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HYPERHOLOMORPHIC FUNCTIONS AND HIGHER ORDER PARTIAL DIFFERENTIAL EQUATIONS IN THE PLANE

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We derive a Taylor formula for matrix-valued functions, in particular for hyperholomorphic functions. The latter functions are matrix-valued functions that satisfy a certain type of first order systems, for which we make no ellipticity assumption. For solutions of higher order linear partial differential equations with constant coefficients in the plane we show the existence of hyperconjugates, an obvious generalization of harmonic conjugates in complex analysis. By way of hyperconjugates we find series expansions for solutions of partial differential equations in terms of polynomial solutions. These polynomials form a basis for real analytic solutions at the origin. An algorithm for obtaining all such polynomials is summarized at the end. This paper continues in the tradition of hypercomplex analysis.

1. Matrix-valued functions. Matrix-valued functions are freely added or multiplied whenever their sizes are compatible. In writing the product FG we automatically assume the number of columns of G to be equal to the number of rows of G. We shall not single out any particular class of matrix-valued functions to form an algebra. The underlying scalars for the matrices can be real, complex, or perhaps even elements of a Banach algebra. Most of the basic concepts in the calculus of scalar-valued functions can be readily extended to matrix-valued functions by means of "componentwise applications". However, certain complications are expected because matrix multiplications are not commutative. Although we need not restrict to two independent variables, we will do so in order to simplify our presentation.

Let F belong to class C^1 in some unspecified domain, namely every component function $f_{ij}(x, y)$ has continuous first order partial derivatives, then the *differential* of F or (df_{ij}) is conveniently expressed as $dF = F_x\,dx + F_y\,dy$ where the subscripts represent componentwise partial differentiation. More generally we consider a *differential form* $P(x, y)\,dx + Q(x, y)\,dy$, which is said to be *exact* if there exists an F such that $dF = P\,dx + Q\,dy$, or equivalently, $P = F_x$ and $Q = F_y$.

A differential form $P\,dx + Q\,dy$ is said to be *divisible* if there exist F and G such that

$$P\,dx + Q\,dy = F(dG) := FG_x\,dx + FG_y\,dy,$$

or alternatively

$$P\,dx + Q\,dy = (dF)G := F_xG\,dx + F_yG\,dy.$$

If the differential of H is divisible, say if

$$dH = F(dG),$$

that is, $H_x = FG_x$ and $H_y = FG_y$, then we say that H is *left-differentiable* with respect to G with a *left-derivative* F, and we may write $dH/dG = F$. Likewise, if H, F, and G are such that

$$dH = (dF)G,$$

that is, $H_x = F_xG$ and $H_y = F_yG$, then we say that H is *right-differentiable* with respect to F with a *right-derivative* G, and we may perhaps write $dF \setminus dH = G$. However, in most cases it will suffice to write H' to denote either of these derivatives.

A differential form is not necessarily exact; neither is a divisible differential form. Nevertheless, the well-known criterion for exactness of differential forms in general may be applied in particular to divisible differential forms to produce a useful criterion.

PROPOSITION 1. *If F belongs to C^1, and G belongs to C^2 in some simply-connected domain Ω, then $F(dG)$ is exact if and only if*

$$F_xG_y = F_yG_x \quad \text{in } \Omega.$$

Proof. As is well known, in a simply-connected domain, $P\,dx+Q\,dy$ is exact if and only if $P_y = Q_x$. Applying this criterion to $F(dG) = FG_x\,dx + FG_y\,dy$, we obtain

$$(FG_x)_y = (FG_y)_x$$

whence follows

$$F_yG_x = F_xG_y$$

since $G_{xy} = G_{yx}$ because G belongs to C^2.

We can likewise show that in a simply-connected domain $(dF)G$ is exact if and only if $F_xG_y = F_yG_x$ assuming that F belongs to C^2 and G belongs to C^1. Although our exactness criterion is valid only in a simply-connected domain, it does not prevent a divisible differential form from being exact in Ω regardless of whether Ω is simply connected or not. If $F(dG)$ is exact in Ω, then $F(dG) = dH$ for some H in Ω. In this case, we say that F is *left-antidifferentiable* with respect to G, and H is a *left-antiderivative* of F with respect to G. We may write $H = \int F(dG)$. Although $\int F(dG)$ is not unique, no statement shall be made about $\int F(dG)$ unless it is valid independently of choices of antiderivatives. Likewise, if $(dG)F = dH$, then H is a *right-antiderivative* of F with respect to G, and we may write $H = \int(dG)F$. However, in most cases it will suffice simply to write $F^{\#}$ for the antiderivative of F.

The line integral of a differential form is defined componentwise:

$$\int_\gamma P\,dx + Q\,dy = \left(\int_\gamma p_{ij}\,dx + q_{ij}\,dy\right)$$

where γ is a *path of integration* (having a continuous tangent vector). We have the following fundamental theorem of line integral, which seems quite obvious.

THEOREM 1. *If a divisible differential form $F(dG)$ is exact in Ω, then*

$$\int_\gamma F(dG) = F^{\#}(x_1, y_1) - F^{\#}(x_0, y_0)$$

where γ is a path of integration connecting (x_0, y_0) to (x_1, y_1) in Ω, and $F^{\#}$ is an antiderivative of F with respect to G.

Proof. Since $F(dG)$ is exact, there exists an $F^{\#}$ such that $F(dG) = dF^{\#}$. Consequently

$$\int_\gamma F(dG) = \int_\gamma dF^{\#} = F^{\#}(x_1, y_1) - F^{\#}(x_0, y_0).$$

In practice it may not be easy to find $F^{\#}$.

Every matrix-valued function Z is differentiable with respect to Z since $dZ = I\,dZ$, but powers of Z need not be differentiable with respect to Z. For example, we can go no further than

$$dZ^2 = d(ZZ) = (dZ)Z + Z(dZ)$$

unless $(dZ)Z = Z(dZ)$, in which case we could go on to

$$dZ^2 = (2Z)(dZ) = (dZ)(2Z).$$

Therefore, following Hile **[8]**, we shall say that Z is *self-commuting* in Ω if

$$Z(x_1, y_1)Z(x_2, y_2) = Z(x_2, y_2)Z(x_1, y_1)$$

for any (x_1, y_1) and (x_2, y_2) in Ω. For example, $Z = Ax + By$ is self-commuting if the constant matrices A and B commute. Needless to say, Z has to be a square matrix in order to be self-commuting.

PROPOSITION 2. *If Z is self-commuting and C^1 in Ω, we have*

(1.1) *Z_x, Z_y and Z commute pairwise, in particular* $Z(dZ) = (dZ)Z$.

(1.2) *Z_x, Z_y, and Z^{-1} commute if Z is invertible, and thus* $Z^{-1}(dZ) = (dZ)Z^{-1}$.

(1.3) $dZ^n = (nZ^{n-1})\,dZ = dZ(nZ^{n-1})$ *for all integers n.*

(1.4) $d(Z - Z_0)^n = n(Z - Z_0)^{n-1}\,dZ = dZ[n(Z - Z_0)^{n-1}]$ *where $Z_0 = Z(x_0, y_0)$ and $(x_0, y_0) \in \Omega$.*

Proof. We show (1.1) by writing out the difference quotients for Z_x and Z_y and applying the self-commuting property of Z. (1.1) leads to (1.2), and together they imply (1.3) and (1.4). We omit the details.

One interesting thing about self-commuting Z is that Z-differentiability and Z-antidifferentiability are not unrelated, and this in turn leads to some nice theorems. Three such theorems, 2 to 4, are stated below though they are not used in the rest of the paper except Theorem 4.

LEMMA 1. *If F is right-differentiable with respect to Z in a simply-connected domain, and Z is self-commuting and C^2, then F is right-antidifferentiable with respect to Z.*

Proof. Suppose $dF = (dZ)G$, or

$$F_x = Z_x G \quad \text{and} \quad F_y = Z_y G.$$

Multiplying with Z_y and Z_x respectively, we have

$$Z_y F_x = Z_y Z_x G \quad \text{and} \quad Z_x F_y = Z_x Z_y G.$$

But since $Z_x Z_y = Z_y Z_x$ by Proposition 2, we obtain

$$Z_x F_y = Z_y F_x,$$

which is the exactness criterion in Proposition 1; hence $dH = (dZ)F$ for some H, and F is right-antidifferentiable.

THEOREM 2 (*Cauchy integral theorem*). *If F is right-differentiable with respect to a C^2 and self-commuting Z in a simply-connected domain Ω, then*

$$\int_\gamma (dZ)F = 0 \tag{1.5}$$

for any closed path of integration γ in Ω.

Proof. By Lemma 1, there exists H such that $(dZ)F = dH$. Consequently

$$\int_\gamma (dZ)F = \int_\gamma dH = 0$$

for any closed γ in Ω.

THEOREM 3. *If F is right-antidifferentiable with respect to a C^2 and self-commuting Z in a simply-connected domain, it is infinitely many times right-antidifferentiable with respect to Z.*

Proof. If F is Z-antidifferentiable with an antiderivative $F^{\#}$, then since $F^{\#}$ is Z-differentiable, by Lemma 1 $F^{\#}$ is Z-antidifferentiable with antiderivative $F^{\#\#}$. Repeating the same argument on $F^{\#\#}$, we show the existence of the next antiderivative, etc.

We now consider the following reversal of the preceding. Here we no longer need the simply-connectedness of the domain, but instead we need the invertibility of either Z_x or Z_y in the domain.

LEMMA 2. *If F is C^1 and right-antidifferentiable with respect to Z, and Z is self-commuting, C^2 and has invertible Z_x or Z_y, then F is right-differentiable with respect to Z.*

Proof. Suppose $(dZ)F = dH$, or

$$Z_x F = H_x \quad \text{and} \quad Z_y F = H_y.$$

Differentiating, we have

$$Z_{xy}F + Z_x F_y = H_{xy} \quad \text{and} \quad Z_{yx}F + Z_y F_x = H_{yx},$$

whence follows $Z_x F_y = Z_y F_x$. Multiplying $(Z_x)^{-1}$, we obtain

$$F_y = (Z_x)^{-1} Z_y F_x.$$

Hence

$$dF = F_x\,dx + (Z_x)^{-1}Z_yF_x\,dy = (I\,dx + Z_x^{-1}Z_y\,dy)F_x$$
$$= (Z_x)^{-1}(dZ)F_x = (dZ)(Z_x^{-1}F_x).$$

THEOREM 4. *If F is C^k, $k \geq 1$, and right-differentiable with respect to Z, and Z is self-commuting, C^k and has invertible Z_x or Z_y, then F is k times right-differentiable with respect to Z.*

Proof. The case $k = 1$ is trivial. So suppose $k \geq 2$. Let $dF = (dZ)F'$; then $F_x = Z_xF'$ and F' is C^{k-1} with $k - 1 \geq 1$. Hence by Lemma 2 F' is Z-differentiable. $dF' = (dZ)F''$ and $F'_x = Z_xF''$, so F'' is C^{k-2}. Continuing thus, we reach $F^{(k)}$, which is $C^{k-k} = C^0$.

We rely on integration-by-parts to derive our Taylor formula. There are actually two parts to this technique, which we formulate separately as Propositions 3 and 4.

PROPOSITION 3. *For F and G of class C^1 in Ω, and any path of integration γ from (x_0, y_0) to (x_1, y_1) in Ω, we have*

$$\int_\gamma (dF)G = FG\Big|_{(x_0,y_0)}^{(x_1,y_1)} + \int_\gamma (-F)\,dG. \tag{1.6}$$

Thus, either both integrals are independent of paths connecting (x_0, y_0) to (x_1, y_1), or neither is.

Proof. Since $d(FG) = (dF)G + F(dG)$,

$$\int_\gamma d(FG) = \int_\gamma (dF)G + \int_\gamma F(dG).$$

Hence,

$$\int_\gamma (dF)G = \int_{(x_0,y_0)}^{(x_1,y_1)} d(FG) - \int_\gamma F(dG).$$

Needless to say, we have likewise

$$\int_\gamma F(dG) = FG\Big|_{(x_0,y_0)}^{(x_1,y_1)} + \int_\gamma dF(-G). \tag{1.7}$$

PROPOSITION 4. *If F is left-differentiable, and G is right-antidifferentiable, both with respect to some Z, then*

$$\int_\gamma (dF)G = \int_\gamma F'(dG^{\#}), \tag{1.8}$$

where F' is the left-derivative, and $G^{\#}$ is a right-antiderivative with respect to Z.

Proof. The formula follows from

$$(dF)G = (F'dZ)G = F'(dZG) = F'dG^{\#}.$$

Again needless to say, we have likewise, under suitable assumptions on F and G,

$$\int_\gamma F(dG) = \int_\gamma (dF^{\#})G'. \tag{1.9}$$

We now attempt to approximate a given F by powers of some self-commuting Z. For simplicity of notation we shall use z to represent the point (x, y) without thinking of z as a complex number. The line integral $\int_\gamma (dF)G$ may be more fully expressed as $\int_\gamma dF(z)G(z)$ or $\int_\gamma dF(\tilde{z})G(\tilde{z})$ with $\tilde{z}$ emphasizing its being a variable of integration. The use of dummy variable $\tilde{z}$ is especially appropriate when we have to consider, for example, $\int_{z_0}^{z} dF(\tilde{z})G(\tilde{z}, z)$, in which G depends on $\tilde{z}$ as well as a fixed z. The upper and lower limits of integration appear only when the line integral is independent of the paths connecting z_0 to z.

THEOREM 5 (*Taylor formula*). *Let F be $(k+1)$-times right-differentiable with respect to a self-commuting Z in Ω. Let z_0 be a fixed point in Ω; then for any z in Ω, we have*

$$\begin{aligned} F(z) = &\sum_{j=0}^{k} [(Z - Z_0)^j / j!] F^{(j)}(z_0) \\ &+ \int_{z_0}^{z} d[-(Z - \widetilde{Z})^{k+1}/(k+1)!] F^{(k+1)}(\tilde{z}), \end{aligned} \tag{1.10}$$

where $\widetilde{Z} = Z(\tilde{z})$, $Z_0 = Z(z_0)$, and the line integral is taken along any path of integration connecting z_0 to z in Ω.

Proof. We give a straightforward derivation, using the integration-by-parts formulas. All the line integrals are independent of the paths of integration since the first one clearly is

$$
\begin{aligned}
F(z)-F(z_0) &= \int_{z_0}^{z} I\,dF(\tilde{z}) = \int_{z_0}^{z} dI^{\#}F'(\tilde{z}) \quad \text{by (1.9)} \\
&= \int_{z_0}^{z} d[-(Z-\widetilde{Z})]F'(\tilde{z}) \quad \text{where we picked } \widetilde{Z}-Z \text{ for } I^{\#} \\
&= [-(Z-\widetilde{Z})F'(\tilde{z})]_{z_0}^{z} + \int_{z_0}^{z} (Z-\widetilde{Z})\,dF'(\tilde{z}) \quad \text{by (1.6)} \\
&= (Z-Z_0)F'(z_0) + \int_{z_0}^{z} d(Z-\widetilde{Z})^{\#}F''(\tilde{z}) \quad \text{again by (1.9).}
\end{aligned}
$$

But now $(Z-\widetilde{Z})^{\#}$ may be chosen to be $[-(Z-\widetilde{Z})^2/2!]$ in view of (1.4) of Proposition 2, and this is where we need the self-commuting property of Z. Thus we have shown the Taylor formula for $k=1$. By repeated applications of formulas (1.6) and (1.9) we eventually arrive at the formula (1.10). One can also write out a formal inductive proof.

We state the following Leibniz formula, which will be needed later (in Theorems 7 and 9 below).

PROPOSITION 5 (*Leibniz formula*). *If $G(z,\tilde{z})$ and $H(z)$ are C^1 in $\Omega\times\Omega$ and Ω respectively, then for any z_0 and z in Ω, we have*

$$
\frac{\partial}{\partial x}\int_{z_0}^{z} G(z,\tilde{z})\,dH(\tilde{z}) = \int_{z_0}^{z} G_x(z,\tilde{z})\,dH(\tilde{z}) + G(z,z)H_x(z), \tag{1.11}
$$

and ditto for $\partial/\partial y$, provided that the line integral on the left is independent of the paths of integration from z_0 to z.

Proof. We need only add the following equalities:

$$
\begin{aligned}
\frac{\partial}{\partial x}\int_{z_0}^{z} G(z,\tilde{z})H_x(\tilde{z})\,d\tilde{x} &= \int_{z_0}^{z} G_x(z,\tilde{z})H_x(\tilde{z})\,d\tilde{x} + G(z,z)H_x(z), \\
\frac{\partial}{\partial x}\int_{z_0}^{z} G(z,\tilde{z})H_y(\tilde{z})\,d\tilde{y} &= \int_{z_0}^{z} G_x(z,\tilde{z})H_y(\tilde{z})\,d\tilde{y} + 0.
\end{aligned}
$$

Both of these follow from the Leibniz formula for scalar-valued functions.

2. Hyperholomorphic functions. Let M be a constant square matrix. A matrix-valued function F is said to be *M-holomorphic* in a domain if it belongs to C^1 and satisfies the first order system

$$
F_y = MF_x \tag{2.1}
$$

in the domain. Among all the M-holomorphic functions for a given M, the key role is played by

$$Z = xI + yM, \tag{2.2}$$

which is clearly self-commuting and satisfies (2.1). It turns out that M-holomorphicity is equivalent to Z-differentiability (see Theorem 6 below). This allows us to apply results of the last section to M-holomorphic functions.

Following complex analysis, we say that a matrix-valued function F is *M-analytic* (or *Z-analytic*) at the origin if it has a series expansion in powers of Z:

$$F = \sum_{j=0}^{\infty}(Z^j/j!)A_j \tag{2.3}$$

in an open disk around the origin, where A_j are constant matrices having the same number of columns as F.

Clearly, if F is M-analytic, it is M-holomorphic as termwise differentiation of (2.3) will verify (2.1). However, the converse is not true (see Theorem 7 below), and here we part company with complex analysis.

THEOREM 6. *F is M-holomorphic if and only if F is differentiable with respect to $Z = xI + yM$. The derivative F' is equal to F_x.*

Proof. If F is M-holomorphic,

$$\begin{aligned} dF &= F_x\,dx + F_y\,dy = F_x\,dx + MF_x\,dy \\ &= (I\,dx + M\,dy)F_x = (dZ)F_x. \end{aligned}$$

Thus, F is Z-differentiable with the derivative equal to F_x, also to be denoted by $D^{(1,0)}F$.

Conversely, if F is Z-differentiable,

$$dF = dZF'$$

so that $F_x = Z_xF' = IF'$ and $F_y = Z_yF' = MF'$. Consequently $F_y = MF_x$, and F is M-holomorphic.

If Z is self-commuting, then Z^n are Z-differentiable by Proposition 2, and hence we have

COROLLARY 6a. *Z^n are M-holomorphic for all integers n..*

COROLLARY 6b. *F is M-holomorphic and C^k if and only if F has a continuous kth order Z-derivative $F^{(k)}$.*

Proof. First assume F to be C^k and M-holomorphic, hence Z-differentiable by Theorem 6. Then in view of Theorem 4, since Z is trivially C^k and Z_x trivially invertible, we have $F^{(k)} = D^{(k,0)}F$, which is continuous because F is C^k.

Conversely, if a continuous $F^{(k)}$ exists, F' easily exists, and so F is M-holomorphic by Theorem 6. To show F is C^k, using the continuity of $F^{(k)} = D^{(k,0)}F$, we see

$$D^{(k-j,j)}F = M^j D^{(k,0)}F$$

are all continuous for $0 \le j \le k$. In other words, F is C^k.

COROLLARY 6c (*Taylor formula*). *If F is M-holomorphic and C^k in a neighborhood around the origin, then*

$$F(z) = \sum_{j=0}^{k-1}(Z^j/j!)F^{(j)}(0) + \int_0^z d[-(Z-\widetilde{Z})^k/k!]F^{(k)}(\tilde{z}). \tag{2.4}$$

The proof follows from Theorem 5 and Corollary 6b.

THEOREM 7 (*Taylor expansion*). *F is M-holomorphic and real analytic at the origin if and only if F is M-analytic at the origin with*

$$F(z) = \sum_{j=0}^{\infty}(Z^j/j!)F^{(j)}(0). \tag{2.5}$$

Proof. First suppose F is M-holomorphic and real analytic at the origin; then in an open disk around the origin we have

$$F(z) = \sum_{j=0}^{\infty} F_j(z), \tag{2.6}$$

where F_j is a matrix consisting of jth degree homogeneous polynomials in x and y and possibly also of zero polynomials. On the other hand we also have the Taylor formula (see Corollary 6c above):

$$\begin{aligned} F(z) = {}& \sum_{j=0}^{k}(Z^j/j!)F^{(j)}(0) \\ &= \int_0^z d[-(Z-\widetilde{Z})^{k+1}/(k+1)!]F^{(k+1)}(\tilde{z}) \quad \text{for } k \ge 0, \end{aligned} \tag{2.7}$$

where we used the fact that F is infinitely many times Z-differentiable in view of Corollary 6b. By combining (2.6) and (2.7) we now show inductively

$$F_j(z) = (Z^j/j!)F^{(j)}(0) \quad \text{for } j \geq 0. \tag{2.8}$$

Letting $z = 0$ in (2.6) and (2.7), we see $F_0 = F(0)$ since all homogeneous polynomials of degree one or higher vanish at $x = y = 0$. Next, assume as induction hypothesis

$$F_j = (Z^j/j!)F^{(j)}(0) \quad \text{for } 0 \leq j \leq k-1. \tag{2.9}$$

Now (2.6), (2.7) and (2.9) imply

$$F_k + \sum_{j=k+1}^{\infty} F_j = (Z^k/k!)F^{(k)}(0) + \int_0^z [(Z - \tilde{Z})^k/k!]\, dF^{(k)}(\tilde{z}), \tag{2.10}$$

where we wrote the integral term in an alternative form via integration-by-parts formula (1.8) in order to apply the Leibniz formula (1.11). Differentiating both sides of (2.10) by applying $\partial^k/\partial x^i \partial y^{k-i} = D^{(i,k-i)}$, for $0 \leq i \leq k$, we obtain

$$\begin{aligned} D^{(i,k-i)}F_k + \sum_{j=k+1}^{\infty} D^{(i,k-i)}F_j &= D^{(i,k-i)}(Z^k/k!)F^{(k)}(0) \\ &\quad + \int_0^z D^{(i,k-i)}[(Z - \tilde{Z})^k/k!]\, dF^{(k)}(\tilde{z}). \end{aligned} \tag{2.11}$$

Note that in applying the Leibniz formula (1.11) the term $G(z, z) = (Z - Z)^k/k!$ vanishes. Setting $z = 0$ in (2.11), we obtain

$$D^{(i,k-i)}F_k = D^{(i,k-i)}(Z^k/k!)F^{(k)}(0) \quad \text{for } 0 \leq i \leq k \tag{2.12}$$

since all homogeneous polynomials of degree one or higher vanish at $z = 0$, and so does the line integral. Having all the kth order partial derivatives equal by (2.12), F_k and $(Z^k/k!)F^{(k)}(0)$ can differ only by a polynomial of degree at most $k - 1$. But since F_k and $(Z^k/k!)F^{(k)}(0)$ are both homogeneous of degree k, they must be equal.

$$F_k = (Z^k/k!)F^{(k)}(0). \tag{2.13}$$

This proves the first half of the theorem.

To prove the converse, assume F is M-analytic at the origin. We have then

$$F = \sum_{j=0}^{\infty} (Z^j / j!) A_j, \tag{2.14}$$

where it is easily checked that $A_j = F^{(j)}(0)$, and (2.14) easily implies that F is real analytic since each component of F has a power series expansion. To see F is M-holomorphic, we differentiate (2.14) termwise to show $F_y = MF_x$. This completes the proof of the theorem.

We shall call an M-holomorphic column-vector-valued function an *M-conjugation*. Note that if F is M-holomorphic, then each column of F is an M-conjugation since equation (2.1) can be split into as many equations as there are columns in F. We shall refer to every M-conjugation as an M-conjugation of its first component, and all lower components as *M-conjugates* of the first component. The existence of such "hyperconjugates" is important if we are to apply any theory of hyperholomorphic functions to solutions of higher order partial differential equations.

3. Partial differential equations. We consider equations in the (x, y)-plane of the form

$$a_0 \frac{\partial^m u}{\partial x^m} + a_1 \frac{\partial^m u}{\partial x^{m-1} \partial y} + \cdots + a_{m-1} \frac{\partial^m u}{\partial x \, \partial y^{m-1}} + \frac{\partial^m u}{\partial y^m} = 0 \tag{3.1}$$

where $m \geq 2$ and the coefficients $a_0, a_1, \ldots, a_{m-1}$ are real or complex constants, with the last coefficient a_m normalized as 1. Letting

$$L = a_0 D^{(m,0)} + a_1 D^{(m-1,1)} + \cdots + D^{(0,m)},$$

we condense (3.1) to $Lu = 0$, and refer to u loosely as a "solution" of L. Letting $A = (a_0, a_1, \ldots, a_{m-1}, 1)$, we can also write (3.1) as

$$A \nabla^{(m)} u = 0, \tag{3.1a}$$

where the column vector $\nabla^{(m)} u$ is the mth order *hypergradient* of u. The above equation in $\nabla^{(m)} u$ can be further rewritten as a first order equation in $\nabla^{(m-1)} u$, namely

$$(a_0, \ldots, a_{m-1})[\nabla^{(m-1)} u]_x + (0, \ldots, 0, 1)[\nabla^{(m-1)} u]_y = 0. \tag{3.1b}$$

This last equation will be put into a first order system satisfied by $\nabla^{(m-1)} u$ (see Theorem 8 below).

In elementary calculus one learns to recover a function from its gradient via a line integral, namely

$$u(x,y) = u(x_0,y_0) + \int_{(x_0,y_0)}^{(x,y)} u_x(\tilde{x},\tilde{y})\,d\tilde{x} + u_y(\tilde{x},\tilde{y})\,d\tilde{y},$$

which can be written more concisely as

$$u(z) = u(z_0) + \int_{z_0}^{z} d\tilde{z}\nabla u(\tilde{z}),$$

or somewhat artificially as

$$u(z) = u(z_0) + \int_{z_0}^{z} d[-(z-\tilde{z})]\nabla u(\tilde{z}).$$

It turns out that this last formula can be generalized so that one can also recover u from its hypergradient $\nabla^{(m)}u$. We state the formula for $m = 2$, and explain the notations.

$$\begin{aligned} u(z) = u(z_0) &+ (z-z_0)\nabla u(z_0) \\ &+ \int_{z_0}^{z} d[-(z-\tilde{z})^2/2!]\nabla^{(2)}u(\tilde{z}). \end{aligned}$$

Here z is not meant to be a complex number, but rather a "hypernumber" obeying the following conventions: $z = (x,y)$ $z^2 = (x^2, 2xy, y^2)$, $z^3 = (x^3, 3x^2y, 3xy^2, y^3)$, and so on; also

$$\begin{aligned} (z-z_0)^2 &= (x-x_0, y-y_0)^2 \\ &= ((x-x_0)^2, 2(x-x_0)(y-y_0), (y-y_0)^2), \\ d(z-\tilde{z})^2 &= (d(x-\tilde{x})^2, 2d(x-\tilde{x})(y-\tilde{y}), d(y-\tilde{y})^2). \end{aligned}$$

The general recovery formula, proved in **[9]**, is as follows:

$$\begin{aligned} u(z) = &\sum_{j=0}^{m-1}[(z-z_0)^j/j!]\nabla^{(j)}u(z_0) \\ &+ \int_{z_0}^{z} d[-(z-\tilde{z})^m/m!]\nabla^{(m)}u(\tilde{z}). \end{aligned} \tag{3.1c}$$

For a later application we will write this formula in yet another form. Let X be an infinite square matrix whose entries are all zero except the diagonal entries consisting of x's and the supradiagonal entries consisting of y's, so that the first few rows of X look like

$$E_1X = (x, y, 0, 0, \dots),$$
$$E_2X = (0, x, y, 0, \dots),$$

where E_i shall always denote the ith unit row vector of appropriate dimension determined by the context in which it appears. We note that the power X^j has rows each of which is essentially a copy of z^j. For example, for $j = 2$ we have

$$\begin{aligned} E_1X^2 &= (x^2, 2xy, y^2, 0, 0, 0, \dots), \\ E_2X^2 &= (0, x^2, 2xy, y^2, 0, 0, \dots), \\ E_3X^2 &= (0, 0, x^2, 2xy, y^2, 0, \dots). \end{aligned}$$

Using E_1X^2 in the place of z^2, we would like to rewrite

$$z^2\nabla^{(2)}u(z_0) \quad \text{as} \quad E_1X^2\nabla^{(2)}u(z_0)$$

except that on the right $\nabla^{(2)}u(z_0)$ as a column vector is too short to match the row vector E_1X^2. Therefore, henceforth whenever $\nabla^{(m)}u$ takes part in a matrix multiplication, we shall automatically assume $\nabla^{(m)}u$ to have been extended to an appropriate length by addition of as many 0's as necessary. With these notational agreements we can now rewrite our recovery formula (3.1c), where for simplicity we take $z_0 = 0$, as follows:

$$\text{(3.1d)} \qquad \begin{aligned} u(z) = E_1 &\sum_{j=0}^{m-1} [X^j/j!]\nabla^{(j)}u(0) \\ &+ E_1 \int_0^z d[-(X - \tilde{X})^m/m!]\nabla^{(m)}u(\tilde{z}). \end{aligned}$$

We now go on to Theorem 8 mentioned after the equation (3.1b).

THEOREM 8. *If u is C^m and $Lu = 0$, then $\nabla^{(m-1)}u$ is C^1 and M-holomorphic, $[\nabla^{(m-1)}u]_y = M[\nabla^{(m-1)}u]_x$, where M is the $m \times m$* associated matrix *of L, consisting of 0's everywhere except the supradiagonal consisting of 1's and the bottom row consisting of $-a_0, -a_1, \dots, -a_{m-1}$ (see M below in the proof).*

Conversely, if an $m \times 1$ column vector $\mathbf{f}$ is C^1 and M-holomorphic in a simply-connected domain, i.e.,

$$\text{(3.2)} \qquad \mathbf{f}_y = M\mathbf{f}_x,$$

then there exists a C^m solution $Lu = 0$ such that $\nabla^{(m-1)}u = \mathbf{f}$. If *furthermore* $\mathbf{f}$ *is* C^k, $k \geq m$, *then every component of* $\mathbf{f}$ *is a* C^k *solution of* L.

Proof. To see $\nabla^{(m-1)}u$ is M-holomorphic we need to check

(3.3)

$$\begin{pmatrix} u_{xx\ldots x} \\ u_{xx\ldots y} \\ \vdots \\ u_{yy\ldots y} \end{pmatrix}_y = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & . & . & \ldots & . \\ . & . & . & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{m-1} \end{pmatrix} \begin{pmatrix} u_{xx\ldots x} \\ u_{xx\ldots y} \\ \vdots \\ u_{yy\ldots y} \end{pmatrix}_x .$$

But the bottom row of (3.3) is just a restatement of $Lu = 0$, and the upper rows merely state the well-known equalities of mixed partial derivatives under the sufficient smoothness condition C^m.

Conversely, if $\mathbf{f}$ satisfies (3.2), the upper rows of (3.2) imply compatibility among the components of $\mathbf{f}$ so that under the assumptions of C^1 and simply-connectedness of domain $\mathbf{f}$ is guaranteed to be "hyperexact", namely $\mathbf{f} = \nabla^{(m-1)}u$ for some u (see [9]). The substitution of $\mathbf{f} = \nabla^{(m-1)}u$ in (3.2) shows $Lu = 0$ from the bottom row of (3.2). If furthermore $\mathbf{f}$ happens to be at least C^m, then

$$L\mathbf{f} = L(\nabla^{(m-1)}u) = \nabla^{(m-1)}(Lu) = 0,$$

where L and $\nabla^{(m-1)}$ commute because u is C^{2m-1}. This completes the proof.

According to the last statement of the theorem just proved, if $\mathbf{f}$ is C^k, $k \geq m$, and M-holomorphic where M is the associated matrix of L, then every component of $\mathbf{f}$ is a C^k solution of L. Can we have *all* the C^k solutions of L by merely looking into the components of all the C^k M-holomorphic $\mathbf{f}$? The following theorem guarantees this. In fact, it turns out that we need only look at just the first component of $\mathbf{f}$. We see, therefore, that every C^k solution of $Lu = 0$ has a C^k M-conjugation, $k \geq m$. The proof is a bit cumbersome.

THEOREM 9. *If* u *is* C^k, $k \geq m$, *and* $Lu = 0$ *in an open disk around the origin, then there exists an* $\mathbf{f}$ *also* C^k *with* $\mathbf{f}_y = M\mathbf{f}_x$ *such that* $u = f_1$ *where* f_1 *is the first component of* $\mathbf{f}$ *and* M *is the associated matrix of* L. *For a real analytic* u, f *will be real analytic.*

Proof. With $Z = xI + yM$ we claim that $\mathbf{f}$ defined below satisfies the requirement of the theorem.

$$\mathbf{f}(z) = \sum_{j=0}^{m-2} (Z^j/j!)\nabla^{(j)}u(0) + \int_0^z d[-(Z-\widetilde{Z})^{m-1}/(m-1)!]\nabla^{(m-1)}u(\tilde{z}). \tag{3.4}$$

Note that we are regarding $\nabla^{(j)}u(0)$ as an $m \times 1$ column through the a forementioned convention of appending $m-j$ zeros if $j \leq m-2$. We point our however that if constants other than 0's are used, we would end up with other M-conjugations of u. The fact that M-conjugations of u are not unique is not altogether unexpected. We justify our claim by checking the following four points.

First. The line integral in (3.4) is independent of the path going from 0 to z. Referring to Proposition 1 we need only show the equality

$$[-(Z-\widetilde{Z})^{m-1}/(m-1)!]_{\tilde{x}}[\nabla^{(m-1)}\tilde{u}]_{\tilde{y}} = [-(Z-\widetilde{Z})^{m-1}/(m-1)!]_{\tilde{y}}[\nabla^{(m-1)}\tilde{u}]_{\tilde{x}},$$

which in view of (3.3) is equivalent to the identity

$$[(Z-\widetilde{Z})^{m-2}I]M[\nabla^{(m-1)}\tilde{u}]_{\tilde{x}} = [(Z-\widetilde{Z})^{m-2}M][\nabla^{(m-1)}\tilde{u}]_{\tilde{x}}.$$

Second. $f_1 = u$. To see this, we note from (3.4)

$$f_1(z) = E_1 \sum_{j=0}^{m-2} [Z^j/j!]\nabla^{(j)}u(0) + E_1 \int_0^z d[-(Z-\widetilde{Z})^{m-1}/(m-1)!]\nabla^{(m-1)}u(\tilde{z}).$$

We also note from (3.1d)

$$u(z) = E_1 \sum_{j=0}^{m-2} [X^j/j!]\nabla^{(j)}u(0) + E_1 \int_0^z d[-(X-\widetilde{X})^{m-1}/(m-1)!]\nabla^{(m-1)}u(\tilde{z}).$$

If we write out the matrices $Z, Z^2, \dots, Z^{m-1}$ (see also Example 1 below), and compare their first rows with those of $X, X^2, \dots, X^{m-1}$, then it becomes clear that the two expressions for $f_1(z)$ and $u(z)$ above are equal.

Third. If u is C^k or real analytic, so is $\mathbf{f}$. To see this we need to differentiate the integral term, call it $R(z)$, in (3.4). But in order to apply the differentiation formula (1.11) we first rewrite $R(z)$ using the integration-by-parts formula (1.6):

$$(3.5)\qquad R(z) = [Z^{m-1}/(m-1)!]\nabla^{(m-1)}u(0) + \int_0^z [(Z-\tilde{Z})^{m-1}/(m-1)!]\, d[\nabla^{(m-1)}u(\tilde{z})].$$

Differentiating according to (1.11), we have for $0 \le j \le m-1$

$$\begin{aligned} D^{(m-1-j,j)} &\int_0^z [(Z-\tilde{Z})^{m-1}/(m-1)!]\, d[\nabla^{(m-1)}u(\tilde{z})] \\ &= \int_0^z M^j\, d[\nabla^{(m-1)}u(\tilde{z})] \\ &= M^j[\nabla^{(m-1)}u(z) - \nabla^{(m-1)}u(0)]. \end{aligned}$$

Now if u is C^k, $\nabla^{(m-1)}u$ is $C^{k-(m-1)}$, which makes **f** $C^{k-(m-1)+(m-1)} = C^k$. Clearly, if u is real analytic, so is **f**.

Fourth. $\mathbf{f}_y = M\mathbf{f}_x$. Since $Z, Z^2, \ldots, Z^{m-1}$ are all M-holomorphic by Corollary 6a, we need only concentrate on the integral term, call it $\widehat{R}(z)$, in (3.5). Using the formula (1.11), we see

$$\begin{aligned} [\widehat{R}(z)]_y &= \int_0^z [(Z-\tilde{Z})^{m-2}/(m-2)!]M\, d[\nabla^{(m-1)}u(\tilde{z})] \\ &= M\int_0^z [(Z-\tilde{Z})^{m-2}/(m-2)!]I\, d[\nabla^{(m-1)}u(\tilde{z})] \\ &= M[\widehat{R}(z)]_x. \end{aligned}$$

This completes the proof of Theorem 9.

THEOREM 10. *If u is real analytic at the origin, and $Lu = 0$, then u has the following series expansion in an open disk around the origin.*

$$u(z) = E_1 \sum_{j=0}^{m-1} (Z^j/j!)\nabla^{(j)}u(0) + E_1 \sum_{k=0}^{\infty} (Z^{m+k}/(m+k)!)D^{(k+1,0)}\nabla^{(m-1)}u(0)$$

where $E_1 = (1, 0, 0, \ldots, 0)$, $Z = xI + yM$, and M is the associated matrix of L.

Proof. Since u is a real analytic solution of L, by Theorem 9 it has a real analytic M-conjugation **f**, which by Theorem 7 has an expansion in powers of Z,

$$\mathbf{f}(z) = \sum_{j=0}^{\infty}(Z^j/j!)\mathbf{f}^{(j)}(0) \tag{3.6}$$

where $\mathbf{f}^{(j)}$ is the jth derivative of $\mathbf{f}$ with respect to Z and has the expression $D^{(j,0)}\mathbf{f}$ (see Theorem 6). Taking only the top row of (3.6), we have

$$u(z) = E_1\sum_{j=0}^{\infty}(Z^j/j!)\mathbf{f}^{(j)}(0). \tag{3.7}$$

Note that all the entries of every Z^j are solutions of L (see the last statement of Theorem 8), and hence (3.7) may be considered as the rearrangement of the ordinary Taylor series of u, which is in terms of powers of x and y, into one which is in terms of polynomial solutions of u. However these two series are distinct only from mth degree terms onward. For $j \leq m-1$ the top of Z^j consists merely of those powers in x and y that appear in the binomial expansion of $(x+y)^j$. In other words we must have

$$E_1[(Z^j/j!)\mathbf{f}^{(j)}(0)] = E_1[(Z^j/j!)\nabla^{(j)}u(0)]$$

for $0 \leq j \leq m-1$. For higher degree terms we will show, for $0 \leq k < \infty$,

$$\begin{aligned} &E_1[(Z^{m+k}/(m+k)!)\mathbf{f}^{(m+k)}(0)] \\ &\quad = E_1[(Z^{m+k}/(m+k)!)D^{k+1,0)}\nabla^{(m-1)}u(0)]. \end{aligned}$$

Now since $\mathbf{f}$ is M-holomorphic, we have by Theorem 6

$$\mathbf{f}^{(m+k)} = D^{(m+k,0)}\mathbf{f}.$$

Also since $\mathbf{f}$ is M-holomorphic, there exists ϕ by Theorem 8 such that

$$\mathbf{f} = \nabla^{(m-1)}\phi.$$

Consequently,

$$\begin{aligned} \mathbf{f}^{(m+k)} &= D^{(m+k,0)}\nabla^{(m-1)}\phi \\ &= D^{(k+1,0)}D^{(m-1,0)}\nabla^{(m-1)}\phi \\ &= D^{(k+1,0)}\nabla^{(m-1)}D^{(m-1,0)}\phi \\ &= D^{(k+1,0)}\nabla^{(m-1)}E_1\nabla^{(m-1)}\phi \\ &= D^{(k+1,0)}\nabla^{(m-1)}E_1\mathbf{f} = D^{(k+1,0)}\nabla^{(m-1)}u \end{aligned}$$

which completes the proof.

THEOREM 11. *The totality of real analytic solutions of $Lu = 0$ at the origin is given by*

$$u = E_1 \sum_{k=0}^{\infty} Z^k \mathbf{c}_k \tag{3.8}$$

where $\mathbf{c}_k$'s are any $m \times 1$ constant column vectors for which the series converges within a certain radius of convergence. $Z = xI + yM$, and M is the associated matrix of L.

Proof. First it is clear from Theorem 10 that every real analytic solution of $Lu = 0$ at the origin is of the form (3.8). Next clearly (3.8) is a solution of $Lu = 0$ within the radius of convergence, for termwise differentiations give

$$Lu = E_1 \sum_{k=0}^{\infty} L(Z^k \mathbf{c}_k) = 0.$$

To check $L(Z^k \mathbf{c}_k) = 0$, note that Z^k is M-holomorphic by Corollary 6a, hence every column of Z^k is also M-holomorphic, and so is their linear combination $Z^k \mathbf{c}_k$. Thus $Z^k \mathbf{c}_k$ is an M-conjugation, and consequently $L(Z^k \mathbf{c}_k) = 0$ by the last statement of Theorem 8. This completes the proof.

THEOREM 12. *For $k \geq m$ the m polynomials appearing in the top row of Z^k constitute a basis for the kth degree homogeneous polynomial solutions of $Lu = 0$.*

Proof. In view of Theorem 11 we need only show the linear independence of polynomials in $E_1 Z^k$. We proceed by induction. First note that polynomials in $E_1 Z^{m-1}$ are just the terms in the binomial expansion $(x+y)^{m-1}$, and they are easily shown to be linearly independent. Next, assuming polynomials in $E_1 Z^k$ are independent, we show the polynomials in $E_1 Z^{k+1}$ are independent. So suppose

$$E_1 Z^{k+1} \mathbf{c}_{k+1} = 0$$

for some constant column vector $\mathbf{c}_{k+1}$. Differentiating with respect to x, we have

$$E_1 (k+1) Z^k \mathbf{c}_{k+1} = 0,$$

which implies by the induction hypothesis that $\mathbf{c}_{k+1} = 0$.

Note that for $k < m$ all kth degree homogeneous polynomials are trivially solutions of $Lu = 0$, and the terms in the binomial expansion $(x+y)^k$ constitute a basis.

We exhibit polynomial bases for real analytic solutions of some familiar partial differential equations.

EXAMPLE 1. Consider the biharmonic equation

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0.$$

Here we have

$$L = D^{(4,0)} + 0D^{(3,1)} + 2D^{(2,2)} + 0D^{(1,3)} + D^{(0,4)},$$

and accordingly,

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{pmatrix}$$

and since $Z = xI + yM$,

$$Z = \begin{pmatrix} x & y & 0 & 0 \\ 0 & x & y & 0 \\ 0 & 0 & x & y \\ -y & 0 & -2y & x \end{pmatrix}.$$

The top rows of Z^0, Z^1, Z^2, Z^3 produce lower degree polynomials, which trivially satisfy the 4th order biharmonic equation. From Z^4 on we begin to obtain all the nontrivial biharmonic polynomials. These polynomials form a basis for the real analytic solutions. We list these polynomials up to degree 5.

$k = 0$	1	0	0	0
$k = 1$	x	y	0	0
$k = 2$	x^2	$2xy$	y^2	0
$k = 3$	x^3	$3x^2y$	$3xy^2$	y^3
$k = 4$	$x^4 - y^4$	$4x^3y$	$6x^2y^2 - 2y^4$	$4xy^3$
$k = 5$	$x^5 - 5xy^4$	$5x^4y - y^5$	$10x^3y^2 - 10xy^4$	$10x^2y^3 - 2y^5$

EXAMPLE 2. For the wave equation $u_{xx} - u_{yy} = 0$, the top rows of Z^k for $0 \leq k \leq 5$ are

$k = 0$	1	0
$k = 1$	x	y
$k = 2$	$2xy$	$x^2 + y^2$
$k = 3$	$x^3 + 3xy^2$	$3x^2y + y^3$
$k = 4$	$4x^3y + 4xy^3$	$x^4 + 6x^2y^2 + y^4$
$k = 5$	$x^5 + 10x^3y^2 + 5xy^4$	$5x^4y + 10x^2y^3 + y^5$

EXAMPLE 3. For the Laplace equation $u_{xx} + u_{yy} = 0$, the top rows of Z^k for $0 \leq k \leq 5$ are

$k=0$	1	0
$k=1$	x	y
$k=2$	$x^2 - y^2$	$2xy$
$k=3$	$x^3 - 3xy^2$	$3x^2y - y^3$
$k=4$	$x^4 - 6x^2y^2 + y^4$	$4x^3y - 4xy^3$
$k=5$	$x^5 - 10x^3y^2 + 5xy^4$	$5x^4y - 10x^2y^3 + y^5$

The examples above, especially Example 1, demonstrate our simple but quite universal algorithm for obtaining all the polynomial solutions of all the equations of the type (3.1): From the coefficients in (3.1) we construct the square matrix M as shown in (3.3), thence the "generating" matrix $Z = xI + yM$; we then find all the basic kth degree homogeneous polynomial solutions of (3.1) in the top row of the matrix Z^k.

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