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NORMAL STRUCTURE IN BOCHNER L^p-SPACES

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It is shown that, for $1 < p < \infty$, the Bochner L^p-space $L^p(\mu, X)$ has normal structure exactly when X has normal structure. With this result, normal structure in Bochner L^p-spaces is completely characterized except in one seemingly simple setting.

The concept of normal structure, a geometric property of sets in normed linear spaces, was introduced in 1948 by M. S. Brodskiĭ and D. P. Mil′man in order to study the existence of common fixed points of certain sets of isometries. Since then, normal structure has been studied both as a purely geometric property of normed linear spaces and as a tool in fixed point theory [**1**, **4**, **5**, **6**]. In 1968, L. P. Belluce, W. A. Kirk, and E. F. Steiner [**1**] proved that the l^∞-direct sum of two normed linear spaces with normal structure has normal structure. They were not however able to decide if normal structure is preserved under an l^p-direct sum of two normed linear spaces for $1 \leq p < \infty$. In 1984, T. Landes [**5**] proved that, if $1 < p < \infty$, normal structure is preserved under finite or infinite l^p-direct sums. In this article, the corresponding theorem is proven in the nondiscrete setting; consequently it is shown that, if (Ω, Σ, μ) is any measure space and $1 < p < \infty$, the Bochner L^p-space $L^p(\mu, X)$ has normal structure exactly when X has normal structure.

A normed linear space X has *normal structure* if, for each closed bounded convex set K in X that contains more than one point, there is a point p in K such that $\sup\{\|p - x\|: x \in K\}$ is less than the diameter of K; such a point p is called a *nondiametral point* in K. Brodskiĭ and Mil′man [**2**] proved that a space X fails to have normal structure if and only if there is a nonconstant bounded sequence (x_n) in X such that the distance from x_{n+1} to the convex hull of $\{x_1, \ldots, x_n\}$ tends to the diameter of the set $\{x_k: k \in N\}$; such a sequence is called a *diametral sequence* in X. One consequence of this fact is that normal structure is a separably-determined property. Several more conditions equivalent to normal structure are given in [**5**] and in Lemma 3 below.

The notation used in this paper is, with perhaps two exceptions, standard. The two exceptions are as follows: if (x_n) is a sequence in a normed linear space X, $\overline{x}_n$ will denote $(x_1 + \cdots + x_n)/n$ and $\Delta\Sigma_n^p$ will denote

$$\sum_{j=1}^{n} \|x_{n+1} - x_j\|^p - n \left\| \frac{1}{n} \sum_{j=1}^{n} (x_{n+1} - x_j) \right\|^p .$$

Note that the convexity of $\|\cdot\|^p$ implies that $\Delta\Sigma_n^p \geq 0$.

The first lemma is probably well-known.

LEMMA 1. *Let $\varepsilon > 0$ and let $x_1, \ldots, x_n$, x_{n+1} be elements in a normed linear space X such that*

$$\left\| x_{n+1} - \frac{1}{n} \sum_{j=1}^{n} x_j \right\| > \operatorname{diam}\{x_1, \ldots, x_{n+1}\} - \varepsilon.$$

Then $\|x_{n+1} - y\| > \operatorname{diam}\{x_1, \ldots, x_{n+1}\} - n\varepsilon$ for each y in the convex hull of $\{x_1, \ldots, x_n\}$.

Proof. Let y be in the convex hull of $\{x_1, \ldots, x_n\}$ and choose non-negative numbers $\alpha_1, \ldots, \alpha_n$ such that $\sum_{j=1}^n \alpha_j = 1$ and $y = \sum_{j=1}^n \alpha_j x_j$. Let

$$\begin{aligned}
y_1 &= y, \\
y_2 &= \alpha_2 x_1 + \alpha_3 x_2 + \cdots + \alpha_n x_{n-1} + \alpha_1 x_n, \\
&\vdots \\
y_n &= \alpha_n x_1 + \alpha_1 x_2 + \cdots + \alpha_{n-2} x_{n-1} + \alpha_{n-1} x_n.
\end{aligned}$$

Note that $\sum_{j=1}^n y_j = \sum_{j=1}^n x_j$. Thus, if $\|x_{n+1} - y\| \leq \operatorname{diam}\{x_1, \ldots, x_{n+1}\} - n\varepsilon$,

$$\begin{aligned}
\operatorname{diam}\{x_1, \ldots, x_{n+1}\} - \varepsilon &< \left\| x_{n+1} - \frac{1}{n} \sum_{j=1}^{n} x_j \right\| \\
&= \left\| x_{n+1} - \frac{1}{n} \sum_{j=1}^{n} y_j \right\| \leq \frac{1}{n} \sum_{j=1}^{n} \|x_{n+1} - y_j\| \\
&\leq \operatorname{diam}\{x_1, \ldots, x_{n+1}\} - \varepsilon,
\end{aligned}$$

a contradiction which completes the proof of Lemma 1.

The next lemma concerning the monotonicity of a certain expression will be frequently used. In the case $p = 1$, this fact can be found in [**5**, p. 131].

LEMMA 2. *Let $\{x_1, \ldots, x_n, x_{n+1}\}$ be a set in a normed linear space X and let $1 \le p < \infty$. If $\{x_{n_1}, \ldots, x_{n_k}\}$ is a subset of $\{x_1, \ldots, x_n\}$, then*

$$\sum_{j=1}^{k} \|x_{n+1} - x_{n_j}\|^p - k \left\| \frac{1}{k} \sum_{j=1}^{k} (x_{n+1} - x_{n_j}) \right\|^p \le \sum_{j=1}^{n} \|x_{n+1} - x_j\|^p - n \left\| \frac{1}{n} \sum_{j=1}^{n} (x_{n+1} - x_j) \right\|^p .$$

Proof. By convexity of $\|\cdot\|^p$,

$$\begin{aligned} &\frac{1}{n} \sum_{j=1}^{n} \|x_{n+1} - x_j\|^p - \left\| \frac{1}{n} \sum_{j=1}^{n} (x_{n+1} - x_j) \right\|^p \\ &\quad \ge \frac{k}{n} \left[\frac{1}{k} \sum_{j=1}^{k} \|x_{n+1} - x_{n_j}\|^p - \left\| \frac{1}{k} \sum_{j=1}^{k} (x_{n+1} - x_{n_j}) \right\|^p \right] \\ &\quad + \frac{n-k}{n} \left[\frac{1}{n-k} \sum_{j \ne n_1, \ldots, n_k} \|x_{n+1} - x_j\|^p \right. \\ &\qquad \left. - \left\| \frac{1}{n-k} \sum_{j \ne n_1, \ldots, n_k} (x_{n+1} - x_j) \right\|^p \right] \\ &\quad \ge \frac{k}{n} \left[\frac{1}{k} \sum_{j=1}^{k} \|x_{n+1} - x_{n_j}\|^p - \left\| \frac{1}{k} \sum_{j=1}^{k} (x_{n+1} - x_{n_j}) \right\|^p \right] + 0. \end{aligned}$$

Multiplying the inequalities by n completes the proof.

In order to study normal structure in a Bochner L^p-space, various characterizations of normal structure involving the index p will prove useful. These characterizations are given in the next lemma.

LEMMA 3. *Let $1 \le p < \infty$ and let X be a normed linear space. The following assertions are equivalent.*

(a) *X fails to have normal structure.*

(b) *For every sequence (ε_n) of positive real numbers converging to 0, there exists a diametral sequence (x_n) in X such that*

$$(*) \qquad \sum_{j=1}^{n} \|x_{n+1} - x_j\|^p - n \left\| \frac{1}{n} \sum_{j=1}^{n} (x_{n+1} - x_j) \right\|^p < \varepsilon_n .$$

(c) *There exist a sequence* (ε_n) *of positive real numbers converging to* 0 *and a sequence* (x_n) *in* X *satisfying* $(*)$ *and, for all* k *in* N,

$$0 < \lim_{n\to\infty} \|x_{n+1} - x_k\| = \operatorname{diam}(x_n) < \infty.$$

Proof. In order to show (a) implies (b), assume that X fails to have normal structure and let (ε_n) be a sequence of positive real numbers tending to 0. From the work of Brodskiĭ and Mil′man [2], there exists a nonconstant bounded sequence (x_n) in X such that $\operatorname{dist}(x_{n+1}, \operatorname{co}\{x_1, \dots, x_n\}) > D - \alpha_n$ where D denotes $\operatorname{diam}(x_n)$ and α_n is such that $(D - \alpha_n)^p = D^p - \varepsilon_n/n$. Then (x_n) is a diametral sequence in X and

$$\begin{aligned} 0 \le \Delta\Sigma_n^p &= \sum_{j=1}^{n} \|x_{n+1} - x_j\|^p - n\|x_{n+1} - \overline{x}_n\|^p \\ &< nD^p - n(D - \alpha_n)^p = \varepsilon_n. \end{aligned}$$

This proves that (a) implies (b) and it is clear that (b) implies (c).

Suppose that sequences (ε_n) and (x_n) are given as in (c). Fix k in N and let $m \ge k$. Lemma 2 implies that

$$0 \le \sum_{j=1}^{k} \|x_{m+1} - x_j\|^p - k\left\|\frac{1}{k}\sum_{j=1}^{k}(x_{m+1} - x_j)\right\|^p \le \Delta\Sigma_m^p.$$

Letting $m \to \infty$ yields $kD^p - k\lim_{m\to\infty} \|x_{m+1} - \overline{x}_k\|^p = 0$. Thus $\lim_{m\to\infty} \|x_{m+1} - \overline{x}_k\| = D$. Either by applying Lemma 1 to obtain a subsequence of (x_n) which is a diametral sequence in X or by applying Proposition 1 in [5], it follows that X fails to have normal structure. This completes the proof of Lemma 3.

As mentioned earlier, Brodskiĭ and Mil′man characterized normal structure in terms of the nonexistence of a diametral sequence. For their characterization, it appears to be crucial that the limit of the distance from x_{n+1} to the convex hull of $\{x_1, \dots, x_n\}$ is the diameter of $\{x_k : k \in N\}$. However, in [5], Landes notes that appearances can be deceiving and that, in fact, any positive number may take the place of the diameter of $\{x_k : k \in N\}$. More specifically, Landes [5, p. 131] proves that a Banach space X has normal structure if and only if there is no bounded sequence (x_n) in X such that $\lim_{n\to\infty} \|x_n - x_k\| = \lim_{n\to\infty} \|x_n - \overline{x}_m\| > 0$ for all k, m in N. Landes' theorem, as well as the next lemma which is based on Landes' idea, will be used in the proof of the main theorem.

LEMMA 4. *Let* $1 \leq p < \infty$ *and let* (x_n) *be a sequence in a normed linear space* X *such that* $\lim_{n\to\infty} \|x_{n+1} - x_k\| = L > 0$ *for all* k *in* N *and*

$$\lim_{n\to\infty} \left[\sum_{j=1}^{n} \|x_{n+1} - x_j\|^p - n \left\| \frac{1}{n} \sum_{j=1}^{n} (x_{n+1} - x_j) \right\|^p \right] = 0.$$

Then

$$\lim_{n\to\infty} \left\| x_{n+1} - \frac{1}{k} \sum_{j=1}^{k} x_j \right\| = L$$

for each k *in* N.

Proof. Fix k in N and let $n \geq k$. By Lemma 2,

$$\frac{1}{k} \sum_{j=1}^{k} \|x_{n+1} - x_j\|^p - \|x_{n+1} - \overline{x}_k\|^p \leq \frac{1}{k} \Delta\Sigma_n^p.$$

Therefore

$$\begin{aligned} \frac{1}{k} \sum_{j=1}^{k} \|x_{n+1} - x_j\|^p - \frac{1}{k} \Delta\Sigma_n^p &\leq \|x_{n+1} - \overline{x}_k\|^p \\ &= \left\| \frac{1}{k} \sum_{j=1}^{k} (x_{n+1} - x_j) \right\|^p \leq \frac{1}{k} \sum_{j=1}^{k} \|x_{n+1} - x_j\|^p \end{aligned}$$

by the convexity of $\|\cdot\|^p$. Holding k fixed and letting $n \to \infty$ yields $\lim_{n\to\infty} \|x_{n+1} - \overline{x}_k\| = L$ and the proof of Lemma 4 is complete.

Since normal structure is a property that is inherited by subspaces and since X and $L^p(\mu)$ are subspaces of $L^p(\mu, X)$ (exclude the trivial cases where $\mu E = \infty$ for every nonempty E in Σ or where $X = \{0\}$), the Bochner L^p-space can only have normal structure whenever both X and $L^p(\mu)$ have normal structure. The main result of this paper is that this necessary condition on X for $L^p(\mu, X)$ to have normal structure is also sufficient in the case that $1 < p < \infty$. Before proceeding to that result, consider the cases $p = 1$ and $p = \infty$. If $p = 1$, then, since l^1 and $L^1(\mu)$, where μ is not purely atomic, do not have normal structure, the Bochner L^p-space $L^1(\mu, X)$ can have normal structure only if X has normal structure and the measure space (Ω, Σ, μ) consists of a finite number of atoms, that is, only if $L^1(\mu, X)$ is a finite l^1-direct sum of X. In this setting it remains unknown whether $L^1(\mu, X)$ has normal structure (see **[5]**). If $p = \infty$, then it follows in like manner that the

Bochner L^p-space $L^\infty(\mu, X)$ can have normal structure only if it is a finite l^∞-direct sum of X. In this case it is known that $L^\infty(\mu, X)$ has normal structure (see [**1**]). Thus, with the theorem below, normal structure in Bocher L^p-spaces is completely characterized except in the seemingly simple setting of a finite l^1-direct sum.

THEOREM. *Let (Ω, Σ, μ) be a measure space and let X be a normed linear space. If $1 < p < \infty$, the Bochner L^p-space $L^p(\mu, X)$ has normal structure if and only if X has normal structure.*

Proof. By the remarks above, it is only required to show that normal structure lifts from X to $L^p(\mu, X)$ whenever $1 < p < \infty$. Note that it suffices to establish this implication in the case that (Ω, Σ, μ) is a nonatomic probability space. Indeed, since normal structure is sequentially determined and elements in $L^p(\mu, X)$ have σ-finite support, there is no loss of generality in assuming that (Ω, Σ, μ) is a σ-finite measure space. Then, since $L^p(\mu, X)$ can be written as the l^p-sum of two spaces (a countable l^p-sum of X's and a Bochner L^p-space over a σ-finite nonatomic measure space), two applications of Landes' l^p-result [**5**, p. 135] show that it suffices to prove the theorem in the case that μ is σ-finite and nonatomic. Since, in this case, $L^p(\mu, X)$ can be written as a countable l^p-sum of Bochner L^p-spaces over finite nonatomic measure spaces, another application of Landes' result shows that it suffices to prove the theorem when μ is finite and nonatomic. Finally, since it is clear that, in this case, $L^p(\mu, X)$ is linearly isometric to $L^p(\nu, X)$ where ν is the probability measure $\mu/\mu\Omega$, it suffices to prove the result for a nonatomic probability measure.

Under the assumptions that μ is a nonatomic probability and $1 < p < \infty$, suppose that $L^p(\mu, X)$ fails to have normal structure. By Lemma 3, there exists a diametral sequence (f_n) in $L^p(\mu, X)$ such that

$$(**) \qquad 0 \le \sum_{j=1}^{n} \|f_{n+1} - f_j\|_p^p - n \left\| \frac{1}{n} \sum_{j=1}^{n} (f_{n+1} - f_j) \right\|_p^p < 1/2^n.$$

By translating and taking scalar multiples, if necessary, it may be assumed that $\operatorname{diam}(f_n) = 1$. Then, in particular, since (f_n) is diametral, $\lim_{n\to\infty} \|f_{n+1} - f_k\|_p = 1$ for all k in N.

Claim 1. $(f_n - f_1)$ does not converge to 0 in measure.

Suppose $(f_n - f_1)$ converges to 0 in measure. Choose ε such that $0 < \varepsilon < (2^{1/p} - 1)/(2^{1/p} + 2)$. Since $\lim_{n\to\infty} \|f_n - f_1\|_p = 1$, choose a natural number $N > 1$ such that $n \geq N$ implies $\|f_n - f_1\|_p > 1 - \varepsilon/2$. Well-known theorems of Riesz and Egoroff yield a subsequence (f_{n_j}) of (f_n) such that $(f_{n_j} - f_1)$ converges to 0 almost uniformly. Using this, the nonatomic nature of (Ω, Σ, μ), and the absolute continuity of $\int_{(\cdot)} \|f_N - f_1\|^p \, d\mu$, choose a measurable set A with $\mu(\Omega \setminus A) > 0$,

$$\|(f_N - f_1)\chi_{\Omega\setminus A}\|_p < \varepsilon/2, \qquad \|(f_N - f_1)\chi_A\|_p > 1 - \varepsilon,$$

and $(f_{n_j} - f_1)$ converging to 0 uniformly on A. Then choose a natural number i so that, with $M = n_i$,

$$\|(f_M - f_1)\chi_A\|_p < \varepsilon/2 < \varepsilon \quad \text{and} \quad \|(f_M - f_1)\chi_{\Omega\setminus A}\|_p > 1 - \varepsilon.$$

Set $g = f_N - f_1$, $h = f_M - f_1$, $g' = g\chi_A$, and $h' = h\chi_{\Omega\setminus A}$. Then, by the choice of ε,

$$\begin{aligned} 1 &\geq \|g - h\|_p \\ &\geq \|g' - h'\|_p - \|g - g'\|_p - \|h' - h\|_p \\ &\geq 2^{1/p}(1 - \varepsilon) - \varepsilon - \varepsilon \\ &> 1, \end{aligned}$$

a contradiction which proves the claim.

By Claim 1 and the fact that $\|f_n - f_1\|_p \leq 1$ for all n in N, there exist $\delta > 0$, $\xi > 0$ and a subsequence of (f_n), called (f_n) again, such that the first term of the subsequence is the original f_1 and such that

$$\mu\{s \in \Omega\colon 1/\delta \geq \|f_n(s) - f_1(s)\|_X \geq \delta\} > \xi \quad \text{for all } n \geq 2.$$

Note that Lemma 2 implies that the new subsequence (f_n) still satisfies $(**)$. *In the remainder of the proof, whenever subsequences of (f_n) are chosen, they will always be chosen so that the first term is the original f_1, and, automatically, $(**)$ is satisfied.*

Claim 2. There exist a subsequence (f_{n_m}) of (f_n) and a measurable set E with $\mu E < \xi/2$ such that

$$|(\|f_{n_k}(s) - f_{n_i}(s)\|_X - \|f_{n_k}(s) - f_{n_j}(s)\|_X)| < 1/2^{k-2}$$

for all i, j, k with $k \geq 3$ and $1 \leq i, j \leq k - 1$ and for all s in $\Omega \setminus E$.

The subsequence (f_{n_m}) and the set E will be constructed inductively. Let $n_1 = 1$ and $n_2 = 2$. Note that

$$\begin{aligned} \|f_n - \tfrac{1}{2}(f_{n_1} + f_{n_2})\|_p \leq \|\tfrac{1}{2}(&\|f_n(\cdot) - f_{n_1}(\cdot)\|_X \\ &+ \|f_n(\cdot) - f_{n_2}(\cdot)\|_X)\|_p \leq 1. \end{aligned}$$

Since (f_n) is a diametral sequence with diameter 1, the left-hand side of the inequality tends to 1 as n increases. The uniform rotundity of $L^p(\mu)$ then implies that

$$\|(\|f_n(\cdot) - f_{n_1}(\cdot)\|_X - \|f_n(\cdot) - f_{n_2}(\cdot)\|_X)\|_p \to 0.$$

Then, by well-known theorems of Riesz and Egoroff, there exists a subsequence (h_n^1) of (f_n) with $h_1^1 = f_{n_1}$ and $h_2^1 = f_{n_2}$ so that the sequence (g_n^1), defined by

$$g_n^1 = \|h_n^1(\cdot) - f_{n_1}(\cdot)\|_X - \|h_n^1(\cdot) - f_{n_2}(\cdot)\|_X,$$

converges to 0 almost uniformly. Thus there exists a measurable set E_1 with $\mu E_1 < \xi/4$ such that (g_n^1) converges uniformly to 0 on $\Omega \setminus E_1$. Choose $M > 2$ such that if $n \geq M$, then $|g_n^1| < 1/2$ on $\Omega \setminus E_1$. Let n_3 be the index such that $h_M^1 = f_{n_3}$ and note $n_3 > n_2$. Define (f_n^1) by $f_1^1 = f_{n_1}$, $f_2^1 = f_{n_2}$, $f_3^1 = f_{n_3}$, and $f_n^1 = h_{M+n-3}^1$ if $n \geq 4$. Then

$$\|f_n^1(\cdot) - f_{n_1}(\cdot)\|_X - \|f_n^1(\cdot) - f_{n_2}(\cdot)\|_X \to 0 \quad \text{uniformly on } \Omega \setminus E_1$$

and, if $n \geq 3$,

$$|(\|f_n^1(\cdot) - f_{n_1}(\cdot)\|_X - \|f_n^1(\cdot) - f_{n_2}(\cdot)\|_X)| < 1/2 \quad \text{on } \Omega \setminus E_1.$$

Assume the following have been chosen:

(i) natural numbers $n_1 < n_2 < \cdots < n_{k+2}$,

(ii) measurable sets $E_1 \subset E_2 \subset \cdots \subset E_k$ with $\mu E_k < \xi/2 - \xi/2^{k+1}$, and

(iii) a subsequence (f_n^k) of (f_n^{k-1}) such that, if $1 \leq i,\, j \leq k+1$,

$$\|f_n^k(\cdot) - f_{n_i}(\cdot)\|_X - \|f_n^k(\cdot) - f_{n_j}(\cdot)\|_X \to 0 \quad \text{uniformly on } \Omega \setminus E_k$$

and, for $n \geq k+2$,

$$|(\|f_n^k(\cdot) - f_{n_i}(\cdot)\|_X - \|f_n^k(\cdot) - f_{n_j}(\cdot)\|_X)| < 1/2^k \quad \text{on } \Omega \setminus E_k$$

and

$$f_j^k = f_{n_j} \qquad \text{if } j = 1, 2, \ldots, k+2.$$

(For the sake of completeness, define $f_n^0 = f_n$.)

Choose a natural number $N > n_{k+2}$ such that, if $n \geq N$,

$$|(\|f_n^k(\cdot) - f_{n_i}(\cdot)\|_X - \|f_n^k(\cdot) - f_{n_j}(\cdot)\|_X)| < 1/2^{k+2} < 1/2^{k+1}$$

on $\Omega \setminus E_k$ for $1 \leq i,\, j \leq k+1$. Note that

$$\begin{aligned}
&\|f_n^k - \tfrac{1}{2}(f_{n_1} + f_{n_{k+2}})\|_p \\
&\leq \|\tfrac{1}{2}(\|f_n^k(\cdot) - f_{n_1}(\cdot)\|_X + \|f_n^k(\cdot) - f_{n_{k+2}}(\cdot)\|_X)\|_p \\
&\leq 1.
\end{aligned}$$

Since the left-hand side approaches 1 as n increases and $L^p(\mu)$ is uniformly rotund,

$$\|(\|f_n^k(\cdot) - f_{n_1}(\cdot)\|_X - \|f_n^k(\cdot) - f_{n_{k+2}}(\cdot)\|_X)\|_p \to 0 \quad \text{as } n \to \infty.$$

Then, by theorems of Riesz and Egoroff again, there exists a subsequence (h_n^{k+1}) of (f_n^k) with $h_j^{k+1} = f_j^k = f_{n_j}$ if $j = 1, \dots, k+2$ so that the sequence (g_n^{k+1}), defined by

$$g_n^{k+1} = \|h_n^{k+1}(\cdot) - f_{n_1}(\cdot)\|_X - \|h_n^{k+1}(\cdot) - f_{n_{k+2}}(\cdot)\|_X,$$

converges to 0 almost uniformly. Thus there exists a measurable set F_{k+1} with $\mu F_{k+1} < \xi/2^{k+2}$ such that (g_n^{k+1}) converges uniformly to 0 on $\Omega \setminus F_{k+1}$ (and hence uniformly to 0 on $\Omega \setminus E_{k+1}$ where $E_{k+1} \equiv E_k \cup F_{k+1}$). Note that $\mu E_{k+1} \le \xi/2 - \xi/2^{k+2}$. Choose $M \ge N$ such that if $n \ge M$, then $|g_n^{k+1}| < 1/2^{k+2}$ on $\Omega \setminus E_{k+1}$. Let n_{k+3} be the index such that $h_M^{k+1} = f_{n_{k+3}}$ and note $n_{k+3} > n_{k+2}$. Define (f_n^{k+1}) by $f_j^{k+1} = h_j^{k+1} = f_{n_j}$ if $j = 1, \dots, k+2$; $f_{k+3}^{k+1} = f_{n_{k+3}}$; and $f_j^{k+1} = h_{M+j-(k+3)}^{k+1}$ if $j \ge k+4$.

Now, by the triangle inequality, for $n \ge k+3$ and $1 \le i, j \le k+2$,

$$\begin{aligned} &|(\|f_n^{k+1}(\cdot) - f_{n_i}(\cdot)\|_X - \|f_n^{k+1}(\cdot) - f_{n_j}(\cdot)\|_X)| \\ &\qquad < 1/2^{k+1} \quad \text{on } \Omega \setminus E_{k+1}. \end{aligned}$$

This completes the induction step. The subsequence (f_{n_m}) of (f_n) and the set $E \equiv \bigcup_{k=1}^{\infty} E_k$ have the properties in Claim 2.

Denote the subsequence obtained from Claim 2 by (f_n) again. With

$$g_n \equiv \sum_{j=1}^{n} \|f_{n+1}(\cdot) - f_j(\cdot)\|_X^p - n \left\| \frac{1}{n} \sum_{j=1}^{n} (f_{n+1}(\cdot) - f_j(\cdot)) \right\|_X^p,$$

the convexity of $\|\cdot\|^p$ and inequality $(**)$ yield that g_n is a non-negative integrable function with $\int_\Omega g_n \, d\mu \le 1/2^n$. Since, by the Monotone Convergence Theorem, $\sum_{n=1}^{\infty} g_n$ is integrable, the sequence (g_n) converges to 0 almost everywhere.

Since

$$\mu \left(\limsup_{n \to \infty} \{s \in \Omega \colon 1/\delta \ge \|f_n(s) - f_1(s)\|_X \ge \delta\} \right) \ge \xi$$

and $\mu E < \xi/2$, choose a point t such that

(1) t is in $\limsup_{n \to \infty} \{s \in \Omega \colon 1/\delta \ge \|f_n(s) - f_1(s)\|_X \ge \delta\}$,
(2) t is not in E, and
(3) $\lim_{n \to \infty} g_n(t) = 0$.

Using (1), choose a subsequence of (f_n), called (f_n) again, so that $\lim_{n\to\infty} \|f_{n+1}(t) - f_1(t)\|_X$, defined to be L, exists and is positive. By Claim 2 and (2), it follows that $\lim_{n\to\infty} \|f_{n+1}(t) - f_k(t)\|_X = L$ for each k in N. Also, by Lemma 2 and (3), the (sub-)sequence $(f_n(t))$ satisfies

$$\lim_{n\to\infty} \left[\sum_{j=1}^{n} \|f_{n+1}(t) - f_j(t)\|_X^p - n \left\| \frac{1}{n} \sum_{j=1}^{n} (f_{n+1}(t) - f_j(t)) \right\|_X^p \right] = 0.$$

Thus, with $x_n = f_n(t)$, Lemma 4 combines with Landes' theorem (stated prior to Lemma 4) to prove that X fails to have normal structure. This completes the proof of the theorem.

As a corollary, note that if $1 < p < \infty$ and X is a reflexive Banach space with normal structure, $L^p(\mu, X)$ is also a reflexive space [**3**, p. 100] with normal structure and hence, by a well-known theorem of W. A. Kirk [**4**], the space $L^p(\mu, X)$ has the fixed point property for nonexpansive mappings.

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