\operatorname{Ker} L$ is isomorphic, because of the snake lemma construction, to the $A$-module $H^0(U,\mathscr{J})/I(U)$. In fact, $H^0(U,\mathscr{J}) \simeq \operatorname{Ker} L \oplus I(U)$. We obtain also the exact sequence,

$$0 \to I(U) \to H^0(U,\mathscr{J}) \xrightarrow{T_a^U} A \xrightarrow{L} H^0(U,\mathscr{A}) \to H^1(U,\mathscr{J}) \to 0$$

and therefore, $H^1(U,\mathscr{J}) \simeq H^0(U,\mathscr{A})/\operatorname{Im} L$.

On the other hand, the exact sequence of sheaves

$$0 \to \mathscr{N}_a \to (\mathscr{O}^A)^n \xrightarrow{\lambda_a} \mathscr{J} \to 0$$

produces the exact cohomology sequence

$$\begin{gathered} 0 \to \mathscr{N}_a(U) \to \mathscr{O}(U,A) \xrightarrow{n} H^0(U,\mathscr{J}) \to H^1(U,\mathscr{N}_a) \to 0 \to \cdots \\ \cdots \to 0 \to H^{p-1}(U,\mathscr{J}) \to H^p(U,\mathscr{N}_A) \to 0 \to \cdots \end{gathered}$$

We then have

$$H^1(U,\mathscr{N}_a) \simeq H^0(U,\mathscr{J})/I(U) \simeq \operatorname{Ker} L$$

and

$$H^p(U,\mathscr{N}_a) \simeq H^{p-1}(U,\mathscr{J}), \quad \text{for } p > 1.$$

We have proved:

PROPOSITION 2.4. *The morphism $L\colon A \to H^0(U,\mathscr{A})$ is*
(i) *a monomorphism iff $H^0(U,\mathscr{J}) = 0$ iff $H^1(U,\mathscr{N}_a) = 0$*
(ii) *an epimorphism iff $H^1(U,\mathscr{J}) = 0$ iff $H^2(U,\mathscr{N}_a) = 0$.*

Note that $\operatorname{Ker} L$ consists of the elements $x$ in $A$ whose local analytic spectrum is empty. Therefore, $L(x) = 0$ implies $x^{n+1} = 0$. If $A$ has no nilpotent elements, $\operatorname{Ker} L = 0$ and $H^0(U,\mathscr{J}) = I(U)$.

DEFINITION. We shall say that $A$ is $a$-representable if
(i) $\operatorname{sp}(a)$ is holomorphically convex, and
(ii) $H^1(\operatorname{sp}(a),\mathscr{N}_a) = 0$, $H^2(\operatorname{sp}(a),\mathscr{N}_a) = 0$.

Note that the first condition ensures the existence of a basis for neighborhoods of $\operatorname{sp}(a)$ made up of holomorphically convex open sets, while the second says that $L$ is an isomorphism for a basis of neighborhoods of $\mathrm{sp}(a)$. Hence, $A = H^0(\mathrm{sp}(a), \mathscr{A})$.

If $n = 1$, and $\mathrm{sp}(a)$ has no interior, $A$ is $a$-representable: in this case, $\mathscr{N}_a = 0$, for if $g \in \mathscr{N}_a(V)$, then $g|_{V \cap (\mathbb{C} - \mathrm{sp}(a))} = 0$, and hence, $g = 0$.

Finally, we wish to compare $a$-representability and the unique extension property [**4**].

THEOREM 2.5. *Suppose that* $\mathrm{sp}(a)$ *is holomorphically convex, and that the n-tuple* $a = (a_1, \ldots, a_n)$ *(considered as a family of operators from A to A) has the unique extension property. Then A is a-representable.*

*Proof*. Consider the sheaf complex $K = K(\mathscr{O}, \alpha)$, where, for each open set $V$,

$$K^r(V) = \mathscr{O}(V, \Lambda^r_A(A^n))$$

consists of analytic $A$-valued $r$-forms over $V$, and

$$\alpha_r \colon K^r \to K^{r+1}$$

is induced by the exterior product $\eta \to \sum_{j=1}^n (z_j - a_j)\, dz_j \wedge \eta$. For $n - 1$, $\alpha$ may be written as

$$\alpha_{n-1}\left(\sum_{i=1}^n f_i\, dz_1 \cdots d\hat{z}_i \cdots dz_n\right) = \sum_{i=1}^n (-1)^{i+1} f_i (z_i - a_i)\, dz_1 \cdots dz_n$$

so that $\operatorname{Ker} \alpha_{n-1}(z)$ is the stalk $\mathscr{N}_{a_z}$ (save a sign), and $\operatorname{Ker} \alpha_{n-1} = \mathscr{N}_a$.

Now the unique extension property expresses that cohomology $H^r(K) = 0$, for $r = 0, \ldots, n-1$, that is, the sequence of sheaves

$$0 \to K^0 \underset{\alpha_0}{\to} K^1 \underset{\alpha_1}{\to} \cdots \to K^{n-2} \underset{\alpha_{n-2}}{\to} \mathscr{N}_a \to 0$$

is exact. Since the sheaves $K^r$ are acyclic, that is, $H^i(U, K^r) = 0$ for holomorphically convex $U$ and $i > 0$, we obtain $H^p(U, \mathscr{N}_a) = 0$ for all $p > 0$.

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Received October 20, 1987 and in revised form September 26, 1988.

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