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EXAMPLES OF FOLIATIONS WITH FOLIATED GEOMETRIC STRUCTURES

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We present examples of foliated compact nilmanifolds, whose foliations are neither simple nor given by suspensions, admitting various foliated geometric structures. For example, we construct foliations which are:

1. transversely symplectic but not transversely Kähler,
2. transversely symplectic but not transversely holomorphic,
3. transversely Sasakian but not transversely cosymplectic.

1. Foliated structures on foliated manifolds. Let $(M, \mathscr{F})$ be a foliated manifold. The foliation $\mathscr{F}$ is called an (N, G)-structure (cf. [**14**]) if $\mathscr{F}$ is given by a cocycle $\mathscr{U} = \{U_i, f_i, g_{ij}\}$ where:

1. $\{U_i\}$ is an open covering of M,
2. $f_i\colon U_i \to N$ are submersions with connected fibres,
3. $g_{ij}\colon f_j(U_i \cap U_j) \to f_j(U_i \cap U_j)$ are local diffeomorphisms of N for which
   (a) $f_i|_{U_i \cap U_j} = g_{ij} \circ f_j|_{U_i \cap U_j}$,
   (b) there exists $h_{ij} \in G$ such that $h_{ij}|_{f_i(U_i \cap U_j)} = g_{ij}$.

If the group G acts quasi-analytically (i.e. if for some element h of G there exists an open subset of N on which h is the identity transformation then h itself is the identity), then we have the following

LEMMA (*Thurston* [14]). *If the group G acts quasi-analytically, then any (N, G)-structure is developable, i.e. there exists a covering $\widehat{M}$ of M and a developing mapping $D\colon \widehat{M} \to M$ such that the lifted foliation $\widehat{\mathscr{F}}$ is given by the fibres of D. Moreover, there is a homomorphism $h\colon \pi_1(M) \to G$, called the holonomy homomorphism, such that the mapping D is $\pi_1(M)$-equivariant for these two actions of the group.*

Any foliated geometric structure of $(M, \mathscr{F})$ defines the corresponding structure on the transverse manifold (cf. [**16**, **17**]), which in this case can be chosen to be an open submanifold of N. If the developing mapping is surjective and its fibres are connected, then the transverse manifold can be identified with N. The holonomy pseudogroup of $(M, \mathscr{F})$ has as its representative a pseudogroup generated by a subgroup $\operatorname{Im} h$ of G (cf. [**10**, **9**]). Therefore, a transverse geometric structure of $(M, \mathscr{F})$ is a geometric structure on N invariant by the holonomy pseudogroup $\operatorname{Im} h$, and any geometric structure on N invariant by the holonomy group defines a transverse, and hence foliated, geometric structure.

Transversely holomorphic and transversely symplectic foliations have been studied from many points of view by various authors, among others by: D. Burns, T. Duchamp, A. El Kacimi-Alaoui, J. Girbau, A. Haefliger, H. Holman, M. Kalka, S. Morita, M. Nicolau, A. Sundararaman and T. Suwa. We refer the reader to the recent book by Ph. Tondeur, cf. [**15**], for the most extensive and up-to-date reference list. As the examples presented in this note have very precise transverse geometric structure, they can be useful in the study of the behavior of additional transverse structures (Kähler, symplectic) under deformations of transversely holomorphic or symplectic foliations, (cf. remark at the end of §2.2), the nontriviality and rigidity of their characteristic classes, (cf. §3), as well as in the study of the behavior of leaves of such foliations. Moreover, being modelled on nilpotent Lie groups, they are transversely affine foliations and therefore they can serve as test examples for various hypotheses concerning the interplay and influence of these various transverse structures.

The paper [**4**] contains a very useful survey of the known results on Kähler structures and nilmanifolds.

2. Construction of the examples. Let N be a simply connected nilpotent Lie group and Γ a torsionfree, finitely generated subgroup of N. Then according to [**12**], or [**13**], Theorems 2.11 and 2.18, there exist a simply connected nilpotent Lie group U containing Γ as a uniform subgroup and a homomorphism $u \colon U \to N$ such that u is the identity on Γ (if we identify the subgroups of U and N isomorphic to Γ). So we have the following commutative diagram:

$$\begin{array}{ccc} \Gamma \subset U & & \\ \cong\uparrow & \searrow_{u} & \\ \Gamma & \rightarrow & N \end{array}$$

The homomorphism u is a surjective submersion with connected fibres since both manifolds U and N are contractible. The foliation defined by the submersion u is Γ-invariant and therefore it projects to a foliation $\mathscr{F}(\Gamma, U, u)$ on the compact manifold $M(\Gamma) = \Gamma \setminus U$. The foliation $\mathscr{F}(\Gamma, U, u)$ is an (N, Γ)-structure, a developable one, and the submersion u is its developing mapping. Therefore, foliated geometric structures on $(M(\Gamma), \mathcal{F}(\Gamma, U, g))$ correspond bijectively to Γ-invariant ones on N.

If the subgroup Γ contains a uniform subgroup Γ_0 of N, then any foliated geometric structure on $(M(\Gamma), \mathcal{F}(\Gamma, U, u))$ defines a geometric structure of the same type on the compact manifold $E(\Gamma_0) = \Gamma_0 \setminus N$. The following diagram presents this correspondence:

$$\begin{array}{ccc} \Gamma \subset U & \xrightarrow{u} & N \supset \Gamma \supset \Gamma_0 \\ \downarrow & & \downarrow \\ \Gamma \setminus U & \dashrightarrow & \Gamma_0 \setminus N \end{array}$$

In fact, any foliated geometric structure on $(M(\Gamma), \mathcal{F}(\Gamma, U, u))$ lifts to a Γ-invariant foliated structure on U. This one, in its turn, defines a Γ-invariant structure on N which projects to a geometric structure on $E(\Gamma_0) = \Gamma_0 \setminus N$.

For example, if $(M(\Gamma), \mathcal{F}(\Gamma, U, u))$ is

— transversely symplectic then $E(\Gamma_0)$ is symplectic,

— transversely holomorphic then $E(\Gamma_0)$ is complex,

— transversely Kähler then $E(\Gamma_0)$ is Kähler,

— transversely Hermitian then $E(\Gamma_0)$ is Hermitian, and so on.

We are now going to present examples of foliations on compact nilmanifolds which are

1. transversely symplectic but never transversely Kähler;
2. transversely symplectic and holomorphic but never transversely Kähler, on complex and non-complex nilmanifolds;
3. transversely symplectic but not transversely holomorphic;
4. transversely Sasakian but never transversely cosymplectic.

Our examples are based on the first author's joint work with M. Fernández, A. Gray and others on geometric structures on compact nilmanifolds.

The general scheme for the constructions is the following. First, we consider a simply connected nilpotent group N of uppertriangular matrices and $\Gamma_0 \subset N$ the uniform subgroup of matrices with integral entries. Next, we take a subgroup Γ of N whose matrices have some entries of the form $a_i + sb_i$ where $s \notin \mathbf{Q}$ and $a_i, b_i \in \mathbf{Z}$. This subgroup Γ can be represented as a uniform subgroup of some group of uppertriangular matrices, which will be the group U of the construction described above.

Let us pass to the precise examples.

2.1. A transversely symplectic but not transversely Kähler foliation. Let us consider $N = (\mathbf{R}^4, *)$ with the following group operation:

$$(a, b, c, d) * (x, y, z, t) = (a + x, b + y, c + z + ay, d + t).$$

In the matrix form the group $N = (\mathbf{R}^4, *)$ can be represented as follows:

$$\begin{pmatrix} 1 & x & t & z \\ & 1 & 0 & y \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}.$$

As the group Γ_0 we take $(\mathbf{Z}^4, *)$, and as $\Gamma \supset \Gamma_0$ we take the group of matrices of the form

$$\begin{pmatrix} 1 & x_1 + sx_2 & t & z_1 + sz_2 \\ & 1 & 0 & y \\ & & 1 & 0 \\ & & & 1 \end{pmatrix},$$

where $s \notin \mathbf{Q}$ and $x_1, x_2, y, z_1, z_2, t \in \mathbf{Z}$.

In this case the group U is $\mathbf{R}^6$ with the following group operation:

$$\begin{aligned}(a_1, a_2, b, c_1, c_2, d)\square(x_1, x_2, y, z_1, z_2, t) \\ = (a_1 + x_1, a_2 + x_2, b + y, c_1 + z_1 + a_1 y, c_2 + z_2 + a_2 y, d + t),\end{aligned}$$

and the subgroup Γ is isomorphic to $(\mathbf{Z}^6, \square)$.

The submersion $u: U = (\mathbf{R}^6, \square) \to N = (\mathbf{R}^4, *)$ is given by the correspondence

$$(x_1, x_2, y, z_1, z_2, t) \mapsto (x_1 + sx_2, y, z_1 + sz_2, t).$$

The foliation obtained in this way cannot be transversely Kähler because $E(\Gamma_0) = \Gamma_0 \backslash N$ is Kähler if and only if N is commutative (cf. [**2**, **4**]). This foliation is transversely symplectic as the form

$$\Omega = dx \wedge (dz - xdy) + dy \wedge dt$$

is a closed left invariant 2-form on N of maximal rank.

Summing up, we have constructed a transversely sumplectic foliation $\mathscr{F}(\Gamma, U, u)$ of codimension 4 on a real compact manifold $M(\Gamma)$ of dimension 6, and such a foliation cannot be made transversely Kähler.

2.2. Transversely symplectic and transversely holomorphic but not transversely Kähler foliations on real and complex manifolds. Let $E(\Gamma_0) = \Gamma_0 \setminus N$ be the Iwasawa manifold, i.e. N is the complex Lie group of complex matrices of the form

$$\begin{pmatrix} 1 & z_1 & z_3 \\ & 1 & z_2 \\ & & 1 \end{pmatrix},$$

and Γ_0 is the subgroup of N of those matrices whose entries are Gauss integers.

A basis of holomorphic left invariant 1-forms on N given by

$$\begin{aligned} \alpha &= dz_1, \\ \beta &= dz_2, \\ \gamma &= dz_3 - z_1\, dz_2, \end{aligned}$$

and it verifies

$$d\alpha = 0, \quad d\beta = 0, \quad d\gamma = -\alpha \wedge \beta.$$

Let us put

$$\begin{aligned} \alpha &= \alpha_1 + \sqrt{-1}\alpha_2, \\ \beta &= \beta_1 + \sqrt{-1}\beta_2, \\ \gamma &= \gamma_1 + \sqrt{-1}\gamma_2. \end{aligned}$$

Then

$$\Omega = \alpha_1 \wedge \gamma_1 - \alpha_2 \wedge \gamma_2 + \beta_1 \wedge \beta_2$$

is a left invariant symplectic form on N.

Let $\Gamma_1 \supset \Gamma_0$ be the subgroup of N of matrices of the form

$$\begin{pmatrix} 1 & x_1 + \sqrt{-1}(y_1 + sy_1') & x_3 + sx_3' + \sqrt{-1}(y_3 + sy_3') \\ & 1 & x_2 + \sqrt{-1}y_2 \\ & & 1 \end{pmatrix},$$

where $s \notin \mathbf{Q}$, and $x_i, x_i', y_i, y_i' \in \mathbf{Z}$.

Then Γ_1 can be considered as a uniform subgroup of $\mathbf{R}^9$ with the group operation:

$$\begin{aligned} &(a_1, \ldots, a_9) \square (x_1, \ldots, x_9) \\ &\quad = (a_i + x_i, a_6 + x_6 + a_1x_4 - a_2x_5, \\ &\qquad a_7 + x_7 - a_3x_5, a_8 + x_8 + a_1x_5 + a_2x_4, a_9 + x_9 + a_3x_4). \end{aligned}$$

The matricial form of $U = (\mathbf{R}^9, \square)$ is the following:

$$\begin{pmatrix} 1 & x_3 & x_7 & x_9 & x_1 & x_2 & x_6 & x_8 \\ & 1 & -x_5 & x_4 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 & 0 \\ & & & & 1 & 0 & x_4 & x_5 \\ & & & & & 1 & -x_5 & x_4 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{pmatrix}.$$

The submersion $u\colon U = (\mathbf{R}^9, \square) \to N$ is given by the correspondence:

$$(x_1, \dots, x_9) \mapsto (x_1 + \sqrt{-1}(x_2 + sx_3), x_4 + \sqrt{-1}x_5, x_6 + sx_7 + \sqrt{-1}(x_8 + sx_9)).$$

In this case, the resulting foliation $\mathcal{F}(\Gamma_1, U, u)$ is transversely symplectic and transversely holomorphic of complex codimension 3 on a real compact manifold $M(\Gamma_1)$ of real dimension 9, and it cannot be made transversely Kähler (cf. [**3**]).

A completely different example arises if we consider the group $\Gamma_2 \supset \Gamma_0$ of matrices of the form

$$\begin{pmatrix} 1 & z_1 + sz_1' & z_3 + sz_3' \\ & 1 & z_2 \\ & & 1 \end{pmatrix}$$

where $s \notin \mathbf{Q}$ and $z_1, z_1', z_2, z_3, z_3'$ are Gauss integers.

The subgroup Γ_2 can be considered as a uniform subgroup of $\mathbf{C}^5$ with the following group operation:

$$(u_1, \dots, u_5)\square(z_1, \dots, z_5) = (z_i + u_i, z_4 + u_4 + u_1 z_3, z_5 + u_5 + u_2 z_3).$$

The submersion $u\colon U = (\mathbf{C}^5, \square) \to N$ is given by the correspondence

$$(z_1, z_2, z_3, z_4, z_5) \mapsto (z_1 + sz_2, z_3, z_4 + sz_5).$$

The foliation $\mathcal{F}(\Gamma_2, U, u)$ obtained in this way is holomorphic and transversely symplectic of complex codimension 3 on a compact complex nilmanifold $M(\Gamma_2)$ of complex dimension 5, and it cannot be made transversely Kähler either.

The group $(\mathbf{C}^5, \square)$ can be represented in the matrix form as

$$\begin{pmatrix} 1 & z_1 & z_4 & z_2 & z_5 \\ & 1 & z_3 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & z_3 \\ & & & & 1 \end{pmatrix}.$$

and the submersion $u\colon U = (\mathbf{C}^5, \square) \to N$ is given by

$$\begin{pmatrix} 1 & z_1 & z_4 & z_2 & z_5 \\ & 1 & z_3 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & z_3 \\ & & & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & z_1 + s z_2 & z_4 + s z_5 \\ & 1 & z_3 \\ & & 1 \end{pmatrix}.$$

REMARK. Examples of this type have been considered, from another point of view, by A. El Kacimi and M. Nicolau in [**7**]; there, only the non-existence of transverse Kähler structures is studied. Nevertheless, the existence of foliated symplectic structures has a significant influence on the cohomological structure of the foliated manifold (cf. [**1**]).

2.3. A transversely symplectic but not transversely Kähler foliation on a compact non-complex nilmanifold. Let N be the real nilpotent Lie group of complex matrices of the form

$$\begin{pmatrix} 1 & \bar z_1 & z_2 \\ & 1 & z_1 \\ & & 1 \end{pmatrix}$$

where $z_1, z_2 \in \mathbf{C}$, and let Γ_0 be the uniform subgroup of matrices with Gauss integer entries. Then $E(\Gamma_0) = \Gamma_0 \setminus N$ is the well-known Kodaira-Thurston manifold (cf. [**8**]).

Let us consider the left invariant 1-forms over N given by:

$$\alpha = dz_1,$$
$$\beta = dz_2 - \bar z_1\, dz_2;$$

they define real 1-forms

$$\alpha_1 = dx_1,$$
$$\alpha_2 = dy_1,$$
$$\beta_1 = dx_2 - x_1\, dx_1 - y_1\, dy_1,$$
$$\beta_2 = dy_2 - x_1\, dy_1 + y_1\, dx_1,$$

where $z_1 = x_1 + \sqrt{-1}y_1, z_2 = x_2 + \sqrt{-1}y_2$. Then

$$d\alpha_1 = d\alpha_2 = d\beta_1 = 0, \qquad d\beta_2 = -2\alpha_1 \wedge \alpha_2.$$

Therefore, the 2-form

$$\Omega = \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2$$

is a left invariant symplectic form on N.

As the group $\Gamma \supset \Gamma_0$ we take the group of matrices of the form

$$\begin{pmatrix} 1 & \bar{z}_1 + s\bar{z}'_1 & z_2 + sz'_2 + s^2 z''_2 \\ & 1 & z_1 + sx'_1 \\ & & 1 \end{pmatrix},$$

where $s \notin \mathbf{Q}$ and z_1, z'_1, z_2, z'_2 and z''_2 are Gauss integers.

The group Γ can be considered as the uniform subgroup of Gauss integers 5-tuples in $(\mathbf{C}^5, \square)$, where $\square$ is the following group operation:

$$\begin{aligned}(a_1, \dots, a_5)\square(z_1, \dots, z_5) = (a_i + z_i, a_3 + z_3 + \bar{a}_1 z_1, \\ a_4 + z_4 + \bar{a}_1 z_2 + \bar{a}_2 z_1, a_5 + z_5 + \bar{a}_2 z_2).\end{aligned}$$

The group $(\mathbf{C}^5, \square)$ can be represented in matricial form by

$$\begin{pmatrix} 1 & \bar{z}_1 & \bar{z}_2 & z_3 & z_4 & z_5 \\ & 1 & 0 & z_1 & z_2 & 0 \\ & & 1 & 0 & z_1 & z_2 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}.$$

The submersion $u\colon U = (\mathbf{C}^5, \square) \to N$ is given by the correspondence

$$(z_1, z_2, z_3, z_4, z_5) \mapsto (z_1 + sz_2, z_3 + sz_4 + s^2 z_5).$$

The foliation $\mathscr{F}(\Gamma, U, u)$ constructed in this example is transversely symplectic and transversely holomorphic of complex codimension 2 on a complex manifold $M(\Gamma)$ of complex dimension 5 (which is a real nilmanifold but not a complex nilmanifold, cf. **[4]**).

2.4. A transversely symplectic but not transversely holomorphic foliation. To construct this example we shall consider a compact 4-dimensional nilmanifold which is symplectic but does not admit any complex structure (cf. **[8]** and **[6]**).

Let N be the 4-dimensional Lie group of real matrices of the form

$$\begin{pmatrix} 1 & y & 2x & -(z/n) & -(t/q) \\ & 1 & 0 & 0 & 0 \\ & & 1 & 2y & ny^2/2 \\ & & & 1 & ny/2 \\ & & & & 1 \end{pmatrix},$$

where $n, q \in \mathbf{Z}$ are nonzero and fixed. Then $N = (\mathbf{R}^4, *)$, where

$$(a,b,c,d)*(x,y,z,t) = (a+x, b+y, c+z-2nay, d+t-nqay^2+qcy).$$

As the subgroup Γ_0 we take the integer lattice, and then the compact nilmanifold $E(\Gamma_0) = \Gamma_0 \setminus N$ is symplectic but not complex (cf. **[8]**).

Now, let us consider the group Γ of the matrices of the form

$$\begin{pmatrix} 1 & b & 2(a+sa') & -2(c+sc')/n & -(d+sd')/q \\ & 1 & 0 & 0 & 0 \\ & & 1 & 2b & nb^2/2 \\ & & & 1 & nb/2 \\ & & & & 1 \end{pmatrix}$$

where $s \notin \mathbf{Q}$ and $a, a', b, c, c', d, d' \in \mathbf{Z}$.

Then Γ can be imbedded as a uniform subgroup of $\mathbf{R}^7$ with the following group operation:

$$\begin{aligned}(a_1, \dots, a_7)\Box(x_1, \dots, x_7) \\ = (a_i + x_i, a_4 + x_4 - 2na_1x_3, a_5 + x_5 - 2na_2x_3, \\ a_6 + x_6 - nqa_1x_3^2 + qa_4x_3, a_7 + x_7 - nqa_2x_3^2 + qa_5x_3).\end{aligned}$$

The group $U = (\mathbf{R}^7, \Box)$ can be represented as the following group of matrices:

$$\begin{pmatrix} 1 & x_3 & 2x_1 & 2x_2 & -2x_4/n & -2x_5/n & -x_6/q & -x_7/q \\ & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 2x_3 & 0 & nx_3^2/2 & 0 \\ & & & 1 & 0 & 2x_3 & 0 & nx_3^2/2 \\ & & & & 1 & 0 & nx_3/2 & 0 \\ & & & & & 1 & 0 & nx_3/2 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{pmatrix}.$$

The submersion $u \colon U \to N$ is given by the correspondence

$$(x_1, \dots, x_7) \mapsto (x_1 + sx_2, x_3, x_4 + sx_5, x_6 + sx_7).$$

The foliation $\mathcal{F}(\Gamma, U, u)$ is, in this case, transversely symplectic of real codimension 4 on a real compact nilmanifold $M(\Gamma)$ of real dimension 7, and it cannot be made transversely holomorphic (since the results in **[8]** ensure that $E(\Gamma_0) = \Gamma_0 \setminus N$ is never a complex manifold).

2.5. A transversely Sasakian but not transversely cosymplectic foliation. Let $N = H(r, 1), r \geq 1$, be the Heisenberg group of real matrices of the form

$$\begin{pmatrix} 1 & A & c \\ & I_r & B \\ & & 1 \end{pmatrix}$$

where A is a $1 \times r$ matrix, B is an $r \times 1$ matrix and c is a real number. As the subgroup Γ_0 we take the group of matrices of the same form with integer entries, and as the subgroup Γ the group of matrices of the form

$$\begin{pmatrix} 1 & A + sA' & c + sc' \\ & I_r & B \\ & & I \end{pmatrix}$$

where A, A', B are matrices with integer entries, c, c' are integers too, and $s \notin \mathbf{Q}$.

The group Γ can be identified with the integer lattice of the group U of real matrices of the form

$$\begin{pmatrix} 1 & a_1 & \cdots & a_r & a'_1 & \cdots & a'_r & c & c' \\ & 1 & \cdots & 0 & 0 & \cdots & 0 & b_1 & 0 \\ & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & 1 & 0 & \cdots & 0 & b_r & 0 \\ & & & & 1 & \cdots & 0 & 0 & b_1 \\ & & & & & \ddots & \vdots & \vdots & \vdots \\ & & & & & & 1 & 0 & b_r \\ & & & & & & & 1 & 0 \\ & & & & & & & & 1 \end{pmatrix}.$$

The submersion $u \colon U \to N$ is given by the correspondence

$$(a_i, a'_i, c, c', b_i) \mapsto (a_i + sa'_i, c + sc', b_i).$$

From [**5**] it is known that $E(\Gamma_0) = \Gamma \setminus N$ admits a Sasakian structure. Then the resulting foliation $\mathscr{F}(\Gamma, U, u)$ is transversely Sasakian of codimension $2r+1$ on a compact real manifold of dimension $4r+1$; it cannot be made transversely cosymplectic (cf. [**5**]).

2.6. Other examples. Let $N = H(1, r)$, $r \geq 1$, be the Heisenberg group of real matrices of the form

$$\begin{pmatrix} I_r & A & C \\ & 1 & b \\ & & 1 \end{pmatrix}$$

where A, C are $r \times 1$ matrices and b is a real number.

Let Γ_0 be the integer lattice of N, and let Γ be the subgroup of matrices of the form

$$\begin{pmatrix} I_r & A + sA' & C + sC' \\ & 1 & b \\ & & 1 \end{pmatrix}$$

where A, A', C, C' are integer matrices, b is an integer too, and $s \notin \mathbf{Q}$.

Then the group Γ can be considered as a uniform subgroup of the group U of matrices of the form

$$\begin{pmatrix} & A & c \\ I_{2r} & A' & C' \\ & 1 & b \\ & & 1 \end{pmatrix}$$

The submersion $u\colon U \to N$ is given by the correspondence

$$\begin{pmatrix} & A & C \\ I_{2r} & A' & C' \\ & 1 & b \\ & & 1 \end{pmatrix} \mapsto \begin{pmatrix} I_r & A+sA' & C+sC' \\ & 1 & b \\ & & 1 \end{pmatrix}.$$

According to the results of **[5]** the foliation $\mathscr{F}(\Gamma, U, u)$ of $M(\Gamma)$ admits

1. a foliated nonnormal almost cosymplectic structure, but no foliated cosymplectic structure (for $r \geq 1$);
2. if $r = 2p$ or $r = 4p + 1$, no foliated Sasakian structure;
3. if $r = 2p$, a foliated semi-cosymplectic normal structure;
4. if $r = 2p + 1 (p \geq 0)$, a foliated normal structure.

In particular, if $r = 1$ the foliation modeled on $H(1,1)$ always admits a foliated Sasakian structure.

3. Final remarks. Let us recall the following result of E. Macías (cf. **[11]**):

THEOREM. *If a dense subgroup Γ of a simply connected nilpotent Lie group N contains a uniform subgroup Γ_0 then the mapping*

$$H^*(M) \cong H^*(\Gamma_0 \setminus N) \to H^*(\Gamma \setminus U) \cong H^*(\Gamma),$$

induced by the mapping $u\colon U \to N$ defined by Γ, is injective. (U *is the Mal'cev completion of* Γ.)

This theorem ensures that any invariant symplectic form on N is mapped to a non-zero cohomology class in $H^*(\Gamma \setminus U)$. This means that the characteristic classes defined by the elements ω^k of the complex $W(\mathrm{sp}(q), 2)_{2q}$ (cf. **[1]**) are non-zero for such foliations.

Therefore, the example constructed in 2.3 is a new example of a transversely symplectic foliation with non-trivial characteristic classes. The other examples can be reworked so that the subgroups Γ would be dense, and therefore they would provide examples of transversely symplectic foliations with non-trivial characteristic classes.

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