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A GEOMETRIC BOUND FOR MAXIMAL FUNCTIONS ASSOCIATED TO CONVEX BODIES

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For a convex symmetric body B in $\mathbb{R}^n$ let M_B denote the centered maximal operator

$$M_B f(x) = \sup_{t>0} \frac{1}{\operatorname{Vol} B} \int |f(x+ty)|\,dy$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We associate with B two linear invariants $\sigma(B)$ and $Q(B)$, and show that for $p > 1$ the norm of the operator M_B on $L^p(\mathbb{R}^n)$ is bounded by a constant which may depend on $p, \sigma(B)$ and $Q(B)$, but not explicitly on the dimension n. In particular, if B_q denotes the unit ball in $\mathbb{R}^n$ with respect to the l^q-norm, we can prove that M_{B_q} has a bound on $L^p(\mathbb{R}^n)$ which is independent of n, provided that $1 \leq q < \infty$.

The behaviour of maximal functions associated to convex bodies has been studied by various authors during recent years. When B is the Euclidean ball, i.e. $B = B_2$, Stein [**9**] has shown that M_B is bounded on $L^p(\mathbb{R}^n)$ uniformly in n for every $p > 1$, and Bourgain [**2, 3, 4**] and Carbery [**6**] have shown that the analogue of this holds for any convex body B, provided $p > 3/2$. Moreover, by a result of Stein and Strömberg [**11**] it is known that the L^p operator norm $\|M_B\|_{p,p}$ of M_B grows at most linearly in the dimension n for any $p > 1$.

Since the general estimates for convex bodies in [**2**] do not imply that $\|M_B\|_{p,p}$ has a bound independent of n, if $p \leq 3/2$, it is well possible that for $p \leq 3/2$ one can only hope for estimates of $\|M_B\|_{p,p}$ which depend on additional geometric invariants associated with the body B. In this article, we shall show that one can in fact prove an estimate of this kind:

We associate with B the following two linear invariants $\sigma(B)$ and $Q(B)$: There exists a regular linear transformation S of $\mathbb{R}^n$, which is unique modulo orthogonal transformations, and a unique constant $L(B)$ such that $\operatorname{Vol}_n S(B) = 1$ and

$$\int_{S(B)} |\langle x, \xi\rangle|^2\,dx = L(B)^2$$

for all unit vectors $\xi \in \mathbb{R}^n$. Let $1/\sigma(B)$ be the minimum of all $(n-1)$-dimensional volumes of all sections of $S(B)$ by hyperplanes, and $Q(B)$

the maximum of the $(n-1)$-dimensional volumes of all orthogonal projections of $S(B)$ onto hyperplanes (we note that $\sigma(B) \approx L(B)$). Then, for $p > 1$, the operator norm $\|M_B\|_{p,p}$ can be estimated by a constant depending only on p, $\sigma(B)$ and $Q(B)$.

This criterion suffices for example to prove the uniform boundedness in n of the maximal function M_{B_q}, where B_q denotes the unit ball with respect to the l^q-norm on $\mathbb{R}^n$, $1 \leq q < \infty$. This extends a result of Bourgain **[4]** who proved it for $q \in 2\mathbb{N}$ by making use of an "extra" decay of the Fourier transform of $\chi_{B_{2k}}$, $\chi_{B_{2k}}$ denoting the characteristic function of B_{2k}. However, this extra decay depends on some "smoothness" of B_q for $q \in 2\mathbb{N}$, which can easily be destroyed by cutting off a small piece of B_{2k} along an affine hyperplane, whereas our result is invariant under such operations.

Moreover, since one can show that $Q(B_\infty) = \sqrt{n}$, this might indicate that the norm of the "cubic" maximal operator M_{B_∞} associated with the unit cube of L^p is possibly growing with the dimension, if $p \leq 3/2$, and our results give some hints how one might try to prove this.

I would like to express my gratitude to the Mathematical Sciences Research Institute in Berkeley for the warm hospitality during my stay there by which this paper was completed, and especially to E. M. Stein for hints concerning multipliers of Laplace-transform type.

2. The main theorem. Let B be a convex symmetric body in $\mathbb{R}^n$. Arguing as in **[2]**, we see that there exist a linear transformation $S \in \mathrm{GL}(\mathbb{R}^n)$ and a constant $L(B) > 0$ such that

$$\text{(1)} \qquad \mathrm{Vol}_n\, S(B) = 1 \quad \text{and} \quad \int_{S(B)} |\langle x, \xi\rangle|^2\, dx = L(B)^2$$

for all unit vectors $\xi \in S^{n-1} = \{\xi \in \mathbb{R}^n : |\xi|^2 = \sum_j |\xi_j|^2 = 1\}$. It is easy to see that $L(B)$ is determined uniquely by (1), and that S is unique up to multiplication by an orthogonal transformation from the left.

For $\xi \in S^{n-1}$, we define similarly as in **[2]**

$$\text{(2)} \qquad \varphi(u) := \varphi_\xi(u) := \mathrm{Vol}_{n-1}(\{x \in S(B) : \langle x, \xi\rangle = u\}), \qquad u \in \mathbb{R}.$$

Moreover, let π_ξ denote the orthogonal projection of $\mathbb{R}^n$ onto the hyperplane perpendicular to ξ. Then the constants

$$\text{(3)} \qquad \begin{aligned} 1/\sigma(B) &:= \max\{\varphi_\xi(0) : \xi \in S^{n-1}\}, \\ Q(B) &:= \max\{\mathrm{Vol}_{n-1}(\pi_\xi(S(B)) : \xi \in S^{n-1}\} \end{aligned}$$

are obviously linear invariants for B, i.e. $\sigma(U(B)) = \sigma(B)$ and $Q(U(B)) = Q(B)$ for all $U \in \mathrm{GL}(\mathbb{R}^n)$.

Since also $\|M_B\|_{p,p}$ is a linear invariant for B, we therefore may and shall assume in the sequel (except for §3) that $S(B) = B$. Then, by **[2]**, Lemma 1, there exist two universal constants $0 < a, A < \infty$, such that

$$\varphi(u) \leq A\varphi(0)e^{-a\varphi(0)|u|}, \qquad u \in \mathbb{R}. \tag{4}$$

Moreover, there is a universal constant $a_1 > 0$, such that with $L = L(B)$

$$a_1^{-1} \leq L \cdot \varphi_\xi(0) \leq a_1, \quad \xi \in S^{n-1}. \tag{5}$$

This implies in particular $\sigma(B) \approx L(B)$.

THEOREM 1. *Let $p > 1$. Then for all $f \in L^p(\mathbb{R}^n)$*

$$\|M_B f\|_p \leq C(p, \sigma(B), Q(B))\|f\|_p, \tag{6}$$

where the constant $C = C(p, \sigma, Q)$ [1] *is independent of n and grows with σ and Q.*

Note that, for $p > 3/2$, C can even be chosen to be independent of σ and Q by **[3]** or **[6]**.

Let us fix some notation. We denote by m the multiplier

$$m(\xi) = \hat{\chi}_B(\xi) = \int_{\mathbb{R}^n} \chi_B(x)e^{-2\pi i\langle \xi, x\rangle}\, dx \tag{7}$$

associated to χ_B. If $w \in L^\infty(\mathbb{R}^n)$ is any multiplier, we define the corresponding multiplier operator T_w as

$$T_w(f) = \mathcal{F}^{-1}(w\hat{f}), \tag{8}$$

$\mathcal{F}^{-1}$ denoting the inverse Fourier transform.

For $\rho \in \mathbb{R}$ with $\rho > 1/2$ let us define the ρth fractional derivative $(\xi \cdot \nabla)^\rho m$ of m as in **[6]** by

$$\begin{aligned}(\xi \cdot \nabla)^\rho m(\xi) &= \left(\frac{d}{dr}\right)^\rho \bigg|_{r=1} m(r\xi) \\ &= \int (-2\pi i\langle x, \xi\rangle)^\rho K(x)e^{-2\pi i\langle x, \xi\rangle}\, dx,\end{aligned} \tag{9}$$

where $K = \chi_B$. Then, by the results of **[6]**, expecially Theorem 2 and Proposition (ii), our Theorem 1 will be an immediate consequence of

[1] Here and in the sequel constants will frequently be denoted by C, with the understanding that they may be different from statement to statement.

PROPOSITION 1. *Let $1/2 < p < 1$. Then for all $f \in L^p(\mathbb{R}^n)$*

$$\|T_{(\xi\cdot\nabla)^\rho m} f\|_p \leq C_\rho(p, \sigma(B), Q(B))\|f\|_p$$

if $1 < p < \infty$, where the constant C_p is again independent of n.

This proposition is closely related to the question raised in **[6]**, whether it is possible to find a bound for $T_{(\xi\cdot\nabla)m}$ which is independent of n.

The proof of Proposition 1 will be based on analytic interpolation. We define a family of operators $T_\alpha = T_{m_\alpha}$, $\alpha \in \mathbb{C}$, by

$$m_\alpha(\xi) = (1 + |\xi|)^{1-\alpha}[I^{-\alpha} m(r\xi)]|_{r=1}, \qquad \xi \neq 0. \tag{10}$$

Here, $I^{-\alpha}$ denotes the αth fractional Riesz derivative with base point 2, that is

$$I^{-\alpha} f(r) = \frac{-1}{\Gamma(-\alpha)} \int_r^2 (s-r)^{-\alpha-1} f(s)\, ds, \qquad \operatorname{Re}\alpha < 0, \tag{11}$$

if $f \in C^\infty(]0, 2])$.

It is well known that $I^{-\alpha}$ can be extended analytically to the whole complex plane, and that $I^{-k} = (d/dr)^k$ is the usual kth derivative for $k = 0, 1, \ldots$. Note that $I^{-\alpha}$ and $(d/dr)^\alpha$ as defined in (9) do not agree. However, we shall show later that the difference of these two is unimportant for our problem. We also define $T_\alpha^\varepsilon = T_{m_\alpha^\varepsilon}$ by

$$m_\alpha^\varepsilon(\xi) = (1 + |\xi|)^{-\varepsilon} m_\alpha(\xi), \qquad \varepsilon > 0. \tag{12}$$

The proof of Proposition 1 will essentially be contained in the Lemmas 2 and 4 to follow, which deal with the two endpoint cases for the interpolation. Lemmas 1 and 3 are more of a technical nature.

LEMMA 1. *Let $0 \leq \operatorname{Re}\alpha < 1$, $k \in \mathbb{N}$. Then for $u > 1$*

$$\left| \int_0^u \frac{s^{-\alpha}}{(1 + s/u)^k} e^{-2\pi i s}\, ds - \frac{e^{\frac{\pi}{2}\alpha i}}{i} \Gamma(1-\alpha) \right| \leq C_k e^{(\pi/2)|\operatorname{Im}\alpha|} u^{-\operatorname{Re}\alpha}.$$

The proof of Lemma 1 is an easy consequence of Cauchy's integral theorem and follows by changing the path of integration from the interval $[0, u]$ to $-i[0, u]$, connecting those two paths by quarter circles of radii u and ε, $\varepsilon \to 0$. We shall omit the technical details.

LEMMA 2. *Fix $N > 0$ and $0 < \varepsilon < 1/2$. Then*
(i) $\|m_\alpha\|_\infty \le C_N(\sigma(B), Q(B))e^{2\pi|\operatorname{Im}\alpha|}$, $0 \le \operatorname{Re}\alpha < N$,
(ii) $\|m_\alpha^\varepsilon\|_\infty \le C_N(\sigma(B), Q(B))e^{2\pi|\operatorname{Im}\alpha|}$, $-\varepsilon \le \operatorname{Re}\alpha \le N$.

Proof. Assume $\operatorname{Re}\alpha \ge -\varepsilon$, and let $k = [\operatorname{Re}\alpha]$ be the integer part of $\operatorname{Re}\alpha$. Then it follows easily by partial integration from (10) that

$$(13)\quad m_\alpha(\xi) = \sum_{j=0}^{k} \frac{(-1)^{j+1}}{\Gamma(j+1-\alpha)}(1+|\xi|)^{1-\alpha}\left(\frac{d}{dr}\right)^j m(r\xi)|_{r=2} + \frac{(-1)^k(1+|\xi|)^{1-\alpha}}{\Gamma(k+1-\alpha)}\int_1^2 (s-1)^{k-\alpha}\left(\frac{d}{ds}\right)^{k+1} m(s\xi)\,ds.$$

By (1), with $\varphi = \varphi_{\xi/|\xi|}$, we have

$$(14)\qquad m(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i|\xi|u}\varphi(u)\,du,$$

hence

$$(15)\qquad \left(\frac{d}{dr}\right)^j m(r\xi)|_{r=2} = (-2\pi i|\xi|)^j \int_{-\infty}^{\infty} e^{-4\pi i|\xi|u}u^j\varphi(u)\,du.$$

By partial integration this implies

$$(16)\quad \left(\frac{d}{dr}\right)^j m(r\xi)|_{r=2} = \frac{1}{2}(-2\pi i|\xi|)^{j-1}\int_{-\infty}^{\infty} e^{-4\pi i|\xi|u}(u^j\varphi)'(u)\,du.$$

(2) and (15) imply for $0 \le j \le N$

$$\left|\left(\frac{d}{dr}\right)^j m(r\xi)|_{r=2}\right| \le C_N|\xi|^j\int_0^\infty u^j\varphi(0)e^{-a\varphi(0)u}\,du \le C_N\varphi(0)^{-j}|\xi|^j \le C_N\sigma(B)^j|\xi|^j.$$

Moreover, since $(u^j\varphi)'(u) = ju^{j-1}\varphi(u) + u^j\varphi'(u)$, and since $\varphi'(u)$ has constant sign for $u \ge 0$ resp. $u \le 0$, (16) and (4) yield

$$\left|\left(\frac{d}{dr}\right)^j m(r\xi)|_{r=2}\right| \le C_N\varphi(0)^{-(j-1)}|\xi|^{j-1} \le C_N\sigma(B)^{j-1}|\xi|^{j-1}.$$

Together, we obtain

$$\left|\left(\frac{d}{dr}\right)^j m(r\xi)|_{r=2}\right| \le C_N(\sigma(B))|\xi|^j/(1+|\xi|),$$

at least for $j \ge 1$. However, for $j = 0$, (15) and (16) easily imply

$$|m(\xi)| \le C(1+\varphi(0))/(1+|\xi|) \le C\cdot Q(B)/(1+|\xi|).$$

So, together we get

$$\left|\left(\frac{d}{dr}\right)^j m\,(r\xi)|_{r=2}\right| \le C_N(\sigma(B), Q(B))\frac{|\xi|^j}{1+|\xi|}, \qquad 0 \le j < N.$$

This implies, for $j = 0, \dots, k$,

$$\left|\frac{(1+|\xi|)^{\alpha-1}}{\Gamma(j+1-\alpha)}\left(\frac{d}{dr}\right)^j m(r\xi)|_{r=2}\right| \le C_N(\sigma, Q)e^{\pi|\operatorname{Im}\alpha|}, \tag{17}$$

where we made use of the well known asymptotics [**8**, p. 79]

$$|\Gamma(x+iy)| \sim e^{-(\pi/2)|y|}|y|^{(x-1/2)} \cdot \sqrt{2\pi} \quad \text{as } |y| \to \infty. \tag{18}$$

So, it remains to estimate the integral term in (13), which, up to the sign, is given by

$$J(\xi) = \frac{(1+|\xi|)^{1-\alpha}}{\Gamma(k+1-\alpha)}(-2\pi i|\xi|)^{k+1}\int_{-\infty}^{\infty} F(|\xi|u)u^{k+1}\varphi(u)\,du,$$

where

$$F(t) = \int_1^2 (s-1)^{k-\alpha}e^{-2\pi i t s}\,ds.$$

The estimate of $J(\xi)$ requires more technique, but is essentially based again on (4), so that the rest of the proof of the lemma could be skipped for a first reading. We set

$$G(u) = \int_0^u t^{k+1}F(t)\,dt, \qquad u \in \mathbb{R}.$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} & F(|\xi|u)u^{k+1}\varphi(u)\,du \\ &= |\xi|^{-k-2}\int_{-\infty}^{\infty} F(u)u^{k+1}\varphi(u/|\xi|)\,du \\ &= -|\xi|^{-k-3}\int_{-\infty}^{\infty} G(u)\varphi'(u/|\xi|)\,du,\end{aligned}$$

and hence

$$\begin{aligned}|J(\xi)| \le\, & C_N|\xi|^{-2}(1+|\xi|)^{1-\operatorname{Re}\alpha} \\ & \times \left|\frac{1}{\Gamma(k+1-\alpha)}\int_{-\infty}^{\infty} G(u)\varphi'(u/|\xi|)\,du\right|.\end{aligned} \tag{19}$$

Now

$$(20)\quad \int_0^u t^{k+1}e^{-2\pi its}\,dt$$
$$= \left(\frac{i}{2\pi}\right)^{k+1}\Bigg\{(-1)^{k+1}(k+1)!s^{-(k+2)}(e^{-2\pi ius}-1)$$
$$+\sum_{j=0}^{k}\binom{k+1}{j}(-1)^j j!(-2\pi i)^{k-j}$$
$$\times s^{-(j+1)}u^{k+1-j}e^{-2\pi ius}\Bigg\}.$$

Let

$$(21)\qquad G_j(u) = u^{k+1-j}\int_1^2 (s-1)^{k-\alpha}s^{-(j+1)}e^{-2\pi ius}\,ds,$$
$$j=0,\ldots,k+1,$$

and

$$(21)'\quad G_{k+2}(u) = G_{k+2} = \int_1^2 (s-1)^{k-\alpha}s^{-(k+2)}\,ds = \int_0^1 \frac{s^{k-\alpha}}{(s+1)^{k+2}}\,ds,$$

and define for $j=0,\ldots,k+2$

$$(22)\quad J_j(\xi) = \frac{(1+|\xi|)^{1-\mathrm{Re}\,\alpha}}{|\xi|^2}\left|\frac{1}{\Gamma(k+1-\alpha)}\int_{-\infty}^{\infty} G_j(u)\varphi'(u/|\xi|)\,du\right|.$$

By (20), G is a linear combination of the G_j, and so it remains only to show that all functions J_j have an estimate of the desired type.

For $j=0,\ldots,k+1$,

$$G_j(u) = u^{\alpha-j}e^{-2\pi iu}\int_0^u \frac{s^{k-\alpha}}{(1+s/u)^{j+1}}e^{-2\pi is}\,ds,$$

so Lemma 1 implies for $|u|>1$

$$(23)\qquad G_j(u) = \pm\, ie^{(\pi/2)(\alpha-k)i}\Gamma(k+1-\alpha)u^{\alpha-j}e^{-2\pi iu}$$
$$+\,O(e^{(\pi/2)|\,\mathrm{Im}\,\alpha|}|u|^{k-j}).$$

Moreover, if $|u|\le 1$, then

$$G_j(u) = u^{\alpha-j}\frac{e^{-2\pi iu}}{k+1-\alpha}\Bigg\{\frac{u^{k+1-\alpha}}{2^{j+1}}e^{-2\pi iu}$$
$$-\int_0^u s^{k+1-\alpha}\frac{d}{ds}\left[\frac{e^{-2\pi is}}{(1+s/u)^{j+1}}\right]ds\Bigg\},$$

which easily implies

$$(23)' \qquad |G_j(u)| \le C_N \frac{|u|^{k+1-j}}{|k+1-\alpha|}, \qquad |u| \le 1.$$

(23) and (23)′ imply

$$(24) \quad |J_j(\xi)| \le -C_N \frac{(1+|\xi|)^{1-\operatorname{Re}\alpha}}{|\xi|^2} \times \left\{ \frac{1}{|\Gamma(k+2-\alpha)|} \int_0^1 u^{k+1-j} \varphi'(u/|\xi|)\, du + e^{(\pi/2)|\operatorname{Im}\alpha|} \times \int_1^\infty \left[u^{\operatorname{Re}\alpha - j} + \frac{u^{k-j}}{|\Gamma(k+1-\alpha)|} \right] \varphi'(u/|\xi|)\, du \right\}.$$

However, if $j \le k+1$, then

$$(25) \qquad \left| \int_0^1 u^{k+1-j} \varphi'(u/|\xi|)\, du \right| \le -\int_0^1 \varphi'(u/|\xi|)\, du \le 2\varphi(0)|\xi|,$$

and similarly one shows by (4) that

$$\begin{aligned} \left| \int_1^\infty u^{\operatorname{Re}\alpha - j} \varphi'(u/|\xi|)\, du \right| &= -|\xi|^{1+\operatorname{Re}\alpha - j} \int_{1/|\xi|}^\infty u^{\operatorname{Re}\alpha - j} \varphi'(u)\, du \\ &\le |\xi|^{1+\operatorname{Re}\alpha - j} \left\{ |\xi|^{j - \operatorname{Re}\alpha} \varphi(1/|\xi|) + |\operatorname{Re}\alpha - j| \varphi(0) \int_{1/|\xi|}^1 u^{\operatorname{Re}\alpha - j - 1}\, du \right. \\ &\qquad \left. + |\operatorname{Re}\alpha - j| \int_1^\infty u^{\operatorname{Re}\alpha - j} \varphi(u)\, du \right\} \\ &\le C_N(\sigma, Q) |\xi|^{1+\operatorname{Re}\alpha - j} (1 + |\xi|^{j - \operatorname{Re}\alpha}); \end{aligned}$$

hence

$$(26) \quad \left| \int_1^\infty u^{\operatorname{Re}\alpha - j} \varphi'(u/|\xi|)\, du \right| \le \begin{cases} C_N(\sigma, Q)|\xi|^{1+\operatorname{Re}\alpha}, & j \le k, \\ C_N(\sigma, Q)|\xi|, & j = k+1. \end{cases}$$

Of course $|\int_1^\infty u^{k-j} \varphi'(u/|\xi|)\, du|$ is even dominated by (26). (24), (25) and (26) imply, for $|\xi| \ge 1$,

$$(27) \quad |J_j(\xi)| \le C_N(\sigma, Q) e^{2\pi|\operatorname{Im}\alpha|} (1 + |\xi|^{-\operatorname{Re}\alpha}), \qquad j = 0, \ldots, k+1.$$

Moreover, since obviously $|G_{k+2}| \le C_N / |k+1-\alpha|$, we have

$$\begin{aligned} (27)' \quad |J_{k+2}(\xi)| &\le C_N |\xi|^{-1-\operatorname{Re}\alpha} \frac{-1}{|\Gamma(k+2-\alpha)|} \int_0^\infty \varphi'(u/|\xi|)\, du \\ &\le C_N(\sigma) e^{\pi|\operatorname{Im}\alpha|} |\xi|^{-\operatorname{Re}\alpha}, \qquad |\xi| \ge 1. \end{aligned}$$

The last two estimates imply the desired uniform estimates of $m_\alpha(\xi)$ and $m_\alpha^\varepsilon(\xi)$ for $|\xi| \geq 1$.

There remains the case $|\xi| < 1$, which is easy: By partial integration

$$J(\xi) = \frac{(1+|\xi|)^{1-\alpha}}{\Gamma(k+2-\alpha)}\left\{\left(\frac{d}{ds}\right)^{k+1} m(s\xi)|_{s=2} - \int_1^2 (s-1)^{k+1-\alpha}\left(\frac{d}{ds}\right)^{k+2} m(s\xi)\,ds\right\}$$

which, together with (15) and (2), implies

$$|J(\xi)| \leq C_N(\sigma, Q)e^{\pi|\operatorname{Im}\alpha|}|\xi|^{k+1};$$

this settles the case $|\xi| < 1$. □

LEMMA 3. *For each unit vector $\eta \in S^{n-1}$ define a distribution $\mu_\eta = \partial\chi_B/\partial_\eta = (\eta\cdot\nabla)\chi_B$. Then μ_η is even a bounded measure, and*

$$\|\mu_\eta\|_{M(\mathbb{R}^n)} = 2\operatorname{Vol}_{n-1}(\pi_\eta(B)).$$

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\|\varphi\|_\infty = 1$. After rotating coordinates, we may assume that η is the nth coordinate vector. Writing $\mathbb{R}^n = \mathbb{R}^{n-1}\times\mathbb{R}$ with coordinates (x,u), we then have

$$\langle\mu_\eta, \varphi\rangle = -\int_B \frac{\partial\varphi}{\partial\eta} = -\int_{\pi_\eta(B)}\int_{B_x}\frac{\partial\varphi}{\partial u}(x,u)\,du\,dx,$$

where B_x is the interval $B_x = \{u \in \mathbb{R} : (x,u) \in B\}$, with endpoints say $a(x) \leq b(x)$, unless $B_x = \varnothing$. So

$$|\langle\mu_\eta, \varphi\rangle| = \left|\int_{\pi_\eta(B)}[\varphi(b(x)) - \varphi(a(x))]\,dx\right| \leq 2\operatorname{Vol}_{n-1}(\pi_\eta(B));$$

hence $\|\mu_\eta\|_M \leq 2\operatorname{Vol}_{n-1}(\pi_\eta(B))$. Moreover, choosing φ to be linear on each section B_x such that $\varphi(b(x)) = 1$ and $\varphi(a(x)) = -1$ immediately also gives $\|\mu_\eta\|_M \geq 2\operatorname{Vol}_{n-1}(\pi_\eta(B))$. □

LEMMA 4. *Let $0 < \varepsilon < 1/2$. Then*

(28) $\quad \|T^\varepsilon_{-\varepsilon+i\nu}f\|_p \leq C_\varepsilon(p, \sigma(B), Q(B))e^{(\pi/2)|\nu|}\|f\|_p, \qquad f \in L^p(\mathbb{R}^n),$

for every $1 < p < \infty$.

Proof. Let $\alpha = -\varepsilon + i\nu$. Since

$$m_\alpha^\varepsilon(\xi) = -\frac{1}{\Gamma(-\alpha)}\int_1^2 (s-1)^{-\alpha-1}(1+|\xi|)^{1-\varepsilon-\alpha}m(s\xi)\,ds,$$

it clearly suffices to prove that the multiplier operator corresponding to $(1+|\xi|)^{1-\varepsilon-\alpha}m(s\xi)$ satisfies (28) uniformly for $1 \le s \le 2$.

Consider the multiplier $M_\nu(\xi) = (1+|\xi|)^{-i\nu}$. This multiplier is of Laplace-transform type in the sense of [**8**, Ch. II, §4], since one easily checks that

$$(1+\lambda)^{-i\nu} = \lambda \int_0^\infty a(t)e^{-\lambda t}\,dt, \qquad \lambda \ge 0, \quad \text{where}$$

$$a(t) = \frac{1}{\Gamma(1+i\nu)}\left[t^{i\nu}e^{-t} + \int_0^t s^{i\nu}e^{-s}\,ds\right].$$

Since $\|a\|_\infty \le Ce^{(\pi/2)|\nu|}$, the general theory of heat-diffusion semigroups [**8**] implies for $1 < p < \infty$

$$\|T_{M_\nu}f\|_p \le C_p e^{(\pi/2)|\nu|}\|f\|_p, \qquad f \in L^p(\mathbb{R}^n), \tag{29}$$

where C_p is a constant depending only on p.

Since $(1+|\xi|)^{1-\varepsilon-\alpha}m(s\xi) = (1+|\xi|)^{-i\nu}(1+|\xi|)m(s\xi)$, and since $\|T_{m(s\cdot)}\|_{p,p} = \|T_m\|_{p,p} \le |B| = 1$ for all p, (29) reduces the proof of (28) finally to estimating the multiplier operator corresponding to

$$m_0(\xi) = -2\pi|\xi|m(\xi). \tag{30}$$

Define measures μ_j by $\mu_j = \partial\chi_B/\partial x_j$, $j = 1, \ldots, n$. Since

$$m_0(\xi) = \sum_{j=1}^n \left(-i\frac{\xi_j}{|\xi|}\right)(-2\pi i\xi_j m(\xi)),$$

we have

$$T_{m_0}f = \sum_{j=1}^n R_j(\mu_j * f), \tag{31}$$

where R_j denotes the jth Riesz transform. By a result of Stein [**10**] (see also [**7**]), it is known that

$$\left\|\left(\sum_j |R_j f|^2\right)^{1/2}\right\|_p \le A_p\|f\|_p, \qquad 1 < p < \infty, \tag{32}$$

where A_p is independent of n. Using a simple duality argument, (31) and (32) imply

$$\|T_{m_0}f\|_p \le A_{p'}\left\|\left(\sum_J |\mu_k * f|^2\right)^{1/2}\right\|_p, \qquad 1 < p < \infty, \tag{33}$$

where $1/p + 1/p' = 1$. Let $g(f)^2(x) = \sum_j |\mu_j * f(x)|^2$. We want to estimate the L^p-operator norm of the sublinear operator g.

If $p = 2$, we obtain from (17)

$$\|g(f)\|_2 = \|T_{m_0} f\|_2 \le \|m_0\|_\infty \|f\|_2 \le C(\sigma, Q)\|f\|_2. \tag{34}$$

For $p = \infty$, we observe that

$$|g(f)(x)| = |(\nabla \chi_B) * f(x)| = \sup_{\eta \in S^{n-1}} |\mu_\eta * f(x)|, \tag{35}$$

where μ_η is defined as in Lemma 4. This in combination with Lemma 4 implies

$$\|g(f)\|_\infty \le \sup_\eta \|\mu_\eta\|_M \|f\|_\infty = 2Q(B)\|f\|_\infty. \tag{36}$$

Interpolation between (34) and (36) yields

$$\|g(f)\|_p \le C(p, \sigma, Q)\|f\|_p, \qquad 2 \le p \le \infty;$$

hence, by (33), also

$$\|T_{m_0} f\|_p \le C(p, \sigma, Q)\|f\|_p, \tag{37}$$

at least for $2 \le p < \infty$, but by passing to the adjoint operator $T^*_{m_0}$, we get (37) also for $1 < p < 2$. This concludes the proof of Lemma 4. □

Proof of Proposition 1. Let $\rho = 1 - \varepsilon \in \,]1/2, 1[$. From Lemma 2 (ii) and (13) it follows easily that the family $\{T^\varepsilon_\alpha\}$ in an admissible family (in the sense of [**12**, Ch. V]) on every strip $-\varepsilon \le \operatorname{Re}\alpha \le N$, $N > 0$. Thus, choosing N sufficiently large and interpolating the estimates in Lemma 2 and Lemma 4 between $\operatorname{Re}\alpha = -\varepsilon$ and $\operatorname{Re}\alpha = N$, we obtain

$$\|T^\varepsilon_{1-\varepsilon} f\|_p \le C_\varepsilon(p, \sigma(B), Q(B))\|f\|_p \tag{38}$$

for any $1 < p \le 2$, hence, by duality, for any $1 < p < \infty$. But,

$$\begin{aligned} m^\varepsilon_{1-\varepsilon}(\xi) &= [I^{-\rho} m(r\xi)]\big|_{r=1} \\ &= -\frac{1}{\Gamma(\varepsilon)} m(\xi) + \frac{1}{\Gamma(\varepsilon)} \int_1^2 (s-1)^{-\rho} \frac{dm(s\xi)}{ds}\, ds. \end{aligned} \tag{39}$$

Moreover, $(\xi \cdot \nabla)^\alpha m(\xi)$ is given by [**5**, p. 51]

$$(\xi \cdot \nabla)^\alpha m(\xi) = \frac{-1}{\Gamma(-\alpha)} \int_1^\infty (s-1)^{-\alpha-1} m(s\xi)\, ds \quad \text{if } -1 < \alpha < 0.$$

By partial integration, we see that

$$(\xi \cdot \nabla)^\alpha m(\xi) = \frac{1}{\Gamma(1-\alpha)} \int_1^\infty (s-1)^{-\alpha} \frac{dm(s\xi)}{ds}\, ds$$

for $0 < \alpha < 1$. A comparison with (39) shows that

$$\begin{aligned}(\xi \cdot \nabla)^\rho m(\xi) &= m_{1-\varepsilon}^{\varepsilon}(\xi) + \frac{1}{\Gamma(\varepsilon)} m(\xi) + \frac{1}{\Gamma(\varepsilon)} \int_2^\infty (s-1)^{-\rho} \frac{dm(s\xi)}{ds}\, ds \\ &= m_{1-\varepsilon}^{\varepsilon}(\xi) - \frac{1}{\Gamma(-\rho)} \int_2^\infty (s-1)^{-\rho-1} m(s\xi)\, ds.\end{aligned}$$

Since $\int_2^\infty (s-1)^{-\rho-1}\, ds < \infty$, this together with (38) implies

$$\|T_{(\xi\cdot\nabla)^\rho} f\|_p \le C_\rho(p, \sigma, Q) \|f\|_p. \qquad \square$$

3. Examples: The l^q-unit balls. In the sequel, let $1 \le q \le \infty$ be fixed, and let

$$B_q = B_q^n = \{x \in \mathbb{R}^n : |x|_q \le 1\}$$

be the unit ball with respect to the l^q-norm $|x|_q = (\sum |x_j|^q)^{1/q}$ (resp. $|x|_\infty = \max |x_j|$, if $q = \infty$).

Let $\kappa(n) = \kappa_q(n)$ denote the volume of B_q^n. A straight-forward calculation, using induction on n, easily yields ($q < \infty$)

$$\kappa_q(n) = 2\Gamma\left(\frac{1}{q} + 1\right) \left[\frac{2}{q} \cdot \Gamma\left(\frac{1}{q}\right)\right]^{n-1} \Big/ \Gamma\left(\frac{n}{q} + 1\right). \tag{40}$$

Choose $m = m_q(n) > 0$ so, that the body $\widetilde{B}_q = mB_q$ has volume 1. (40) implies $m \sim n^{1/q}$ up to a constant a_q (see [**4**]). Of course, if $q = \infty$, we have $\kappa_\infty(n) = 2^n$, and $m = 1/2$. Let us determine the constant L mentioned in (5):

Because of the symmetry properties of $\widetilde{B}_q$, we have for any $\xi \in S^{n-1}$

$$\int_{\widetilde{B}_q} \langle \xi, x\rangle^2\, dx = \sum_j \int_{\widetilde{B}_q} \xi_j^2 x_j^2\, dx = \left(\sum_j \xi_j^2\right) \int_{\widetilde{B}_q} x_n^2\, dx = \int_{\widetilde{B}_q} x_n^2\, dx,$$

and so we may choose $S(B_q)$ to be $\widetilde{B}_q$, and obtain for $L = L(B_q)$

$$\begin{aligned}L^2 = \int_{\widetilde{B}_q} x_n^2\, dx &= 2 \int_0^m x_n^2 (m^q - |x_n|^q)^{(n-1)/q} \kappa_q(n-1)\, dx_n \\ &= 2m^{n+2} \kappa_q(n-1) \mathrm{B}\left(\frac{3}{q}, \frac{n-1}{q} + 1\right),\end{aligned}$$

where B denotes the Beta-function.

Since $m^n \kappa_q(n) = 1$, this yields

$$L^2 = 2m^2 \frac{\kappa_q(n-1)}{\kappa_q(n)} \mathrm{B}\left(\frac{3}{q}, \frac{n-1}{q} + 1\right) \sim A_q^2$$

by Stirling's formula, and so by (5)

$$a_1^{-1}A_q \lesssim \sigma(B_q^n) \lesssim a_1 A_q, \tag{41}$$

at least for $q < \infty$. However, for $q = \infty$ clearly $L^2 = 1/2$, hence $\sigma(B_\infty^n) \approx (2\sqrt{3})^{-1}$.

In order to estimate $Q(\widetilde{B}_q^n)$, we adapt an idea from **[4]**: Let $\tau : [0,\infty[\to [0,1]$ be a smooth function satisfying the conditions ($q < \infty$)

$$\tau = 1 \quad \text{on } [0, m^q], \tag{42}$$

$$\tau = 0 \quad \text{on } [m^q + 1, \infty[, \tag{42$'$}$$

$$-2 \le \tau' \le 0, \tag{42$''$}$$

and set $K(x) = \tau(\sum |x_j|^q)$, $x \in \mathbb{R}^n$. Note that by (42) $\chi_{\widetilde{B}_q} \le K$, and by (42)$'$ $(m^q+1)^{1/q} B_q \subset (1 + c/n)\widetilde{B}_q = \widetilde{\widetilde{B}}_q$, hence $\|K\|_{L^1} \le C$. Moreover, we have

$$\mathrm{Vol}_{n-1}(\pi_\xi(\widetilde{B}_q)) = \frac{1}{2}\left\|\frac{\partial K}{\partial \xi}\right\|_{L^1} \quad \text{for all } \xi \in S^{n-1}. \tag{43}$$

This is in fact true if $B = \widetilde{B}_q$ is any convex body and K any function which is 1 on B, non-increasing with growing distance from B, and such that $\partial K/\partial \xi$ is integrable: We may assume without restriction that $\xi = e_n$. Then, adapting the notations from the proof of Lemma 4,

$$\begin{aligned}\int_{B_x} \left|\frac{\partial K}{\partial \xi}(x,t)\right| dt &= \int_{b(x)}^{\infty} \left|\frac{\partial K}{\partial t}(x,t)\right| dt + \int_{-\infty}^{a(x)} \left|\frac{\partial K}{\partial t}(x,t)\right| dt \\ &= K(x, b(x)) + K(x, a(x)) = 2;\end{aligned}$$

hence

$$\left\|\frac{\partial K}{\partial \xi}\right\|_{L^1} = 2\,\mathrm{Vol}_{n-1}(\pi_\xi(B)).$$

In order to estimate $\|\partial K/\partial \xi\|_{L^1}$, observe that

$$\partial K/\partial \xi = q\tau'\left(\sum |x_j|^q\right) \cdot \sum_j \xi_j \,\mathrm{sgn}(x_j)|x_j|^{q-1},$$

and hence

$$\begin{aligned}\|\partial K/\partial \xi\|_{L^1} &\le 2q \int_{\widetilde{\widetilde{B}}_q} \left|\sum_j \xi_j \,\mathrm{sgn}(x_j)|x_j|^{q-1}\right| dx \\ &= \frac{2q}{2^n} \sum_{\varepsilon_j = \pm 1} \int_{\widetilde{\widetilde{B}}_q} \left|\sum_j \varepsilon_j \xi_j \,\mathrm{sgn}(x_j)|x_j|^{q-1}\right| dx.\end{aligned}$$

However, Khintchine's inequality

$$2^{-n}\sum_{\varepsilon_i=\pm 1}\left|\sum_{j=1}^{n}\varepsilon_j\alpha_j\right| \leq C\left(\sum_j \alpha_j^2\right)^{1/2}, \qquad \alpha_j \in \mathbb{R},$$

implies

$$\|\partial K/\partial\xi\|_{L^1} \leq Cq\int_{\tilde{\tilde{B}}_q}\left[\sum_j \xi_j^2|x_j|^{2(q-1)}\right]^{1/2} dx$$

$$\leq C'q\cdot\left[\int_{\tilde{\tilde{B}}_q}\sum_j \xi_j^2|x_j|^{2(q-1)}\,dx\right]^{1/2}$$

by Hölder's inequality, since $\mathrm{Vol}_n(\tilde{\tilde{B}}_q) \leq C$. Because of the symmetry of $\tilde{\tilde{B}}_q$, this yields

$$\|\partial K/\partial\xi\|_{L^1} \leq C''q\cdot\left[\int_{\tilde{\tilde{B}}_q}|x_n|^{2(q-1)}\,dx\right]^{1/2},$$

and hence, because of (4), (5), (41) and (43),

$$Q(B_q^n) \leq C_q, \qquad 1 \leq q < \infty, \tag{44}$$

independently of n. So Theorem 1 implies

COROLLARY 1. *Let $1 \leq q < \infty$. Then for all $f \in L^p(\mathbb{R}^n)$*

$$\|M_{B_q^n}f\|_p \leq C(p,q)\|f\|_p, \qquad 1 < p \leq \infty,$$

independently of n.

What can be said about the case $q = \infty$?

In this case, an easy geometric consideration shows that for any $\xi \in S^{n-1}$ (see also [**1**], pp. 41, 45)

$$\mathrm{Vol}_{n-1}(\pi_\xi(\tilde{B}_\infty)) = \sum_F \mathrm{Vol}_{N-1}(F)\cdot\langle\xi, n(F)\rangle,$$

where summation is over all faces F of the cube $\tilde{B}_\infty$ whose outward normal $n(F)$ satisfies $\langle\xi, n(F)\rangle \geq 0$. So, if we choose $\xi = n^{-1/2}(1,1,\ldots,1)$, we get

$$\mathrm{Vol}_{n-1}(\pi_\xi(\tilde{B}_\infty)) = \sum_j \xi_j = \sqrt{n}.$$

The same argument easily shows that $\mathrm{Vol}_{n-1}(\pi_\eta(\widetilde{B}_\infty)) \leq \sqrt{n}$ for any $\eta \in S^{n-1}$, and we get

$$Q(B_\infty^n) = \sqrt{n}. \tag{45}$$

So, our criterion gives a bound for $\|M_{B_\infty^n}\|_{p,p}$ which grows with n.

Let us conclude with a direct consequence of our results, which appears a bit surprising at the first glance (we do, however, not claim originality for this result). Let $\Sigma(\widetilde{B}_q^n)$ denote the surface area of $\widetilde{B}_q^n$.

COROLLARY 2. *If* $1 \leq q < \infty$, *then* $c\sqrt{n} \leq \Sigma(\widetilde{B}_q^n) \leq Cq\sqrt{n}$, *whereas* $\Sigma(\widetilde{B}_\infty^n) = 2n$.

Proof. By Cauchy's surface formula [**1**, p. 48]

$$\begin{aligned}\Sigma(\widetilde{B}_q^n) &= \frac{1}{\kappa_2(n-1)} \int_{S^{n-1}} \mathrm{Vol}_{n-1}(\pi_\xi(\widetilde{B}_q^n))\, d\xi \\ &\leq \frac{\Sigma(B_2^n)}{\kappa_2(n-1)} Q(\widetilde{B}_q^n) = \frac{n\kappa_2(n)}{\kappa_2(n-1)} Q(\widetilde{B}_q^n);\end{aligned}$$

hence, by (40), (44), for $q < \infty$

$$\Sigma(\widetilde{B}_q^n) \leq C_q\sqrt{n}.$$

Moreover, it is well known [**1**, p. 104] that the Euclidean ball has minimal surface area among all convex bodies of given volume, and $\Sigma(\widetilde{B}_2^n) = m^{n-1}\Sigma(B_2^n) = m^{n-1}n \cdot \kappa_2(n) = n/m \sim c \cdot \sqrt{n}$ by (40), where $m = m_2(n)$. So we also obtain

$$\Sigma(\widetilde{B}_q^n) \geq \Sigma(\widetilde{B}_2^n) \sim c \cdot \sqrt{n}. \qquad \square$$

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