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WEIERSTRASS POINTS ON GORENSTEIN CURVES

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On a nonsingular projective curve, there are several equivalent ways to define a Weierstrass point. On an irreducible, projective Gorenstein curve, we define Weierstrass points by using a "wronskian" formed from dualizing differentials. We then investigate whether other conditions in the singular case are equivalent to this definition.

Let Y denote a smooth, projective curve of genus g defined over $\mathbf{C}$. A point $P \in Y$ is a Weierstrass point if there exists a rational function on Y with a pole only at P of order at most g, or if there exists a regular differential on Y which vanishes at P to order at least g, or if the divisor gP is special. However, the most "functorial" way to define Weierstrass points is as the zeros of the wronskian, a section of the $(g(g+1)/2)$th tensor power of the canonical bundle on Y. It is this last definition that we use as the foundation for defining Weierstrass points on singular curves. What is essential is that the sheaf of dualizing differentials should be locally free and this is exactly the property satisfied by Gorenstein curves.

Let X be an integral, projective Gorenstein curve of arithmetic genus $g > 0$ over $\mathbf{C}$. Let ω denote the bundle of dualizing differentials on X and let $\mathscr{L}$ denote an invertible sheaf on X. Put $s = \dim H^0(X, \mathscr{L}) = h^0(\mathscr{L})$. Assume $s > 0$ and choose a basis $\phi_1, \ldots, \phi_s$ for $H^0(X, \mathscr{L})$. We will define a section of $\mathscr{L}^{\otimes s} \otimes \omega^{\otimes (s-1)s/2}$ as follows: Suppose that $\{U^{(\alpha)}\}$ is a covering of X by open subsets such that $\mathscr{L}(U^{(\alpha)})$ and $\omega(U^{(\alpha)})$ are free $\mathscr{O}_X(U^{(\alpha)})$-modules generated by $\psi^{(\alpha)}$ and $\tau^{(\alpha)}$, respectively. Define $F_{i,j}^{(\alpha)} \in \Gamma(U^{(\alpha)}, \mathscr{O}_X)$ inductively by

$$\begin{aligned} \phi_j|_{U^{(\alpha)}} &= F_{1,j}^{(\alpha)} \psi^{(\alpha)} \quad \text{for } j = 1, \ldots, s, \\ dF_{i-1,j}^{(\alpha)} &= F_{i,j}^{(\alpha)} \tau^{(\alpha)} \quad \text{for } i = 2, \ldots, s \text{ and } j = 1, \ldots, s. \end{aligned}$$

Here, we consider $F_{i-1,j}$ as an element of the rational function field of X and $dF_{i-1,j}$ is the differential of this rational function. Then $dF_{i-1,j}$ is a rational differential on X and its restriction to $U^{(\alpha)}$, namely

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$dF^{(\alpha)}_{i-1,j}$, is regular on $U^{(\alpha)}$, hence $dF^{(\alpha)}_{i-1,j} \in \Gamma(U^{\alpha}, \omega(U^{(\alpha)}))$ by [8]. Put

$$\rho^{(\alpha)} = \det\left[F^{(\alpha)}_{i,j}\right](\psi^{(\alpha)})^{s}(\tau^{(\alpha)})^{(s-1)s/2}, \qquad i,j = 1,\ldots,s.$$

It is not hard to see, as in the classical case (cf. [1, p. 85]), that $\rho^{(\alpha)} = \rho^{(\beta)}$ in $U^{(\alpha)} \cap U^{(\beta)}$. Hence the $\rho^{(\alpha)}$ determine a section ρ in $H^0(X, \mathscr{L}^{\otimes s} \otimes \omega^{\otimes(s-1)s/2})$. It is easy to see that a different choice of a basis for $H^0(X, \mathscr{L})$ would result in ρ being multiplied by a nonzero scalar. Therefore, the order of vanishing of ρ at P is independent of the choice of basis of $H^0(X, \mathscr{L})$. By the order of vanishing of ρ at P, we mean the following. If ψ generates $\mathscr{L}_P$ (the stalk of $\mathscr{L}$ at P), and τ generates ω_P (the stalk of ω at P), then we may write

$$\rho = f\,\psi^{s}\tau^{(s-1)s/2}.$$

We define $\operatorname{ord}_P \rho$ to be equal to $\operatorname{ord}_P f$. We then have

$$\operatorname{ord}_P \rho = \operatorname{ord}_P f = \dim \mathscr{O}_P/(f)\mathscr{O}_P = \dim \tilde{\mathscr{O}}_P/(f)\tilde{\mathscr{O}}_P,$$

where $\mathscr{O}_P$ denotes the local ring at P and $\tilde{\mathscr{O}}_P$ is its normalization.

DEFINITIONS. Suppose that $P \in X$. The *$\mathscr{L}$-Weierstrass weight of P*, denoted $W_{\mathscr{L}}(P)$, is defined to be $\operatorname{ord}_P \rho$. We call P a *Weierstrass point of $\mathscr{L}$* if $W_{\mathscr{L}}(P) > 0$. We call P a *Weierstrass point of order n of $\mathscr{L}$* if P is a Weierstrass point of $\mathscr{L}^{\otimes n}$. By a *Weierstrass point of X*, we mean a Weierstrass point of ω.

PROPOSITION 1. *The number of Weierstrass points of $\mathscr{L}$, counting multiplicities, is $s \cdot \deg(\mathscr{L}) + (s-1)s(g-1)$.*

Proof. This is immediate from the definitions and a calculation of the degree of $\mathscr{L}^{\otimes s} \otimes \omega^{\otimes(s-1)s/2}$. □

The theory of Weierstrass points, as far as smooth points are concerned, is quite similar to the theory on nonsingular curves. At a smooth point, one may define a sequence of gaps and, as in [4], we have

PROPOSITION 2. *Suppose that P is a smooth point of X. Then P is a Weierstrass point of $\mathscr{L}$ if and only if $h^0(\mathscr{L}(-sP)) \neq 0$.*

Put $\delta_P = \dim \tilde{\mathscr{O}}_P/\mathscr{O}_P$. We recall that P is singular if and only if $\delta_P > 0$.

PROPOSITION 3. $W_{\mathscr{L}}(P) \geq \delta_P \cdot s \cdot (s-1)$.

Proof. [5].

COROLLARY 1. *If P is a singular point of X and $\mathscr{L}$ is an invertible sheaf on X such that $h^0(\mathscr{L}) > 1$, then P is a Weierstrass point of $\mathscr{L}$.*

Thus singular points are almost always Weierstrass points and have high Weierstrass weight. This may be viewed as saying that as a smooth curve degenerates to a singular curve, then many of the smooth Weierstrass points must approach the singularities (see [6] for precise results).

The notion of gaps does not appear to extend to singular points. If P is a singular point, then one is interested not in the (Weil) divisors nP, but rather in all (0-dimensional) subschemes supported at P. The problem here is that as a smooth curve degenerates to a singular one, the limit of a divisor may be a subscheme that is not a divisor (since the divisors on a singular curve form an open subscheme of the Hilbert scheme of the curve). To establish the relationship between Weierstrass points and special subschemes, we need a Riemann-Roch type theorem for subschemes on Gorenstein curves (cf. [2]).

DEFINITION. Suppose Z is a proper closed subscheme of X defined by the coherent sheaf of ideals $\mathscr{I}$. Then the *degree of Z*, denoted $d(Z)$, is defined by

$$d(Z) = \sum_{Q \in \operatorname{Supp}(Z)} \dim_{\mathbf{C}} \mathscr{O}_Q / \mathscr{I}_Q.$$

THEOREM 1. *Suppose Z is a proper closed subsheme of X defined by the coherent sheaf of ideals $\mathscr{I}$. Then*

$$\dim_{\mathbf{C}} \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{I}, \mathscr{O}_X) - \dim_{\mathbf{C}} H^0(X, \mathscr{I} \otimes_{\mathscr{O}_X} \omega) = d(Z) + 1 - g.$$

Proof. We have the short exact sequence $0 \to \mathscr{I} \to \mathscr{O}_X \to \mathscr{O}_Z \to 0$. Tensor this sequence with ω. Since ω is locally free, this gives us the short exact sequence $0 \to \mathscr{I} \otimes \omega \to \omega \to \mathscr{O}_Z \otimes \omega \to 0$. Taking cohomology then yields the following long exact sequence:

$$(*) \qquad \begin{aligned} 0 &\to H^0(X, \mathscr{I} \otimes \omega) \to H^0(X, \omega) \to H^0(X, \mathscr{O}_Z \otimes \omega) \\ &\to H^1(X, \mathscr{I} \otimes \omega) \to H^1(X, \omega) \to H^1(X, \mathscr{O}_Z \otimes \omega). \end{aligned}$$

Since Supp(Z) is a finite set of points, it follows that $H^1(X, \mathscr{O}_Z \otimes \omega) = 0$, and since ω is invertible, we have that $\dim_{\mathbf{C}} H^0(X, \mathscr{O}_Z \otimes \omega) = d(Z)$. Since X is Cohen-Macaulay, $H^1(X, \mathscr{I} \otimes \omega)$ is dual to $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{I} \otimes \omega, \omega)$ and since ω is invertible, the latter vector space is isomorphic to $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{I}, \mathscr{O}_X)$. The theorem then follows by taking dimensions in $(*)$. □

The following definition is due to Kleiman [3].

DEFINITION. Suppose Z is a proper closed subscheme of X defined by the sheaf of ideals $\mathscr{I}$. Then Z is called *r-special* if $\dim_{\mathbf{C}} \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{I}, \mathscr{O}_X) > r$.

Remarks. (1) The elements of $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{I}, \mathscr{O}_X)$ may be identified with rational functions on X. Indeed,

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{I}, \mathscr{O}_X) = \bigcap_{P \in X} \mathrm{Hom}_{\mathscr{O}_P}(\mathscr{I}_P, \mathscr{O}_P)$$

and for each $P \in X$, $\mathrm{Hom}_{\mathscr{O}_P}(\mathscr{I}_P, \mathscr{O}_P)$ is an $\mathscr{O}_P$-submodule of the field of rational functions on X. (In fact, $\mathrm{Hom}_{\mathscr{O}_P}(\mathscr{I}_P, \mathscr{O}_P)$ is the fractional ideal $(\mathscr{O}_P : \mathscr{I}_P)$ of $\mathscr{O}_P$—cf. [7, p. 37].) Thus Z is 1-special if and only if there is a nonconstant rational function $f \in \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{I}, \mathscr{O}_X)$.

(2) If $d(Z) > g + r - 1$, then by Theorem 1, Z is r-special. In particular, if $d(Z) > g$, then Z is 1-special.

(3) If Z is supported at a smooth point P, then Z is simply the divisor $d(Z)P$ and $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{I}, \mathscr{O}_X) = H^0(X, d(Z)P)$.

LEMMA 1. *Let Z denote the closed subscheme of X of degree one and support P. If $g > 0$, then Z is not* 1*-special.*

Proof. Note that $d(Z) = 1$ implies that $\mathscr{I}_P = m_P$, where $\mathscr{I}$ is the ideal sheaf defining Z and m_P is the maximal ideal of $\mathscr{O}_P$. If P is a smooth point, then the result is well-known. If P is a singular point, then note that $m_P \supseteq c_P$, where c_P denotes the conductor of $\mathscr{O}_P$ in $\tilde{\mathscr{O}}_P$. If $f \in \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{I}, \mathscr{O}_X)$, then $f \in \mathscr{O}_Q$ for all $Q \neq P$ and $f \in \mathrm{Hom}_{\mathscr{O}_P}(m_p, \mathscr{O}_P) \subseteq \mathrm{Hom}_{\mathscr{O}_P}(c_p, \mathscr{O}_P)$. But we claim that $\mathrm{Hom}_{\mathscr{O}_P}(c_P, \mathscr{O}_P) = \tilde{\mathscr{O}}_P$. From the definition of c_P, it is easy to see that $\tilde{\mathscr{O}}_P \subseteq \mathrm{Hom}_{\mathscr{O}_P}(c_P, \mathscr{O}_P)$. Now, suppose h generates c_P in $\tilde{\mathscr{O}}_P$ and $f \in \mathrm{Hom}_{\mathscr{O}_P}(c_P, \mathscr{O}_P)$. Then $fh\tilde{\mathscr{O}}_P = fc_P \subseteq \mathscr{O}_P$. In particular, $fh = fh \cdot 1 \in \mathscr{O}_P$. Hence, from the definition of c_P, we have $fh \in c_P = h\tilde{\mathscr{O}}_P$ and so $f \in \tilde{\mathscr{O}}_P$. It follows that if $f \in \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{I}, \mathscr{O}_X)$, then f is a global regular function on the normalization of X, hence must be constant. □

Our main result is:

THEOREM 2. *Suppose $P \in X$. The following statements are equivalent.*

(1) *P is a Weierstrass point of X.*

(2) *There is a nonzero $\sigma \in H^0(X, \omega)$ such that $\mathrm{ord}_P \sigma \geq g$.*

(3) *There is a 1-special subcheme of X with support P and degree equal to g.*

(4) *There is a 1-special subscheme of X with support P and degree at most g.*

Proof. (1) $\Rightarrow$ (2): We may assume $g > 1$. If P is a smooth point, then the proof proceeds as in the classical case, so assume P is a singular point of X. Let $P_1, P_2, \ldots, P_n$ be the points on the normalization of X that lie over P.

Let σ_1 be a dualizing differential on X whose image in ω_P generates ω_P. We may choose a basis $\sigma_1, \sigma_2 = f_2\sigma_1, \ldots, \sigma_g = f_g\sigma_1$ of $H^0(X, \omega)$ such that $0 < \mathrm{ord}_{P_1} f_2 < \cdots < \mathrm{ord}_{P_1} f_g$. If P is a unibranch singularity (i.e. if $n = 1$), then we must have $\mathrm{ord}_{P_1} f_2 \geq 2$. Therefore, we would have $\mathrm{ord}_{P_1} f_g = \mathrm{ord}_P \sigma_g \geq g$. If P is not a unibranch singularity, then we have $\mathrm{ord}_{P_1} f_g \geq g - 1$ and $\mathrm{ord}_{P_i} f_g \geq 1$ for $i = 2, \ldots, n$. Hence, in this case we also have

$$\mathrm{ord}_P \sigma_g = \sum_{i=1}^{n} \mathrm{ord}_{P_i} f_g \geq g.$$

(2) $\Rightarrow$ (3): Let τ generate ω_P and suppose $\sigma = f\tau$ is a nonzero dualizing differential such that $\mathrm{ord}_P \sigma \geq g$. Then $\dim_{\mathbf{C}} \mathcal{O}_P/(f) \geq g$.

We claim that there exists an ideal I of $\mathcal{O}_P$ such that $f \in I$ and $\dim_{\mathbf{C}} \mathcal{O}_P/I = g$. This is a consequence of the following lemma.

LEMMA 2. *Let J be a proper ideal of $\mathcal{O}_P$ with $\dim_{\mathbf{C}} \mathcal{O}_P/J = n$. Then there exists an ideal J' with $J \subset J'$ and $\dim_{\mathbf{C}} \mathcal{O}_P/J' = n - 1$.*

Proof. Let J' be an ideal strictly containing J such that $\dim_{\mathbf{C}} \mathcal{O}_P/J'$ is as large as possible (necessarily less than n). It suffices to show that J'/J is a one-dimensional subspace of the vector space $\mathcal{O}_P/J$.

Choose $x \in J'$ with $x \notin J$. By the choice of J', we have $J' = J + (x)$. Suppose $y \in J'$. Then there exists $z \in \mathcal{O}_P$ such that $y - zx \in J$. Write $z = a + t$, where $a \in \mathbf{C}$ and $t \in m_P$. We claim that $xt \in J$. Indeed, if $xt \notin J$, then we have $J' = J + (xt)$ and so there would exist $u \in \mathcal{O}_P$ such that $x - xtu = x(1 - tu) \in J$. But since $t \in m_P$, the element $1 - tu$

is a unit, so this would imply $x \in J$, which is a contradiction. Thus $xt \in J$. Since $y - zx = y - (a+t)x = y - ax - xt \in J$, it follows that $y - ax \in J$. Thus the image of x in J'/J generates J'/J as a vector space over $\mathbf{C}$. □

Returning to the proof of Theorem 2, put $\mathcal{J}$ equal to the ideal sheaf with support P defined by $\mathcal{J}_P = I$ (where $(f) \subseteq I$ and $\dim_{\mathbf{C}} \mathcal{O}_P/I = g$) and let Z denote the closed subscheme defined by $\mathcal{J}$. Then $d(Z) = g$ and $\sigma \in H^0(X, \mathcal{J} \otimes \omega)$. Thus by Theorem 1, we have $\dim_{\mathbf{C}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{J}, \mathcal{O}_X) \geq 2$, and so Z is 1-special.

(3) $\Rightarrow$ (4): This implication is trivial.

(4) $\Rightarrow$ (1): We note that Lemma 1 implies that $g > 1$. If P is a singular point, then P is a Weierstrass point of X by Corollary 1. If P is a smooth point, then the proof proceeds as in the classical case. □

PROPOSITION 4. *There is a morphism $\phi\colon X \to \mathbb{P}^1_{\mathbf{C}}$ of degree at most g such that $\phi^{-1}(\phi(P)) = \{P\}$ if and only if there exists a rational function $f \in K(X)$ with $f \in \Gamma(X - P, \mathcal{O}_X)$, $f \notin \mathcal{O}_P$, $f^{-1} \in \mathcal{O}_P$ and $0 < \mathrm{ord}_P f^{-1} \leq g$.*

Proof. A morphism $\phi'\colon X - P \to \mathbf{C}$ with associated field homomorphism $\theta\colon \mathbf{C}(T) \to K(X)$ given by $\theta(T) = f$ will extend to give a morphism ϕ from X to $\mathbb{P}^1_{\mathbf{C}}$ if and only if regular functions at infinity pull back to regular functions at P; i.e., if and only if

$$\theta(\mathbf{C}[1/T]_{(1/T)}) \subseteq \mathcal{O}_P.$$

If this condition is satisfied, then clearly $f^{-1} = \theta(1/T) \in \mathcal{O}_P$ and $\mathrm{ord}_P f^{-1} = \deg \phi$. Conversely, suppose $f^{-1} \in \mathcal{O}_P$ and suppose $\alpha = a_0 + a_1/T + \cdots + a_n/T^n \in \mathbf{C}[1/T]$. Then

$$\theta(\alpha) = a_0 + a_1 f^{-1} + \cdots + a_n (f^{-1})^n \in \mathcal{O}_P.$$

Furthermore, since $f \notin \mathcal{O}_P$, we have that $f^{-1} \in m_P$, and so if $a_0 \neq 0$, then $\theta(\alpha)$ is a unit in $\mathcal{O}_P$. The degree of ϕ will equal $\mathrm{ord}_P f^{-1}$. □

Of course, if P is a smooth point and a rational function f does not belong to $\mathcal{O}_P$, then $f^{-1} \in \mathcal{O}_P$ since $\mathcal{O}_P$ is a discrete valuation ring; but at a singular point it is possible for neither a rational function nor its inverse to be in the local ring.

COROLLARY 2. *If there exists a morphism $\phi\colon X \to \mathbb{P}^1_{\mathbf{C}}$ of degree at most g such that $\phi^{-1}(\phi(P)) = \{P\}$, then there exists a locally principal* *subscheme (i.e. Cartier divisor) with support P of degree at most g which is 1-special.*

Proof. The rational function f^{-1} from Proposition 4 defines a principal ideal I of $\mathscr{O}_P$ such that f is a nonconstant element of $\mathrm{Hom}_{\mathscr{O}_P}(I, \mathscr{O}_X)$. □

Now consider the following three statements.

(A) There exists a morphism $\phi\colon X \to \mathbb{P}^1_{\mathbf{C}}$ of degree at most g such that $\phi^{-1}(\phi(P)) = \{P\}$.

(B) There exists a locally principal subscheme (Cartier divisor) supported at P of degree at most g that is 1-special.

(C) P is a Weierstrass point of X.

At a smooth point of X, these three statements are equivalent, just as in the classical case. At a singular point, we have (A)⇒(B), by Corollary 2, and (B)⇒(C), by Theorem 2. However, we will now give examples to show that the reverse implications fail.

PROPOSITION 5. *There exists an integral, projective Gorenstein curve X of arithmetic genus 7 and a singular point P of X such that there is a locally principal 1-special subscheme with support P of degree at most 7, but there is no morphism $\phi\colon X \to \mathbb{P}^1_{\mathbf{C}}$ of degree at most 7 such that $\phi^{-1}(\phi(P)) = \{P\}$.*

Proof. Let Y be a smooth hyperelliptic curve of genus 3 and let Q be a Weierstrass point of Y. Let $h \in \Gamma(Y - Q, \mathscr{O}_Y)$ satisfy $\mathrm{ord}_Q h = -2$. We note that if $f \in \Gamma(Y - Q, \mathscr{O}_Y)$ satisfies $\mathrm{ord}_Q f \geq -6$, then, since the nongaps at Q are $0, 2, 4, 6, \ldots$, we may write $f = a_1 + a_2 h + a_3 h^2 + a_4 h^3$ with $a_i \in \mathbf{C}$ for $i = 1, 2, 3, 4$. We may choose $s \in K(Y)$ with $\mathrm{ord}_Q s = 1$ and $\mathrm{ord}_Q(s^2 h - 1) > 1$. Put $t = s + s^2$. Then t is a rational function on Y such that $\mathrm{ord}_Q t = 1$ and $t^2 h = 1 + bt + kt^2$ where $b \in \mathbf{C}$, $b \neq 0$, and $k \in \mathscr{O}_Q$.

We now construct a Gorenstein curve X of arithmetic genus 7 and normalization Y as follows (cf. [8]). Take $X_{\mathrm{sing}} = \{P\}, X - P \cong Y - Q$, and

$$\mathscr{O}_P = \mathbf{C} + t^3\mathbf{C} + (t^3/h)\mathbf{C} + t^6\mathbf{C} + t^8\mathscr{O}_Q.$$

Note that $\mathrm{ord}_Q(t^3/h) = 5$. We have that the conductor of $\mathscr{O}_P$ in $\mathscr{O}_Q$ is generated by t^8 in $\mathscr{O}_Q$ and $\delta_P = 4$. Thus X is Gorenstein of arithmetic genus 7.

Let Z denote the locally principal subscheme with support P with ideal sheaf $\mathscr{I}$ defined by $\mathscr{I}_P = (t^3/h)\mathscr{O}_P$. Since $h \in \Gamma(X - P, \mathscr{O}_X)$ and $h(t^3/h) \in \mathscr{O}_P$, h is a (nonconstant) element of $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{I}, \mathscr{O}_X)$. Therefore, Z is a locally principal 1-special subscheme of degree 5.

Now suppose there exists $f \in \Gamma(X - P, \mathscr{O}_X)$ with $f^{-1} \in \mathscr{O}_P$ and $\mathrm{ord}_P f^{-1} \leq 7$. By the definition of $\mathscr{O}_P$, the order of f^{-1} at P cannot be 2, 4, or 7. Also the order of f^{-1} at P cannot be 1, 3, or 5 since these are gaps at Q. Therefore, we must have $\mathrm{ord}_P f = -6$, hence $f = a_1 + a_2 h + a_3 h^2 + a_4 h^3$ for some $a_i \in \mathbf{C}, i = 1, 2, 3, 4$. But, $h = (1 + bt + kt^2)/t^2$, so $f = (1 + 3b^3 t + k_1 t^2)/t^6$ with $k_1 \in \mathscr{O}_Q$. As a result, $f^{-1} = t^6 - 3b^3 t^7 + k_2 t^8$ for some $k_2 \in \mathscr{O}_Q$. But then $f^{-1} - t^6$ would be an element of $\mathscr{O}_Q$ which vanishes to order 7 at P, a contradiction. Thus no such f can exist and hence there does not exist a morphism $\phi\colon X \to \mathbf{P}^1_{\mathbf{C}}$ of degree at most 7 with $\phi^{-1}(\phi(P)) = \{P\}$. □

PROPOSITION 6. *There exists an integral, projective Gorenstein curve X of arithmetic genus* 3 *and a singular point P of X such that there is no locally principal* 1*-special subscheme supported at P of degree at most* 3.

Proof. Let Y be an integral, projective Gorenstein curve of arithmetic genus two and let Q_1 and Q_2 be two smooth Weierstrass points of Y. (We could take Y to be nonsingular.) Let X be the Gorenstein curve of arithmetic genus 3 obtained by identifying the points Q_1 and Q_2 of Y to form an ordinary node P. There exist $\sigma_1, \sigma_2 \in H^0(Y, \omega_Y)$ such that $\mathrm{ord}_{Q_i}\sigma_i = 2$ and $\mathrm{ord}_{Q_j}\sigma_i = 0$ for $i, j = 1, 2$ and $i \neq j$. Let $\tau \in \mathrm{H}^0(X, \omega_X)$ be a generator of $\omega_{X,P}$. Note that $\tau = \sigma/h$, where σ is a generator of ω_{Y,Q_1} and ω_{Y,Q_2} and h is a generator of the conductor of $\mathscr{O}_P$ in its integral closure. In particular, $\mathrm{ord}_{Q_i} h = 1$ for $i = 1, 2$. Then τ, σ_1, σ_2 are a basis for $H^0(X, \omega_X)$ and if we write $\sigma_i = f_i \tau$, then

$$\mathrm{ord}_{Q_i} f_i = 3 \quad \text{and} \quad \mathrm{ord}_{Q_j} f_i = 1 \quad \text{for } i, j = 1, 2 \text{ and } i \neq j.$$

Now suppose $\rho \in H^0(X, \omega_X)$, $\rho \neq 0$, and write $\rho = f\tau$ with $f \in \mathscr{O}_P$. There are then four possibilities:

$$(*) \qquad \begin{aligned} &(1)\ \mathrm{ord}_{Q_1} f = 0 \text{ and } \mathrm{ord}_{Q_2} f = 0 \\ &(2)\ \mathrm{ord}_{Q_1} f = 1 \text{ and } \mathrm{ord}_{Q_2} f = 1 \\ &(3)\ \mathrm{ord}_{Q_1} f = 3 \text{ and } \mathrm{ord}_{Q_2} f = 1 \\ &(4)\ \mathrm{ord}_{Q_1} f = 1 \text{ and } \mathrm{ord}_{Q_2} f = 3. \end{aligned}$$

Suppose $k \in m_P$, with $k \neq 0$, and let $\mathscr{I}$ be the (invertible) ideal sheaf with support P defined by $\mathscr{I}_P = (k)$. Suppose $\mathrm{ord}_{Q_1} k = \mathrm{ord}_{Q_2} k = 1$. Suppose μ is a nonzero element of $H^0(X, \mathscr{I} \otimes \omega_X)$ and write $\mu = lk\tau$, where $l \in \mathscr{O}_P$. Then l must be a unit in $\mathscr{O}_P$ or else lk would not satisfy any of the four possibilities in $(*)$. Therefore, $\dim_{\mathbf{C}} H^0(X, \mathscr{I} \otimes \omega_X) = 1$ and so $\dim_{\mathbf{C}} \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{I}, \mathscr{O}_X) = 1$ by Theorem 1. Thus, $\mathscr{I}$ does not define a 1-special subscheme.

Now suppose $\mathrm{ord}_{Q_1} k = 2$ and $\mathrm{ord}_{Q_2} k = 1$. Again, suppose $\mu \in H^0(X, \mathscr{I} \otimes \omega_X)$, where $\mathscr{I}$ is the ideal sheaf with support P defined by $\mathscr{I}_P = (k)$, and write $\mu = lk\tau$. Then lk cannot satisfy any of the possibilites in $(*)$, so we must have $l = 0$. Thus $H^0(X, \mathscr{I} \otimes \omega_X) = 0$ and we have $\dim_{\mathbf{C}} \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{I}, \mathscr{O}_X) = 1$ again. The same argument applies if $\mathrm{ord}_{Q_1} k = 1$ and $\mathrm{ord}_{Q_2} k = 2$. Therefore, we may conclude that there does not exist a locally principal 1-special subscheme with support at P and degree at most 3. (We note that the subscheme defined by the ideal $(f_1^3, f_2)\mathscr{O}_P$, or by $(f_1, f_2^3)\mathscr{O}_P$, is a 1-special subscheme of degree 3, but it is not locally principal and the locally principal subscheme defined by $(f_1)\mathscr{O}_P$, or by $(f_2)\mathscr{O}_P$, is 1-special but has degree 4.) □

An interesting example of a curve of the type in Proposition 6 is the rational curve with three nodes obtained from $\mathbb{P}^1_{\mathbf{C}}$ by identifying 0 with ∞, 1 with -1, and i with $-i$. Each of these three nodes has Weierstrass weight 8, so there are no nonsingular Weierstrass points on this curve. This curve may be realized projectively as the plane quartic $X^2Y^2 + Y^2Z^2 = X^2Z^2$, which has biflecnodes at the points $(1,0,0), (0,1,0)$, and $(0,0,1)$.

Finally, we show that if one restricts to singularities with $\delta_P = 1$ (i.e. simple cusps and ordinary nodes), then the only case of a Weierstrass point P that does not have a corresponding locally principal 1-special subscheme of degree at most g occurs when the arithmetic genus is 3 and P is a node obtained by identifying two Weierstrass points on the partial normalization at P (i.e. the situation in Proposition 6).

THEOREM 3. *Suppose that X has arithmetic genus $g > 1$, that $P \in X_{\mathrm{sing}}$ satisfies $\delta_P = 1$, and that $\theta \colon Y \to X$ is the partial normalization at P.*

(1) *If P is a simple cusp, then there is a morphism $\phi \colon X \to \mathbb{P}^1_{\mathbf{C}}$ of degree at most g such that $\phi^{-1}(\phi(P)) = \{P\}$.*

(2) *If P is an ordinary node with $\theta^{-1}(P) = \{Q_1, Q_2\}$ and Q_1 and Q_2 are not both Weierstrass points of Y, then there is a morphism ϕ as in* (1).

(3) *If P is an ordinary node, then there is a locally principal* 1*-special subscheme with support P and degree at most g, except when $g = 3$ and $\theta^{-1}(P)$ consists of two Weierstrass points of Y.*

Proof. (1) Suppose $\theta^{-1}(P) = \{Q\}$. Since Y has arithmetic genus $g-1$, it follows from the second remark after Theorem 1 that there exists a nonconstant rational function $h \in \Gamma(Y-Q, \mathscr{O}_Y) = \Gamma(X-P, \mathscr{O}_X)$ such that $-1 > \operatorname{ord}_Q h \geq -g$. Since $\operatorname{ord}_Q 1/h \geq 2$ and P is a simple cusp, we have that $1/h \in \mathscr{O}_P$. The existence of ϕ then follows from Proposition 4.

(2) Assume that Q_1 is not a Weierstrass point of Y. Let Z denote the locally principal subscheme of Y with support $\{Q_1, Q_2\}$ defined by the ideal sheaf $\mathscr{I}$ such that $\dim_{\mathbf{C}} \mathscr{O}_{Q_1}/\mathscr{I}_{Q_1} = g-1$ and $\dim_{\mathbf{C}} \mathscr{O}_{Q_2}/\mathscr{I}_{Q_2} = 1$ (i.e. Z is the divisor $(g-1)Q_1 + Q_2$). Since the arithmetic genus of Y is $g-1$ and $d(Z) = g$, there exists a nonconstant rational function $h \in \operatorname{Hom}_{\mathscr{O}_Y}(\mathscr{I}, \mathscr{O}_Y)$. If $\operatorname{ord}_{Q_1} h \geq 0$, then we would have $\operatorname{ord}_{Q_2} h = -1$, which is impossible since Y has positive arithmetic genus. If $\operatorname{ord}_{Q_2} h \geq 0$, then we would have $0 > \operatorname{ord}_{Q_1} h \geq -g+1$, which contradicts the fact that Q_1 is not a Weierstrass point of Y. Thus we must have $\operatorname{ord}_{Q_1} h < 0$ and $\operatorname{ord}_{Q_2} h < 0$. Since P is an ordinary node, this implies that $1/h \in \mathscr{O}_P$ and we are through by Proposition 4.

(3) It remains for us to show that if $g \geq 4$ and $\theta^{-1}(P) = \{Q_1, Q_2\}$ with Q_1 and Q_2 both Weierstrass points of Y, then there exists a locally principal 1-special subscheme of X with support P. Let $\tau = \sigma/h$ denote a generator for $\omega_{X,P}$, where σ is a generator for ω_{Y,Q_1} and for ω_{Y,Q_2} and where h generates the conductor (in $\tilde{\mathscr{O}}_P$) of $\mathscr{O}_P$ in $\tilde{\mathscr{O}}_P$ (so, in particular h vanishes to order 1 at Q_1 and at Q_2). It is not hard to see that we may choose a basis $\sigma_1, \sigma_2, \ldots, \sigma_g$ for $H^0(X, \omega_X)$, with $\sigma_i = f_i \tau$ for $i = 1, 2, \ldots, g$, such that

$$\operatorname{ord}_{Q_1} f_1 < \operatorname{ord}_{Q_1} f_2 < \cdots < \operatorname{ord}_{Q_1} f_g \quad \text{and} \quad \operatorname{ord}_{Q_2} f_i \neq \operatorname{ord}_{Q_2} f_j \quad \text{for } i \neq j.$$

Note that since Q_1 is a Weierstrass point of Y and $\sigma_2, \ldots, \sigma_g$ are a basis for $H^0(Y, \omega_Y)$, we have $\operatorname{ord}_{Q_1} f_g/h \geq g-1$, hence $\operatorname{ord}_{Q_1} f_g \geq g$. We consider four cases.

Case (1): Assume there is a nonzero $\rho \in H^0(X, \omega_X)$ with $\rho = s\tau$ such that $\operatorname{ord}_{Q_1} s \geq g$ and $\operatorname{ord}_{Q_2} s > 1$. Pick $f \in \mathscr{O}_P$ such that $\operatorname{ord}_{Q_1} f = g-1$ and $\operatorname{ord}_{Q_2} f = 1$ and let Z be the locally principal subscheme of X with support P and degree g defined by the ideal sheaf $\mathscr{I}$ such that $\mathscr{I}_P = (f)$. Then using the fact that P is an ordinary node, it is not hard to see that $\rho \in H^0(X, \mathcal{I} \otimes \omega_X)$. By Theorem 1, Z is 1-special.

Case (2): Assume $\mathrm{ord}_{Q_1} f_{g-1} \geq g$. Then for some $a, b \in \mathbf{C}$, $\rho = a\sigma_{g-1} + b\sigma_g$ satisfies $\rho = s\tau$ where $\mathrm{ord}_{Q_1} s \geq g$ and $\mathrm{ord}_{Q_2} s > 1$. Thus this case reduces to the previous one.

Case (3): Assume $\mathrm{ord}_{Q_1} f_{g-1} = g - 1$. By case (1), we may assume that $\mathrm{ord}_{Q_2} f_g = 1$. Then for some $a, b \in \mathbf{C}$, $\rho = a\sigma_{g-1} + b\sigma_g$ satisfies $\rho = s\tau$ where $\mathrm{ord}_{Q_1} s = g - 1$ and $\mathrm{ord}_{Q_2} s = 1$. Let Z be the locally principal subcheme of X with support P defined by the ideal sheaf $\mathcal{I}$ such that $\mathcal{I}_P = (s)$. Then $d(Z) = 1$ and $\rho \in H^0(X, \mathcal{I} \otimes \omega_X)$, so Z is 1-special by Theorem 1.

Case (4): We are now reduced to assuming

$$\mathrm{ord}_{Q_1} f_i = i - 1 \quad \text{for } 1 \leq i \leq g - 1 \quad \text{and} \quad \mathrm{ord}_{Q_2} f_g = 1.$$

Note that by reversing the roles of Q_1 and Q_2, we may assume that $\mathrm{ord}_{Q_2} f_j = 2$ for some j, $2 \leq j \leq g - 1$. (It is at this point that we use the assumption that $g \geq 4$.) Let Z be the locally principal subscheme of X with support P defined by the ideal sheaf $\mathcal{I}$ such that $\mathcal{I}_P = (f_j)$. Then $d(Z) = 2 + (j - 1) \leq g$. Note that $\mathrm{ord}_{Q_2} f_k > 2$ if $2 \leq k \leq g - 1$ and $k \neq j$ so, using the fact that P is an ordinary node, we have that $\sigma_j, \sigma_{j+1}, \ldots, \sigma_{g-1}$ all belong to $H^0(X, \mathcal{I} \otimes \omega_X)$. Therefore, by Theorem 1,

$$\begin{aligned}\dim_{\mathbf{C}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}, \mathcal{O}_X) &= j + 1 - g + 1 + \dim_{\mathbf{C}} H^0(X, \mathcal{I} \otimes \omega_X) \\ &\geq j + 1 - g + 1 + (g - 1 - j + 1) \geq 2.\end{aligned}$$

Hence Z is 1-special, completing the proof. □

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