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# ON THE RESULTANT HYPERSURFACE

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**The *resultant* $R(f,g)$ of two polynomials $f$ and $g$ is an irreducible polynomial such that $R(f,g)=0$ if and only if the equations $f=0$ and $g=0$ have one common root.**

**When $g=f'/p$, then $D(f)=R(f,g)$ is called the *discriminant* of $f$ and the *discriminant hypersurface* $D_p=\{f\in \mathbf{C}^p, D(f)=0\}$ can be identified to the discriminant of a versal deformation of the simple hypersurface singularity $A_{p-1}: x^p=0$. In particular, the fundamental group $\pi=\pi_1(\mathbf{C}^p\backslash D_p)$ is the famous *braid group* and $\mathbf{C}^p\backslash D_p$ in fact a $K(\pi,1)$ space.**

**Here we prove the following.**

THEOREM. $\pi_1(\mathbf{C}^{p+q}\backslash R_{p,q})=Z$.

**As $\mathbf{C}^p\backslash D_p$ can be regarded as a linear section of $\mathbf{C}^{p+q}\backslash R_{p,q}$, this theorem shows that by a nongeneric linear section the fundamental group may change drastically, in contrast with the case of generic section.**

Let $f=x^p+a_1x^{p-1}+\cdots+a_p$ and $g=x^q+b_1x^{q-1}+\cdots+b_q$ be two monic polynomials with complex coefficients of degree $p$ and $q$ respectively.

The *resultant* of them $R(f,g)$ is an irreducible polynomial in the coefficients $a_i, b_j$ such that $R(f,g)=0$ if and only if the equations $f=0$ and $g=0$ have at least one common root. Explicitly, the resultant is given by the next formula (see for instance **[5]**, p. 136):

$$R(f,g)=R(a,b)=\begin{vmatrix} 1 & a_1 & \cdots & \cdots & a_p & \cdots & 0 & \cdots & 0 \\ & 1 & a_1 & \cdots & \cdots & \cdots & a_p & \cdots & 0 \\ &&&&&& 1 & \cdots & a_p \\ 1 & b_1 & \cdots & b_q & & & 0 & \cdots & 0 \\ & 1 & b_1 & \cdots & \cdots & \cdots & b_q & \cdots & 0 \\ &&&&&& 1 & \cdots & b_q \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} q \text{ lines} \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} p \text{ lines}\end{matrix}$$

When $g=f'/p$, then $D(f)=(f,g)$ is called the *discriminant* of the polynomial $f$ and the *discriminant hypersurface* $D_p=\{f\in\mathbf{C}^p, D(f)=0\}$ has occurred several times in Singularity Theory, since it can be identified to the discriminant of a versal deformatioin of the simple hypersurface singularity $A_{p-1}: x^p=0$, see for instance **[1]**, **[3]**, **[9]**. In particular, the fundamental group $\pi = \pi_1(\mathbf{C}^p \backslash D_p)$ is the famous *braid group* **[1]** (with $p$ strings) and $\mathbf{C}^p \backslash D_p$ is in fact a $K(\pi, 1)$ space.

In this note we consider the analogous *resultant hypersurface*

$$R_{p,q} = \{(f, g) \in \mathbf{C}^{p+q}; R(f, g) = 0\}$$

and prove the following.

THEOREM. $\pi_1(\mathbf{C}^{p+q} \backslash R_{p,q}) = Z$.

Since $\mathbf{C}^p \backslash D_p$ can be regarded as a linear section of $\mathbf{C}^{p+q} \backslash R_{p,q}$, this theorem shows that by a nongeneric linear section the fundamental group may change drastically, in contrast with the case of generic section **[4]**.

It is also interesting to note that the complements $F_{p,q} = \mathbf{C}^{p+q} \backslash R_{p,q}$ have already occurred in an important topological problem **[7]**, going back to certain questions in Control Theory **[2]**. In short, consider the space of rational *real* functions of the form

$$\phi = \frac{x^n + \alpha_1 x^{n-1} + \cdots + \alpha_n}{x^n + \beta_1 x^{n-1} + \cdots + \beta_n}$$

with $\alpha_i, \beta_j \in R$ and the numerator and the denominator having no common root. Then $\phi$ induces a continuous map $P^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\} = P^1(\mathbf{C})$ of degree $n$ and its restriction to the equator $R \cup \{\infty\} = S^1 \subset S^2 = P^1(\mathbf{C})$ gives a map $S^1 \to S^1$ having degree $r$ such that $-n \leq r \leq n$ and $n - r \equiv 0 \bmod 2$. Let $E_{n-r}$ denote the space of these mappings with $n$ and $r$ fixed, with the obvious topology. Then Segal has shown in **[7]** that $E_{n,r}$ is homeomorphic to $F_{p,q}$ with $p+q = n$ and $p - q = r$. He has also proved our Theorem in the special case $p = q$, by a method completely different from ours.

We derive our Theorem from some basic properties of the resultant hypersurface (which are also interesting in themselves) combined with a deep result of Lê-Saito **[6]** on the connectivity of the Milnor fiber of non-isolated singularity.

LEMMA 1. *$R \in \mathbf{C}[a, b]$ is a weighted homogeneous polynomial of degree $pq$ with respect to the weights* $\mathrm{wt}(a_i) = \mathrm{wt}(b_i) = i$.

*Proof*. Note that the polynomial $t \cdot f = x^p + ta_1 x^{p-1} + \cdots + t^p a_p$ has as roots the elements $tx_i$, where $x_i$ are the roots of $f$, for any $t \in \mathbf{C}^*$. Then, using **[5]**, p.137, we get $R(t \cdot f, t \cdot g) = \prod_{i,j}(tx_i - ty_j) = t^{pq} \prod_{i,j}(x_i - y_j) = t^{pq} R(f, g)$, where $y_j$ are the roots of $g$. □

The key remark in the proof is that the resultant hypersurface has a *smooth normalization* $\nu$ which can be described explicitly as follows:

$$\nu = \mathbf{C} \times \mathbf{C}^{p-1} \times \mathbf{C}^{q-1} \to R_{p,q} \subset \mathbf{C}^{p+q}$$

$\nu(t, \alpha, \beta) = ((x-t)f_\alpha, (x-t)g_\beta)$, where $f_\alpha = x^{p-1} + \alpha_1 x^{p-2} + \cdots + \alpha_{p-1}$, $g_\beta = x^{q-1} + \beta_1 x^{q-2} + \beta_1 x^{q-2} + \cdots + \beta_{q-1}$. Then $\nu$ is clearly surjective onto $R_{p,q}$ and the cardinal of a fiber $\nu^{-1}(f, g)$ is equal to the number of common roots of the equations $f = 0$, $g = 0$, counted without taking their multiplicities into account. Hence $\nu$ is a finite morphism which is generically one-to-one so that $\nu$ is indeed a normalization for $R_{p,q}$.

We use $\nu$ to investigate the singularities of the hypersurface $R_{p,q}$. To do this, we first compute the differential of $\nu$ at a point $(t_0, \alpha_0, \beta_0)$:

$$\begin{aligned} &d\nu(t_0, \alpha_0, \beta_0)(t, \alpha, \beta) \\ &\quad = ((x - t_0)(f_\alpha - x^{p-1}) - t f_{\alpha_0}, (x - t_0)(g_\beta - x^{q-1}) - t g_{\beta_0}). \end{aligned}$$

Assume that $t_0$ is not a root for $f_{\alpha_0}$ and $g_{\beta_0}$ simultaneously. Then it follows that $d\nu(t_0, \alpha_0, \beta_0)$ is an injective linear map and its image (which is a hyperplane in the vector space $V$ of all the pairs $(A, B)$, with $A, B \in \mathbf{C}[x]$, $\deg A \leq p-1$, $\deg B \leq q-1$) is given by the equation

$$f_{\alpha_0}(t_0)B(t_0) - g_{\beta_0}(t_0)A(t_0) = 0.$$

Let $d(f, g)$ be the greatest common divisor of the polynomials $f$ and $g$. The above computation gives us the next

COROLLARY 2. *The point $(f, g)$ is nonsingular on the hypersurface $R_{p,q}$ if and only if* $\deg d(f, g) = 1$.

*Proof*. Use the fact that a point $(f, g) \in R_{p,q}$ is nonsingular if and only if $\nu^{-1}(f, g)$ consists of one point, say $y$, and the corresponding germ $\nu: (\mathbf{C}^{p+q}, y) \to (R_{p,q}, (f, g))$ is an isomorphism. □

We have also the more general result.

PROPOSITION 3. *Assume that $d(f, g) = (x - t_1) \ldots (x - t_s)$ is a product of $s$ linear distinct factors. Then the germ $(R_{p,q}, (f, g))$ consists of $s$ smooth hypersurface germs passing through $(f, g)$ with normal crossings.*

*Proof*. In this case the fiber $\nu^{-1}(f, g)$ consists of $s$ points, say $y_k$ with $k = 1, \ldots, s$. Moreover, the germs $\nu_i: (\mathbf{C}^{p+q-1}, y_i) \to (R_{p,q}, (f, g)) \subset (\mathbf{C}^{p+q}, (f, g))$ induced by $\nu$ are all imbeddings and $H_i = \mathrm{im}(\nu_i)$ are precisely the (smooth) irreducible components of the germ $(R_{p,q},(f,g))$. The corresponding tangent spaces are $T_k = T_{(f,g)}H_k\colon \overline{f}(t_k)B(t_k) - \overline{g}(t_k)A(t_k) = 0$ for $K-1,\dots,s$ and $\overline{f} = f/d(f,g)$, $\overline{g} = g/d(f,g)$. The condition of normal crossing in this case means that $\operatorname{codim}(\bigcap_{k=1,s} T_k) = s$.

But this intersection corresponds to the kernel of the following linear map. $T\colon V \simeq \mathbf{C}^{p+q} \to \mathbf{C}[x]/(d(f,g)) \simeq \mathbf{C}^s$ such that the $k$th component of $T(A,B)$ is just the evaluation on $t_k$ of $(\overline{f}\cdot B - \overline{g}\cdot A)$, for $k=1,\dots,s$. It is easy to check that $T$ is a surjective map and hence $\operatorname{codim}(\bigcap_{k=1,s} T_k) = \operatorname{codim}(\ker T) = s$.

COROLLARY 4. *The hypersurface $R_{p,q}$ has only normal crossings singularities in codimension* 1 *and hence $\pi_1(\mathbf{C}^{p+q}\backslash R_{p,q}) = Z$.*

*Proof*. The singularities of $R_{p,q}$ which are not normal crossings (as described in Proposition 3) lie in the image of the map

$$\tau\colon \mathbf{C}\times\mathbf{C}^{p-2}\times\mathbf{C}^{q-2} \to R_{p,q},$$

$$\tau(t,\alpha,\beta) = ((x-t)^2\widetilde{f}_\alpha,(x-t)^2\widetilde{g}_\beta)$$

with $\widetilde{f}_\alpha$, $\widetilde{g}_\beta$ having a meaning similar to $f_\alpha$, $g_\beta$. But $\dim(\operatorname{im}\tau) \le p+q-3 = \dim R_{p,q} - 2$ which proves the first assertion above. Next consider the fibration $F \to \mathbf{C}^{p+q}\backslash R_{p,q} \to \mathbf{C}^*$ with $F = F^{-1}(1) = \{(f,g) \in \mathbf{C}^{p+q}; R(f,g) = 1\}$. Using the weighted homogeneity of $R$ given by Lemma 1, we can identify this fibration with the Milnor fibration of the hypersurface singularity $(R_{p,q},(x^p,y^q))$. It follows by [6] that $\prod_1(F) = 0$ and hence we get an isomorphism

$$R_\# = \prod_1(\mathbf{C}^{p+q}\backslash R_{p,q}) \to \prod_1(\mathbf{C}^*) = Z.$$

This ends the proof of this corollary as well as giving a more precise version of our Theorem above.

REMARK 5. There is a natural $\mathbf{C}$-action on $\mathbf{C}^{p+q}$ leaving the resultant hypersurface $R_{p,q}$ invariant. Namely we define the translation of an element $(f,g)$ by the complex number $\lambda$ to be the element $(f^\lambda, g^\lambda)$ where

$$f^\lambda = \prod_{i=1,p}(x-x_i-\lambda), \qquad g^\lambda = \prod_{j=1,q}(x-y_j-\lambda)$$

with $x_i$ (resp. $y_j$) being the roots of $f$ (resp. $g$). Since the hyperplane $a_1 = 0$ is clearly transversal to all the $C$-orbits, it follows that

$$R_{p,q} = \overline{R}_{p,q} \times \mathbf{C} \quad \text{with } \overline{R}_{p,q} = R_{p,q} \cap \{a_1 = 0\}.$$

The first non-trivial case of a resultant hypersurface is for $p = q = 2$. Then $\overline{R}_{2,2}$ is just the Whitney umbrella $W: \overline{b}_2^2 - b_1^2 a_2 = s$, with $\overline{b}_2 = b_2 - a_2$, called also a $D_\infty$-surface singularity for a pinch point. It follows that $\mathbf{C}^4 \backslash R_{2,2} = (\mathbf{C}^3 \backslash W) \times \mathbf{C}$ and the homotopy groups of $\mathbf{C}^3 \backslash W$ can be derived from the Milnor fibration $F_\infty \to \mathbf{C}^3 \backslash W \to \mathbf{C}^*$ associated to the $D_\infty$-singularity **[8]**. It is known that $F_\infty$ has the homotopy type of the 2-sphere $S^2$ and hence

$$\prod_k (\mathbf{C}^4 \backslash R_{2,2}) = \prod_k (S^2) \quad \text{for } k \geq 2.$$

In particular $\mathbf{C}^4 \backslash R_{2,2}$ is not a $K(Z, 1)$ space, since $\Pi_2(\mathbf{C}^4 \backslash R_{2,2}) = Z$.

## References


[1] E. Brieskorn *Sur les groupes de tresses* (d'aprés V. I. Arnold), Séminaire Bourbaki 1971/72, Exposées 400–417, p. 21–44, Lecture Notes in Mathematics 314, Springer 1973.

[2] R. Brockett, *Some geometric questions in the theory of linear systems*, IEEE Trans. of Automatic Control, bf21 (1976), 449–455

[3] A. Dimca and R. Rosianu, *The Samuel stratification of the discriminant is Whitney regular*, Geom. Dedicata, **17** (1984), 181–184.

[4] H. Hamm and D.-T. Le, *Un théorème de Zariski du type de Lefschetz*, Ann. Sci. Ec. Norm. Sup. **6** (1973), 317–366.

[5] S. Lang, *Algebra*, Addison-Wesley (1965), Reading, Massachusetts.

[6] D. T. Lê and K. Saito, *The local* $\prod_1$ *of the complement of a hypersurface with normal crossings in codimension* 1 *is abelian*, Ark. Mat., **22** (1984), 1–24.

[7] G. Segal, *The topology of spaces of rational functions*, Acta Math., **143** (1979), 39–72.

[8] D. Siersma, *Isolated line singularities*, Proc. Symp. Pure Math., **40** (1983) part 2, p. 485–496, Amer. Math. Soc., 1983.

[9] B. Teissier, *The hunting of invariants in the geometry of discriminants*, in *Real and complex Singularities* (Oslo, 1976), Sythoff and Noordhoff (1977), 566–677.


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