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PIECEWISE SMOOTH APPROXIMATIONS TO q-PLURISUBHARMONIC FUNCTIONS

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It is shown that q-plurisubharmonic functions can be approximated by piecewise smooth q-plurisubharmonic functions, and that analytic multifunctions are intersections of analytic multifunctions whose graphs are unions of complex analytic manifolds of the appropriate dimensions.

1. Introduction. This research is an outgrowth of an attempt to answer a question raised by Ted Gamelin in lectures delivered at the University of Washington in the spring of 1986: Can an analytic multifunction on an open set W in $\mathbb{C}$, whose values are subsets of $\mathbb{C}^n$, be approximated from above by analytic multifunctions whose graphs are unions of analytic disks? The question is intimately related to approximating $(n-1)$-plurisubharmonic functions on $W \times \mathbb{C}^n$ by functions of the same type which are smooth enough to allow construction of the disks.

A smooth C^2 function on an open set in $\mathbb{C}^n$ is said to be q-plurisubharmonic $(0 \leq q \leq n-1)$ if its complex Hessian has at least $(n-q)$ non-negative eigenvalues everywhere. The concept was introduced by Andreotti and Grauert [**AG**], who call these functions $(q+1)$-plurisubharmonic. A broader definition that extends the notion to upper semicontinuous functions was given by Hunt and Murray [**HM**], who also seem to be responsible for changing the index q. We will here follow the Hunt and Murray convention to minimize confusion.

The class of q-plurisubharmonic functions is not additive for $q > 0$ and thus standard smoothing techniques available for plurisubharmonic functions do not carry over, a fact which hampered early work on the subject. A breakthrough was achieved by Slodkowski [**S1**] who was able to show that continuous q-plurisubharmonic functions are uniform limits of functions whose second order derivatives exist almost everywhere. We will show that the approximation can be achieved by functions which are locally the maximum of a finite number of smooth (strictly) q-plurisubharmonic functions.

2. The Perron method. We will describe here a general Perron Method which is a distillation of ideas in Bremermann **[B]**, Walsh **[W]**, Hunt and Murray **[HW]**, Kalka **[K]**, and Slodkowski **[S1]**. We will use an axiomatic approach which will allow us to apply the results of this section to several different situations in Sections 3–5.

For each bounded open set $D \subset \mathbb{C}^n$ let there be given a family $\mathscr{P}(\overline{D})$ of continuous real functions on $\overline{D}$ satisfying the following axioms:

(1) $c + \mathscr{P}(\overline{D}) \subset \mathscr{P}(\overline{D})$ *for* $c \in \mathbb{R}$, $c\mathscr{P}(\overline{D}) \subset \mathscr{P}(\overline{D})$ *for* $c \in \mathbb{R}$, $c > 0$.

(2) *If* $D_1 \subset D$ *then* $\mathscr{P}(\overline{D})|\overline{D}_1 \subset \mathscr{P}(\overline{D}_1)$.

(3) $T_y\mathscr{P}(\overline{D}) \subset \mathscr{P}(y + \overline{D})$ *where* $T_y u(z) = u(z - y)$.

(4) *If* $u, v \in \mathscr{P}(\overline{D})$ *then* $\max(u, v) \in \overline{\mathscr{P}(\overline{D})}$ *where the closure is taken in* $C(\overline{D})$ *with respect to the uniform norm.*

(5) *If* g *is a real linear function or* $g = |z|^2$ *then* $g + \mathscr{P}(\overline{D}) \subset \mathscr{P}(\overline{D})$.

Axiom (4) is stated more generally than needed for the examples in this paper since we have future applications in mind. Axioms (1) and (5) imply that for $\varepsilon > 0$ and $x \in \mathbb{C}^n$,

$$\varepsilon|z - x|^2 + \mathscr{P}(\overline{D}) \subset \mathscr{P}(\overline{D}).$$

If D is a bounded open set then a subfamily $\mathscr{A}(\overline{D}) \subset \mathscr{P}(\overline{D})$ is *admissible* for D relative to $\mathscr{P}(\overline{D})$ if the following properties are satisfied:

(A1) $c + \mathscr{A}(\overline{D}) \subset \mathscr{A}(\overline{D})$ *for* $c \in \mathbb{R}$, $c\mathscr{A}(\overline{D}) \subset \mathscr{A}(\overline{D})$ *for* $c \in \mathbb{R}$, $c > 0$.

(A2) *If* $u, v \in \mathscr{A}(\overline{D})$ *then* $\max(u, v) \in \overline{\mathscr{A}(\overline{D})}$ *where the closure is taken in* $C(\overline{D})$.

(A3) *If* D_1 *is an open set with compact closure in* D, $u \in \mathscr{A}(\overline{D})$, $u_1 \in \mathscr{P}(\overline{D}_1)$ *and*

$$u_1(x) < u(x) \quad \text{for } x \in \partial D_1$$

then the function

$$v(z) = \begin{cases} \max(u(z), u_1(z)), & z \in \overline{D}_1, \\ u(z), & z \in \overline{D} \setminus \overline{D}_1, \end{cases}$$

belongs to $\overline{\mathscr{A}(\overline{D})}$.

Let D be an open set in $\mathbb{C}^n$ and assume $\mathscr{A}(\overline{D})$ is admissible for D. If g is any function on $\overline{D}$ with values in $\mathbb{R} \cup \{+\infty\}$, we define

$$\mathscr{A}(\overline{D}, g) = \{u \colon u \in \mathscr{A}(\overline{D}), u < g \text{ on } \overline{D}\},$$
$$E^{\mathscr{A}}(\overline{D}, g) = \sup\{u \colon u \in \mathscr{A}(\overline{D}, g)\}.$$

Note that $E^{\mathscr{A}}(\overline{D}, g)$ is lower semicontinuous at points where it is finite, and $E^{\mathscr{A}}(\overline{D}, g) \leq g$ on $\overline{D}$. Using the above properties (1) through (4) and (A1) through (A3) one can now prove a lemma, which is due to

Walsh [W] in case $\mathscr{A}$ is the class of plurisubharmonic functions. We also note that this is the only place where axiom (3) is explicitly used.

2.1. LEMMA. *Assume D is a bounded open set in $\mathbb{C}^n$ and $\mathscr{A}(\overline{D})$ is admissible for D. Let q be a function on $\overline{D}$ which is either real valued and continuous on $\overline{D}$ or real valued and continuous on ∂D and identically $+\infty$ on D. If $E^{\mathscr{A}}(\overline{D}, g)$, as a function on $\overline{D}$, is continuous at points of the boundary ∂D of D, then $E^{\mathscr{A}}(\overline{D}, g)$ is finite and continuous on all of $\overline{D}$.*

Proof. First we observe that the family $\mathscr{A}(\overline{D}, g)$ is directed under the assumptions made on g. That is, if $u, v \in \mathscr{A}(\overline{D}, g)$ then there is a $w \in \mathscr{A}(\overline{D}, g)$ with

$$w \geq \max(u, v).$$

Indeed, since $\max(u, v) < g$ on $\overline{D}$ there is a $c > 0$ with $\max(u, v) + c < g$, and an application of property (A2) yields the desired w. Now let $\varepsilon > 0$ be given. We can choose finitely many $w_i \in \mathscr{A}(\overline{D}, g)$ such that

$$E^{\mathscr{A}}(\overline{D}, g) - \max\{w_i\} < \varepsilon \quad \text{on } \partial D.$$

By the above remark, there is then a $w \in \mathscr{A}(\overline{D}, g)$ with

$$E^{\mathscr{A}}(\overline{D}, g) - w < \varepsilon \quad \text{on } \partial D.$$

Since $E^{\mathscr{A}}(\overline{D}, g)$ is assumed to be continuous at points of ∂D, we can find a δ such that

$$E^{\mathscr{A}}(\overline{D}, g)(z) - w(z) < \varepsilon \quad \text{for } d(z, \partial D) < 2\delta,$$

$$|w(z) - w(z')| < \varepsilon \quad \text{for } |z - z'| < \delta,$$

$$g(z') \leq g(z) + \varepsilon \quad \text{for } |z - z'| < \delta \text{ and } z, z' \in D,$$

where $d(z, \partial D)$ is the distance of $z \in \overline{D}$ from ∂D. If $u \in \mathscr{A}(\overline{D}, g)$ and $u \geq w$, then the first two inequalities imply

$$|u(z) - u(z')| < 2\varepsilon$$

if $|z - z'| < \delta$ and z and z' are within 2δ of ∂D. For given $x, y \in \overline{D}$ with $|x - y| < \delta$ let

$$D_1 = \{z \in D \colon d(z, \partial D) > \delta\},$$
$$u_1 = T_{y-x}u - 2\varepsilon \quad \text{on } \overline{D}_1.$$

Then the function v defined as in property (A3) satisfies $v < g$ on $\overline{D}$. Note that $u_1 \in \mathscr{P}(\overline{D}_1)$ and therefore by property (A3) there is a $\tilde{v} \in \mathscr{A}(\overline{D}, g)$ with $\tilde{v} \geq v$ and thus $\tilde{v}(x) \geq u(y) - 2\varepsilon$. Taking the supremum over all $u \in \mathscr{A}(\overline{D}, g)$, $u \geq w$, gives

$$E^{\mathscr{A}}(\overline{D}, g)(x) \geq E^{\mathscr{A}}(\overline{D}, g)(y) - 2\varepsilon \quad \text{for } x, y \in \overline{D}, |x - y| < \delta,$$

whence the finiteness and continuity of $E^{\mathscr{A}}(\overline{D}, g)$ on $\overline{D}$.

Recall that a function $u \in C(\overline{D})$ is called a peak function for $x \in \partial D$ if

$$u(x) = 0 \quad \text{and} \quad u(z) < 0 \quad \text{for } z \in \overline{D} \setminus \{x\}.$$

2.2. Lemma. *Assume $D \subset \mathbb{C}^n$ is a bounded open set such that $\mathscr{A}(\overline{D})$ is admissible for D and the closure of $\mathscr{A}(\overline{D})$ in $C(\overline{D})$ contains a peak function for every point of ∂D. If g is lower semicontinuous at points of ∂D then $E^{\mathscr{A}}(\overline{D}, g) = g$ on ∂D.*

Proof (see proof of Theorem 4.1 in **[B]**). Let $x \in \partial D$ and $\varepsilon > 0$ be fixed, and choose $\delta > 0$ so that

$$g(y) > g(x) - \varepsilon \quad \text{if } y \in \overline{D}, \qquad |y - x| < \delta.$$

Let $u \in \overline{\mathscr{A}(\overline{D})}$ be a peak function for x. Then there is $c \in \mathbb{R}$, $c > 0$ so that

$$cu(y) < g(y) - g(x) + \varepsilon \quad \text{for } y \in \overline{D}, \qquad |y - x| \geq \delta,$$

and therefore $v = cu + g(x) - \varepsilon < g$ on $\overline{D}$. Since $v \in \overline{\mathscr{A}(\overline{D})}$ and $v(x) = g(x) - \varepsilon$, we conclude $E^{\mathscr{A}}(\overline{D}, g)(x) = g(x)$.

2.3. Corollary. *Assume D is a bounded open set in $\mathbb{C}^n$ such that $\mathscr{A}(\overline{D})$ is admissible for D and $\overline{\mathscr{A}(\overline{D})}$ contains a peak function for every point of ∂D. Assume further that one of the following conditions holds:*

(a) *g is continuous on $\overline{D}$ or*

(b) *g is continuous on ∂D and $g = +\infty$ on D, and there is an upper semicontinuous extension $\tilde{g}$ of $g|\partial D$ to $\overline{D}$ so that $\mathscr{A}(\overline{D}, g) = \mathscr{A}(\overline{D}, \tilde{g})$. Then $E^{\mathscr{A}}(\overline{D}, g)$ is a continuous real valued function on $\overline{D}$ and $E^{\mathscr{A}}(\overline{D}, g) = g$ on ∂D.*

Proof. We have $E^{\mathscr{A}}(\overline{D}, g) = g$ on ∂D by Lemma 2.2. Note that in case (b), $E^{\mathscr{A}}(\overline{D}, g) \leq \tilde{g}$ with equality on ∂D, whence $E^{\mathscr{A}}(\overline{D}, g)$ is continuous at points of ∂D. Since $E^{\mathscr{A}}(\overline{D}, g) \leq g$, we conclude the same in case (a). Lemma 2.1 implies thus the continuity of $E^{\mathscr{A}}(\overline{D}, g)$ on $\overline{D}$.

For each open set W in $\mathbb{C}^n$ we define $\mathscr{P}'(W)$ as the family of continuous function v on W such that for each open $D \subset\subset W$ and each

$u \in \mathscr{P}(\overline{D})$, $u + v$ attains its maximum on $\overline{D}$ at a point of ∂D. For a bounded open set D we set

$$\mathscr{P}'(\overline{D}) = \mathscr{P}'(D) \cap C(\overline{D}).$$

Note that $\mathscr{P}'(\overline{D})$ may be empty. However, axioms (1) through (5) are satisfied for $\mathscr{P}'(\overline{D})$, and it follows easily from the next lemma that $\mathscr{P}'(\overline{D})$ is admissible relative to $\mathscr{P}'(\overline{D})$, i.e. axiom (A3) is satisfied with $\mathscr{P}'$ in place of $\mathscr{A}$ and $\mathscr{P}$. We note also that $\mathscr{P}'(\overline{D})$ is closed in $C(\overline{D})$ in the uniform norm.

2.4. Lemma. *Assume W is an open set in $\mathbb{C}^n$ and v a continuous function on W such that each point in W has a neighborhood $U \subset\subset W$ so that $v|U \in \mathscr{P}'(\overline{U})$. Then $v \in \mathscr{P}'(W)$.*

Proof. This is essentially the proof of Lemma 2.7 in **[HW]**. Assume $v \notin \mathscr{P}'(W)$. Then there is an open $D \subset\subset W$ and a function $u \in \mathscr{P}(\overline{D})$ so that $v + u$ does not attain its maximum M on $\overline{D}$ at any point of ∂D. Choose $\varepsilon > 0$ so that the function w defined on $\overline{D}$ by

$$w(z) = v(z) + u(z) + \varepsilon|z|^2$$

also satisfies $w < M$ on ∂D. Then w assumes its maximum on $\overline{D}$ at a point $x \in D$. Let $U \subset\subset W$ be a neighborhood of x so that $v|U \in \mathscr{P}'(\overline{U})$. Since axiom (2) is satisfied by $\mathscr{P}'$, we may assume that U is a small ball around x with closure contained in D. Since $\varepsilon(|z|^2 - |z-x|^2)$ is an affine function, we have $u + \varepsilon(|z|^2 - |z-x|^2) \in \mathscr{P}(\overline{D})$. Thus

$$w - \varepsilon|z-x|^2 = v + u + \varepsilon(|z|^2 - |z-x|^2)$$

assumes it maximum on $\overline{U}$ at a point $y \in \partial U$,

$$w(y) - \varepsilon|y-x|^2 \geq w(z) - \varepsilon|z-x|^2, \qquad z \in U.$$

Evaluating at $z = x$ gives the contradiction $w(y) > w(x)$.

The above proof also yields the following lemma (see Proposition 1.1 of **[S1]**).

2.5. Lemma. *Let W be an open subset of $\mathbb{C}^n$ and v a continuous function on W such that $v \notin \mathscr{P}'(W)$. Then there exists $x \in W$, a ball $B(x,r)$ of radius $r > 0$ with closure contained in W, an $\varepsilon > 0$ and an $f \in \mathscr{P}(\overline{B(x,r)})$ such that*

$$f(x) + v(x) = 0,$$
$$f(z) + v(z) \leq -\varepsilon|z-x|^2 \quad \text{on } \overline{B(x,r)}.$$

Proof. Using the construction of the last proof with $B(x,r) = U$ and

$$f(z) = u(z) + \varepsilon(|z|^2 - |z-x|^2) - w(x),$$

we have $f(x) + v(x) = 0$. Since w assumes its maximum on $\overline{B(x,r)}$ at x, we have $w(z) \leq w(x)$ which can be re-written as $f(z) + v(z) \leq -\varepsilon|z-x|^2$.

We define now $\widetilde{\mathscr{P}} = \mathscr{P}''$ (this is the construction made in **[K]**), i.e. for an open set W in $\mathbb{C}^n$, $\widetilde{\mathscr{P}}(W)$ is the family of continuous functions u on W such that for each open $D \subset\subset W$ and each $v \in \mathscr{P}'(\overline{D})$, $u + v$ attains its maximum on $\overline{D}$ at a point of ∂D. Further,

$$\widetilde{\mathscr{P}}(\overline{D}) = \widetilde{\mathscr{P}}(D) \cap D(\overline{D}).$$

We obviously have

$$\overline{\mathscr{P}(\overline{D})} \subset \widetilde{\mathscr{P}}(\overline{D}).$$

In particular, the function $E^{\mathscr{A}}(\overline{D}, g)$ of Corollary 2.3 belongs to $\widetilde{\mathscr{P}}(\overline{D})$. In Section 4, we will investigate conditions under which $\mathscr{P}(\overline{D})$ is dense in $\widetilde{\mathscr{P}}(\overline{D})$.

2.6. LEMMA. *With the notation and assumptions of Corollary* 2.3, *let D_1 be the set of points $z \in D$ for which $E^{\mathscr{A}}(\overline{D}, g)(z) < g(z)$. Then, if $D_1 \neq \emptyset$,*

$$-E^{\mathscr{A}}(\overline{D}, g)|\overline{D}_1 \in \mathscr{P}'(\overline{D}_1).$$

Proof. (The proof given here is an adaptation of the proof of Theorem 6.8 in **[S1]**.) Assume $-E^{\mathscr{A}}(\overline{D}, g)$ does not belong to $\mathscr{P}'(\overline{D}_1)$. By Lemma 2.5 there are $x \in D_1$ and a ball $B(x,r)$ with closure contained in D_1, an $\varepsilon > 0$ and $f \in \mathscr{P}(\overline{B(x,r)})$ such that for $z \in \overline{B(x,r)}$

$$f(x) = E^{\mathscr{A}}(\overline{D}, g)(x),$$
$$f(z) \leq E^{\mathscr{A}}(\overline{D}, g)(z) - \varepsilon|z-x|^2.$$

Choose $\delta > 0$ so that $E^{\mathscr{A}}(\overline{D}, g) + \delta < g$ on $\overline{B(x,r)}$ and $2\delta < \varepsilon r^2$. Let $u \in \mathscr{A}(\overline{D})$ be such that on $\overline{D}$

$$u \leq E^{\mathscr{A}}(\overline{D}, g) \leq u + \delta/2$$

and define

$$u_1(z) = \begin{cases} u(z), & z \in D \setminus B(x,r), \\ \max(u(z), f(z) + \delta), & z \in B(x,r). \end{cases}$$

For $|z-x| = r$ we have

$$f(z) + \delta \leq E^{\mathscr{A}}(\overline{D}, g)(z) + \delta - \varepsilon r^2 \leq u(z) + \tfrac{3}{2}\delta - \varepsilon r^2 < u(z),$$

whence u_1 belongs to the closure of $\mathscr{A}(\overline{D})$ in $C(\overline{D})$ by axiom (A3). Let $\tilde{u}_1 \in \mathscr{A}(\overline{D})$ approximate u_1 to within $\delta/2$ on $\overline{D}$. By axioms (A1) we can achieve $\tilde{u}_1 \leq u_1$. By the choice of δ, we have on $\overline{B(x,r)}$

$$f(z) + \delta \leq E^{\mathscr{A}}(\overline{D}, g) + \delta - \varepsilon|z - z_0|^2 < g(z).$$

Thus u_1, and hence $\tilde{u}_1$, is dominated by g on $\overline{D}$. Therefore $\tilde{u}_1$ is one of the functions used to form the envelope $E^{\mathscr{A}}(\overline{D}, g)$. But

$$u(x) \leq E^{\mathscr{A}}(\overline{D}, g)(x) = f(x) < f(x) + \delta$$

whence $u_1(x) = f(x) + \delta$ and

$$\begin{aligned}\tilde{u}_1(x) &\geq u_1(x) - \delta/2 = f(x) + \delta - \delta/2 \\ &= E^{\mathscr{A}}(\overline{D}, g)(x) + \delta/2 > E^{\mathscr{A}}(\overline{D}, g)(x),\end{aligned}$$

a contradiction.

3. The Dirichlet problem. The Dirichlet problem has essentially been solved by Corollary 2.3. We need only find a condition on D which will imply the assumptions made in (b) of that corollary.

3.1. THEOREM. *Let D be a bounded open set in $\mathbb{C}^n$. Assume $\mathscr{A}(\overline{D}) \subset \mathscr{P}(\overline{D})$ is admissible for D and $\mathscr{A}(\overline{D})$ contains a peak function for every point of ∂D. Assume further that $\mathscr{P}'(\overline{D})$ also contains a peak function for every point of ∂D. Then for every continuous function g on ∂D there is a unique function $w \in \tilde{\mathscr{P}}(\overline{D})$ such that $w|\partial D = g$ and $-w \in \mathscr{P}'(\overline{D})$.*

Proof. We set $g(z) = \infty$ for $z \in D$. As before, we let $\mathscr{A}(\overline{D}, g)$ be the collection of functions in $\mathscr{A}(\overline{D})$ whose boundary values are everywhere less than g. We will show that the upper envelope $E^{\mathscr{A}}(\overline{D}, g)$ of $\mathscr{A}(\overline{D}, g)$ is the desired solution w. Let g_0 be any continuous extension of $g|\partial D$ to $\overline{D}$. Define $\tilde{g}$ for $z \in \overline{D}$ by

$$\begin{aligned}-\tilde{g}(z) &= \sup\{v(z) \colon v \in \mathscr{P}'(\overline{D}), v|\partial D < -g_0\} \\ &= E^{\mathscr{P}'}(\overline{D}, -g_0).\end{aligned}$$

Since $\mathscr{P}'(\overline{D})$ is admissible (relative to $\mathscr{P}'(\overline{D})$) and contains a peak function for every point of ∂D, Corollary 2.3 implies that $-\tilde{g}$ is continuous, and $\tilde{g}|\partial D = g|\partial D$. Since $\mathscr{P}'(\overline{D})$ is closed in $C(\overline{D})$, $-\tilde{g}$ belongs to $\mathscr{P}'(\overline{D})$ and hence $u - \tilde{g}$ will attain its maximum on D at a point of ∂D for every $u \in \mathscr{P}(\overline{D})$. In particular, we obtain

$$u - \tilde{g} \leq 0 \text{ on } \overline{D}, \quad u \in \mathscr{A}(\overline{D}, g).$$

Therefore $\tilde{g}$ is a continuous extension of g to $\overline{D}$ with $\mathscr{A}(\overline{D}, \tilde{g}) = \mathscr{A}(\overline{D}, g)$. Corollary 2.3 implies now that $w = \overline{E^{\mathscr{A}}(\overline{D}, g)}$ is a continuous extension of $g|\partial D$ to $\overline{D}$. By construction, $w \in \mathscr{A}(\overline{D}) \subset \widetilde{\mathscr{P}}(\overline{D})$ and by Lemma 2.6, $-w \in \mathscr{P}'(\overline{D})$. To prove uniqueness, let $\tilde{w} \in \widetilde{\mathscr{P}}(\overline{D})$ be any extension of $g|\partial D$ to $\overline{D}$ so that $-\tilde{w} \in \mathscr{P}'(\overline{D})$. By the definition of $\widetilde{\mathscr{P}}(\overline{D})$, $w - \tilde{w}$ and $\tilde{w} - w$ must attain their maximum at points of ∂D. Since both functions vanish on ∂D, this implies $w \equiv \tilde{w}$.

The question naturally arises whether the hypotheses in Theorem 3.1 are satisfied when $D = B(x, r)$, a ball. If $\mathscr{P}(\overline{D}) \neq \varnothing$ then axiom (1) implies $0 \in \overline{\mathscr{P}(\overline{D})}$. By axiom (5), $\mathscr{P}(\overline{D})$ will then contain all real linear functions, which provides peak functions for $B(x, r)$. To conclude the same for $\mathscr{P}'(\overline{D})$ we need $0 \in \mathscr{P}'(\overline{D})$, which is equivalent to the following:

(6) *If D is a bounded open subset of $\mathbb{C}^n$ then the functions in $\mathscr{P}(\overline{D})$ assume their maximum at points of ∂D.*

3.2. PROPOSITION. *If axiom* (6) *above holds and if the real linear functions belong to $\mathscr{A}((\overline{B(x,r)})$ then Theorem* 3.1 *is valid for $D = B(x, r)$.*

We will next discuss what these results mean in terms of q-plurisubharmonic functions. As mentioned in the introduction, we will use the Hunt and Murray **[HM]** definition of an upper semicontinuous q-plurisubharmonic function (see also **[S1]**). Unless otherwise stated, we will assume $0 \leq q \leq n-1$. For each bounded open set D in $\mathbb{C}^n$ let $\mathrm{CPSH}_q(\overline{D})$ be the family of continuous functions on $\overline{D}$ which are q-plurisubharmonic on D. Then $\mathrm{CPSH}_q(\overline{D})$ satisfies axioms (1) through (5) of Section 2, (see **[HM]** or Proposition 1.2 in **[S1]**).

Let $\mathscr{P}_q(\overline{D}) = \mathrm{CPSH}_q(\overline{D})$ and define as in **[K]** (except that we restrict ourselves here to continuous functions)

$$\widetilde{\mathscr{P}}_{n-q-1}(\overline{D}) = \mathscr{P}'_q(\overline{D}).$$

Then Theorem 3.1 yields unique solutions to the Dirichlet problem in $\mathscr{P}_q(\overline{D}) \cap -\widetilde{\mathscr{P}}_{n-q-1}(\overline{D})$ under appropriate conditions on D (see Theorem 3.7 below). We believe that this is what Kalka established in **[K]** even though some of his formulations are flawed as noted in **[S1]**.

For our further discussion we will need to refer to the following property of q-plurisubharmonic functions.

3.3. PROPERTY (*Theorem* 5.1 *of* [S1].). *If u is q-plurisubharmonic and v is r-plurisubharmonic on D then $u+v$ is $(q+r)$-plurisubharmonic on D.*

3.4. COROLLARY. *If* $\mathscr{P}_q(\overline{D}) = \mathrm{CPSH}_q(\overline{D})$ *then*

$$\mathscr{P}'_q(\overline{D}) = \mathrm{CPSH}_{n-q-1}(\overline{D}).$$

Proof. It follows readily from the definition of q-plurisubharmonic functions that $\mathscr{P}'_q(\overline{D}) \subset \mathscr{P}_{n-q-1}(\overline{D})$ (e.g., see Proposition 1.1 in **[S1]**). By Property 3.3, the sum of a q-plurisubharmonic function and a $(n-q-1)$-plurisubharmonic function is $(n-1)$-plurisubharmonic and thus satisfies the Maximum Principle (see Lemma 2.7 of **[HW]**). The definition of $\mathscr{P}'_q(\overline{D})$ implies now $\mathscr{P}_{n-q-1}(\overline{D}) \subset \mathscr{P}'_q(\overline{D})$.

3.5. DEFINITION. A bounded domain D in $\mathbb{C}^n$ is said to have an r-pseudoconvex barrier at a boundary point $x \in \partial D$ if there are a neighborhood U of x and a peak function for x in $\mathrm{CPSH}_r(\overline{D \cap U})$.

Recall that ∂D is strictly r-pseudoconvex at $x \in \partial D$ if there are a neighborhood V of x and a C^2 strictly r-plurisubharmonic function ρ in V so that

$$V \cap D = \{z \in V : \rho(z) < 0\}.$$

Then, for any neighborhood U of x with compact closure in V, $\rho - \varepsilon|z-x|^2$ will be a peak function for x in $\mathrm{CPSH}_r(\overline{D \cap U})$ if ε is small.

3.6. LEMMA. *If D is a bounded domain in $\mathbb{C}^n$ then there is a peak function in* $\mathrm{CPSH}_r(\overline{D})$ *for every point $x \in \partial D$ for which there is an r-pseudoconvex barrier. In particular this is true for every $x \in \partial D$ where ∂D is strictly r-pseudoconvex.*

Proof. If $u \in \mathrm{CPSH}_r(\overline{D \cap U})$ is a peak function for $x \in \partial D$ then $\tilde{u} = \max(u, -\varepsilon)$ belongs to $\mathrm{CPSH}_r(\overline{D})$ for small ε and clearly $\tilde{u}$ is a peak function for x.

If we specialize Theorem 3.1 now to q-plurisubharmonic functions, we recover a proof of the existence and uniqueness of the solution to the Dirichlet problem for q-plurisubharmonic functions on certain strictly r-pseudoconvex domains. But first we consider an example.

Let $D \subset \mathbb{C}^2$ be defined by

$$D = \{z = (z_1, z_2) : \rho(z) = -\operatorname{Re} z_1 + |z_1|^2 - |z_2|^2 < 0, |z| < 1\}.$$

Then D is a bounded domain with a 1-pseudoconvex barrier at every point of ∂D. Let g be a continuous function on ∂D with a strict maximum value of 0 at 0. We claim there is no function $u \in \mathrm{CPSH}_0(\overline{D})$ with $u|\partial D = g$. If there were such a function u, then u is not constant and therefore does not assume its maximum value of 0 on D. Therefore u is bounded away from 0 on the compact subset $\{z\colon \operatorname{Im} z_1 = 0, 0 \le \operatorname{Re} z_1 \le \frac{1}{2}, |z_2| = \frac{1}{2}\}$ of D. Since u is subharmonic on the disks

$$\Delta_\varepsilon = \{z\colon z_1 = \varepsilon, |z_2| \le \tfrac{1}{2}\} \subset\subset D, \qquad 0 < \varepsilon \le \tfrac{1}{2},$$

it must be bounded away from 0 by the same constant on each of these disks. But

$$0 \in \Delta_0 \subset \overline{\bigcup_{0<\varepsilon\le 1/2} \Delta_\varepsilon}$$

and $u(0) = 0$, a contradiction. However, for any continuous function h on ∂D, there are functions $v \in \mathrm{CPSH}_1(\overline{D})$ with $v|\partial D = h$. To see this, let $\tilde{h}$ be any continuous extension of h to $\overline{D}$ and set $\mathscr{P}(\overline{D}) = \mathrm{CPSH}_1(\overline{D})$. Then $v = E^{\mathscr{P}}(\overline{D}, \tilde{h})$ provides such a function by Corollary 2.3. If $h = -g$, then this solution does not belong to $-\mathrm{CPSH}_0(\overline{D})$ for any extension $\tilde{h}$ as we have argued above.

We are now ready to state a general Dirichlet problem for q-plurisubharmonic functions. For $q = r$, the existence part of the following theorem is due to Hunt and Murray **[HM]**. The uniqueness was shown by Slodkowski in **[S1]**, where one also finds a general formulation of the theorem (Theorem 5.6 in **[S1]**, which contains what appears to be a misprint as the above example shows).

3.7. THEOREM. *Let D be a bounded domain in $\mathbb{C}^n$ with the property that each point on the boundary of D admits an r-pseudoconvex barrier, where $0 \le r \le n-1$. Then for every continuous real valued function g on ∂D and every q with $r \le q \le n-r-1$ there exists a unique $u \in \mathrm{CPSH}_q(\overline{D})$ so that $-u \in \mathrm{CPSH}_{n-q-1}(\overline{D})$ and $u|\partial D = g$.*

Proof. Let $\mathscr{P}_q(\overline{D}) = \mathrm{CPSH}_q(\overline{D})$. Corollary 3.4 yields $\mathscr{P}'_q(\overline{D}) = \mathrm{CPSH}_{n-q-1}(\overline{D})$. The conditions on r and q imply $r \le q$ as well as $r \le n-q-1$ and thus

$$\mathrm{CPSH}_r(\overline{D}) \subset \mathscr{P}_q(\overline{D}) \cap \mathscr{P}'_q(\overline{D}).$$

Since Lemma 3.6 guarantees now peak functions for every $x \in \partial D$ in $\mathscr{P}_q(\overline{D})$ and in $\mathscr{P}'_q(\overline{D})$, Theorem 3.1 can be applied.

4. Piecewise smooth approximations on strictly q-pseudoconvex domains. We start out with the axiomatic setting of Section 2.

4.1. THEOREM 4.1. *Assume D is a bounded open set in $\mathbb{C}^n$ and $\mathscr{A}(\overline{D}) \subset \mathscr{P}(\overline{D})$ is admissible for D. If the closure of $\mathscr{A}(\overline{D})$ in $C(\overline{D})$ contains a peak function for every point of ∂D, then $\mathscr{A}(\overline{D})$ is dense in $\widetilde{\mathscr{P}}(\overline{D})$.*

Proof. Recall that $\widetilde{\mathscr{P}}(\overline{D}) = \mathscr{P}''(\overline{D})$. Let $g \in \widetilde{\mathscr{P}}(\overline{D})$. By Corollary 2.3, $E^{\mathscr{A}}(\overline{D}, g)$ is continuous and agrees with g on ∂D. Assume that

$$D_1 = \{z \in D \colon E^{\mathscr{A}}(\overline{D}, g)(z) < g(z)\}$$

is not empty. Then g and $E^{\mathscr{A}}(\overline{D}, g)$ agree on ∂D_1. By Lemma 2.6, $-E^{\mathscr{A}}(\overline{D}, g) \in \mathscr{P}'(\overline{D}_1)$. Thus, by the definition of $\widetilde{\mathscr{P}}(D_1)$, $g - E^{\mathscr{A}}(\overline{D}, g)$ assumes its maximum on $\overline{D}_1$ at points of ∂D_1, i.e.,

$$g - E^{\mathscr{A}}(\overline{D}, g) \leq 0 \quad \text{on } D_1.$$

This is a contradiction to the definition of D_1. Thus $D_1 = \varnothing$ and $g = E^{\mathscr{A}}(\overline{D}, g) \in \overline{\mathscr{A}(\overline{D})}$.

In trying to approximate q-plurisubharmonic functions by functions from a subclass exhibiting some smoothness properties, the only operation one can work with for $q > 0$ is that of taking suprema. A subclass of relatively smooth functions that is closed under taking maxima, is the class of piecewise smooth functions.

4.2. DEFINITION. A function defined on an open set in $\mathbb{C}^n$ is called piecewise smooth (strictly) q-plurisubharmonic if in some neighborhood of every point in its domain it is the maximum of a finite number of C^2 (strictly) q-plurisubharmonic functions.

For a bounded open set D in $\mathbb{C}^n$, let $\mathscr{P}_q(\overline{D})$ be the collection of functions on $\overline{D}$ which are the restrictions to $\overline{D}$ of functions that are piecewise smooth q-plurisubharmonic in some neighborhood of $\overline{D}$. It is clear that axioms (1) through (5) of §2, and also properties (A1)–(A3), are satisfied with $\mathscr{P} = \mathscr{A} = \mathscr{P}_q$.

4.3. DEFINITION. A domain D in $\mathbb{C}^n$ is said to have piecewise smooth strictly q-pseudoconvex boundary if every point $x \in \partial D$ has a neighborhood V with a piecewise smooth strictly q-plurisubharmonic function ρ defined on V such that

$$D \cap V = \{z \in V, \rho(z) < 0\}.$$

4.4. PROPOSITION. *If the domain D in $\mathbb{C}^n$ has piecewise smooth strictly q-pseudoconvex boundary then $\mathscr{P}_q(\overline{D})$ contains a peak function for every boundary point of D.*

Proof. Let $x \in \partial D$ and let V and ρ be as in the above definition. Let U be a neighborhood of x which is relatively compact in V. Then

$$\rho_0 = \rho - \varepsilon|z - x|^2$$

is strictly q-plurisubharmonic on $\overline{U}$ and bounded away from 0 on $\partial U \cap \overline{D}$ for small $\varepsilon > 0$. Thus, if $\delta > 0$ and

$$-\delta > \max_{z \in \partial U \cap \overline{D}} \rho_0(z),$$

then $\max(\rho_0, -\delta)$ is in $\mathscr{P}_q(\overline{D})$ and peaks at x.

We can therefore apply Theorem 4.1 to a domain D with piecewise smooth strictly q-pseudoconvex boundary and obtain:

4.5. THEOREM. *Assume D is a bounded domain in $\mathbb{C}^n$ and $\mathscr{A}(\overline{D}) \subset \mathscr{P}_q(\overline{D})$ is admissible for D. If the closure of $\mathscr{A}(\overline{D})$ in $C(\overline{D})$ contains a peak function for every point of ∂D, then $\mathscr{A}(\overline{D})$ is dense in* $\mathrm{CPSH}_q(\overline{D})$ *in the uniform norm.*

As an immediate consequence we have

4.6. THEOREM. *Let D be a bounded domain in $\mathbb{C}^n$ with piecewise smooth strictly q-pseudoconvex boundary. Then every function in* $\mathrm{CPSH}_q(\overline{D})$ *is the uniform limit on $\overline{D}$ of a sequence of functions each of which is piecewise smooth q-plurisubharmonic on some neighborhood of $\overline{D}$.*

Another immediate consequence is a Runge type theorem. For an open set U let us denote by $\mathrm{CPSH}_q(U)$ the family of continuous q-plurisubharmonic functions on U, and by $\mathscr{P}_q(U)$ the piecewise smooth functions in $\mathrm{CPSH}_q(U)$. For a compact set K, $\mathrm{APSH}_q(K)$ will denote the subset of $C(K)$ of functions obtained by restrictions of functions in $\mathrm{CPSH}_q(U)$ where U runs through a neighborhood basis of K. For a bounded domain D,

$$\mathrm{CPSH}_q(\overline{D}) = C(\overline{D}) \cap \mathrm{CPSH}_q(D).$$

4.7. THEOREM. *Let W be an open subset of $\mathbb{C}^n$ and $\varphi \in \mathscr{P}_q(W)$. Assume $K = \{z \in W: \varphi(z) \leq 0\}$ is compact in W. Then $\mathscr{P}_q(W)$ is dense in* $\mathrm{APSH}_q(K)$. *If, in addition, φ is strictly q-plurisubharmonic on W, then $\mathscr{P}_q(W)$ is dense in* $\mathrm{CPSH}_q(\overline{D})$, *where $D = \{z \in W: \varphi(z) < 0\}$.*

Proof. We establish first the second part of the theorem under modified assumptions. Assume $\varphi - \varepsilon|z|^2 \in \mathscr{P}_q(W)$ for small $\varepsilon > 0$ and let D be a union of connected components of $\{z \in W: \varphi(z) < 0\}$ such that $\overline{D}$ is compact in W. Let

$$\mathscr{A}(\overline{D}) = \mathscr{P}_q(W)|\overline{D}.$$

Then $\mathscr{A}(\overline{D})$ is admissible for D (satisfies axioms (A1)–(A3)). Also, for $x \in \partial D$, we have

$$\varphi_x = \varphi - \varepsilon|z - x|^2 \in \mathscr{A}(\overline{D}).$$

Since φ_x is a peak function for x, Theorem 4.5 applies, showing that $\mathscr{P}_q(W)$ is dense in $\mathrm{CPSH}_q(\overline{D})$.

We shall now prove the first part of the theorem. Let U be a neighborhood of K which has compact closure in W. Choose $\varepsilon > 0$ such that

$$D_\varepsilon = \{z \in W: \varphi(z) < \varepsilon\} \cap U \subset\subset U,$$

and then select $c > 0$ so that $c(|z|^2 + 1) < \varepsilon$ on $\overline{U}$. Define

$$\varphi_0 = \varphi + c(|z|^2 + 1) - \varepsilon.$$

Then $\varphi_0 \in \mathscr{P}_q(W)$, and also $\varphi_0 - \varepsilon'|z|^2 \in \mathscr{P}_q(W)$ for small ε'. Therefore $\mathscr{P}_q(W)$ is dense in $\mathrm{CPSH}_q(\overline{D}_0)$, where

$$D_0 = \{z \in W: \varphi_0(z) < 0\} \cap U \subset\subset U$$

by what we have already proven. Note that $K \subset D_0 \subset U$. Since U was arbitrary, this shows $\mathscr{P}_q(W)$ is dense in $\mathrm{APSH}_q(K)$.

Now back to the second part of the theorem. Let W' be a neighborhood of K with compact closure in W and set

$$K' = \{z \in W: \varphi(z) \leq \varepsilon'\} \cap W'$$

where $\varepsilon' > 0$ is chosen so small that K' is compact in W'. On W', $\varphi - \varepsilon|z|^2$ will be q-plurisubharmonic for small $\varepsilon > 0$, whence $\mathscr{P}_q(W')$ is dense in $\mathrm{CPSH}_q(\overline{D})$ by what we have shown already. In particular, $\mathrm{APSH}_q(K')$ is dense in $\mathrm{CPSH}_q(\overline{D})$ and, repeating the proof of the first part of the theorem with $U \subset W'$, $\mathscr{P}_q(W)$ is dense in $\mathrm{APSH}_q(K')$.

Note that the smoothness assumption on φ in Theorem 4.7 can be dropped. If only $\varphi \in \mathrm{CPSH}_q(W)$ then $\varphi_x = \varphi - \varepsilon|z - x|^2$ can be approximated uniformly on a neighborhood of $\overline{D}$ by functions in $\mathscr{P}_q(W)$ by Theorem 5.3 of the next section. Thus $\varphi_x \in \overline{\mathscr{A}(\overline{D})}$ and the above proof goes through with obvious modifications.

5. Piecewise smooth approximation on open sets. On an open set W in $\mathbb{C}^n$ we will approximate arbitrary q-plurisubharmonic functions by functions of the class $\mathscr{P}_q(W)$ of piecewise smooth q-plurisubharmonic functions. We shall start by extending the definition of strict q-plurisubharmonicity to upper semicontinuous functions.

5.1. DEFINITION. An upper semicontinuous function u (with values in $[-\infty, \infty)$) on an open set W in $\mathbb{C}^n$ is called strictly q-plurisubharmonic if for every $x \in W$ there is a neighborhood of x in W where $u - \varepsilon|z|^2$ is q-plurisubharmonic for small $\varepsilon > 0$.

If u is piecewise smooth, then this definition agrees with the one previously given. For suppose

$$u = \max\{u_i \colon 1 \le i \le s\}$$

on a neighborhood U of a point x where u_i are C^2 plurisubharmonic. Without loss of generality, we may assume that x belongs to the closure of

$$U_i = \{z \in U \colon u_i(z) > u_j(z), 1 \le j \le s, j \ne i\}$$

for each i, $1 \le i \le s$. (Shrink U and omit the u_i for which this is not true.) If now $u - \varepsilon|z|^2$ is q-plurisubharmonic on U then the Hessian of $u_i = u$ has $n - q$ eigenvalues $\ge \varepsilon$ at every point of U_i and hence at $x \in \overline{U}_i$.

5.2. LEMMA. *Assume u is a continuous strictly q-plurisubharmonic function on the open set W in $\mathbf{C}^n$ and let $x \in W$. Then for a given ball $B(x, r)$ with closure in W and given $\varepsilon > 0$ there is an open neighborhood N_x of x contained in $B(x, r)$ and a continuous strictly q-plurisubharmonic function u_x on W such that $u_x \in \mathscr{P}_q(N_x)$ and*

$$u < u_x < u + \varepsilon \quad \text{on } N_x, \qquad u = u_x \quad \text{on } W \setminus N_x.$$

Proof. Choose a $c > 0$ with $cr^2 < \varepsilon$ and so that the function v defined by

$$v(z) = u(z) - c|z - x|^2 + cr^2$$

is q-plurisubharmonic on $B(x, r)$. By Theorem 4.6, we can approximate v on $\overline{B(x, r)}$ to within $cr^2/2$ by a function $\tilde{v}$ which is piecewise smooth q-plurisubharmonic in a neighborhood of $\overline{B(x, r)}$, and we can arrange $\tilde{v} < v$ on $\overline{B(x,r)}$. By adding a small multiple of $|z|^2$ to $\tilde{v}$ we obtain a piecewise smooth strictly q-plurisubharmonic $\tilde{v}$ with

$$\begin{aligned} &\tilde{v}(z) < v(z) = u(z), && |z-x| = r, \\ &\tilde{v}(z) \leq v(z) \leq u(z) + cr^2 \leq u(z) + \varepsilon, && |z-x| \leq r, \\ &\tilde{v}(x) \geq v(x) - cr^2/2 = u(x) + cr^2/2. && \end{aligned}$$

Thus

$$u_x(z) = \begin{cases} u(z), & z \in W \setminus B(x,r), \\ \max(u(z), \tilde{v}(z)), & z \in B(x,r), \end{cases}$$

defines a q-plurisubharmonic function on W with the desired properties, where $N_x = \{z \in W \colon u_x(z) > u(z)\}$.

5.3. THEOREM. *Assume u is a continuous strictly q-plurisubharmonic function on the open set W in $\mathbb{C}^n$ and g a continuous function such that $u < g$ on W. Then there is a strictly q-plurisubharmonic function $\tilde{u} \in \mathscr{P}_q(W)$ with $u < \tilde{u} < g$. In particular, there is a monotone decreasing sequence of strictly q-plurisubharmonic $u_n \in \mathscr{P}_q(W)$ that converges to u uniformly on W.*

Proof. For each $x \in W$ choose a ball $B(x,r)$ of radius $r < 1$ with closure in W. Let $2\varepsilon_x > 0$ be a lower bound for $g - u$ on $\overline{B(x,r)}$ and construct u_x and N_x as in the lemma (with $\varepsilon = \varepsilon_x$). Since the N_x have small diameter as x approaches the boundary of W, we can select a locally finite subcover $\{N_{x_i} \colon i = 1, 2 \ldots\}$ of W. Define

$$\tilde{u} = \sup\{u_{x_i} \colon i \geq 1\}.$$

In a neighborhood of any point $x \in W$ only the finitely many u_{x_i} with $x \in N_{x_i}$ need to be used in the sup. Thus $\tilde{u} \in \mathscr{P}_q(W)$, and by construction $u < \tilde{u} < g$. To obtain the sequence u_n, pick $u_0 \in \mathscr{P}_q(W)$ with $u < u_0 < u + 1$ and then u_n inductively so that $u < u_n < \min(u_{n-1}, u + 1/n)$.

A closer look at the proof shows that one can insure that the lower bound for the positive eigenvalues of the Hessians of the approximating functions does not deviate much from the amount of strict q-pseudoconvexity (as measured by the ε in Definition 5.1) of the approximated functions. A more delicate problem is that of approximating continuous, but not necessarily strictly, q-plurisubharmonic functions, and the extension of these results to complex manifolds. Let us mention here only that on a bounded domain D, every continuous q-plurisubharmonic function u can be approximated uniformly by functions in $\mathscr{P}_q(D)$ by applying Theorem 5.3 to $u + \varepsilon|z|^2$. On an arbitrary open set we have the following:

5.4. COROLLARY. *Let W be an open set in $\mathbb{C}^n$ and u a continuous q-plurisubharmonic function on W then there is a monotone decreasing sequence $u_n \in \mathscr{P}_q(W)$ that converges to u uniformly on compacta.*

Proof. Choose $u_0 \in \mathscr{P}_q(W)$ with $u + |z|^2 < u_0 < u + |z|^2 + 1$ and then $u_n \in \mathscr{P}_q(W)$ inductively so that

$$u + |z|^2/n < u_n < \min(u_{n-1}, u + (|z|^2 + 1)/n).$$

Theorem 5.3 and Corollary 5.4 can also be established in the axiomatic setting of §2 if $\mathscr{P}(\overline{D})$ satisfies axioms (1) through (5) and (A3) with $\mathscr{A} = \mathscr{P}$. If *strict* functions in $\widetilde{\mathscr{P}}(W)$ are defined analogously to strictly q-plurisubharmonic functions, then approximation of functions in $\widetilde{\mathscr{P}}(W)$ by functions in $\mathscr{P}(W)$ is obtained by the same methods.

6. A characterization of k-maximum sets. Recall that an open set W in $\mathbb{C}^n$ is q-pseudoconvex if the function v defined by

$$v(z) = -\log(\operatorname{dist}(z, \partial W))$$

is q-plurisubharmonic outside some compact subset K of W. If $M > v(z)$ for $z \in K$ then

$$u(z) = \max(v(z), M) + |z|^2$$

defines a strictly q-plurisubharmonic exhaustion function u on W, i.e. the sets $\{z \in W : u(z) < c\}$ are compact in W for $c \in \mathbb{R}$. An application of Theorem 5.3 yields now:

6.1. COROLLARY. *If W is a q-pseudoconvex open subset of $\mathbb{C}^n$ then there is a piecewise smooth strictly q-plurisubharmonic exhaustion function on W.*

More generally, if $U \subset W$ are open in $\mathbb{C}^n$ then U is said to be q-pseudoconvex in W if

$$u(z) = -\log(\operatorname{dist}(z, \partial U \cap W))$$

is q-plurisubharmonic for $z \in U$ near $\partial U \cap W$. Let $U_0 \subset U$ be an open set with $\partial U_0 \cap W \supset \partial U \cap W$ so that u is q-plurisubharmonic on U_0. By Theorem 5.3 applied to $u + |z|^2$, there is thus a piecewise smooth strictly q-plurisubharmonic function v on U_0 with $v(z) \to \infty$

as $z \to \partial U \cap W$. In the following we say that a relatively closed subset Y of W is the union of complex manifolds if each point $x \in Y$ has a neighborhood V_x with a complex submanifold of V_x passing through x and entirely contained in Y.

6.2. LEMMA. *If $U \subset W$ are open subsets of $\mathbb{C}^n$ and U is q-pseudoconvex in W then $X = W \setminus U$ is the intersection of a decreasing sequence of relatively closed subsets X_m of W each of which is the union of complex manifolds of dimension $n - q - 1$.*

Proof. Let U_0 and v be as described before the lemma. By Sard's theorem, there is a strictly increasing sequence of real numbers $c_m \to \infty$, such that in a neighborhood of each point of

$$Y_m = \{z \in U_0 \colon v(z) = c_m\}$$

v is the maximum of a finite number of C^2 strictly q-plurisubharmonic functions with non-varnishing gradient. By a theorem of Basener (Proposition 6 in **[Ba]**), there is then for each $z_0 \in Y_m$ a complex manifold of dimension $n - q - 1$ passing through z_0 and lying entirely in $\{z \in U_0 \colon v(z) > c_m\} \cup \{z_0\}$. Therefore

$$X_m = W \setminus U \cup \{z \in U_0 \colon v(z) \geq c_m\}$$

have the desired property.

A locally closed subset E of $\mathbb{C}^n$ is a k-maximum set if the polynomials have the local maximum modulus property on $L \cap E$ for every affine subspace L of codimension k in $\mathbb{C}^n$ (see **[S2]**). It is not difficult to show that an example is any locally closed set which is the union of a family of analytic sets of pure dimension $k + 1$. Thus the sets X_m of the above lemma are $(n - q - 2)$-maximum sets. Since any $(n - q - 2)$-maximum set X can be obtained as $W \setminus U$ as in the lemma (Theorem 4.2 of **[S2]**), we have the following:

6.3. COROLLARY. *X is a k-maximum set in $\mathbb{C}^n$ if and only if it can be written as an intersection of a decreasing sequence X_m of k-maximum sets each of which is a union of complex manifolds of dimensions $k + 1$.*

Let D be an open subset of $\mathbb{C}^k$ and assume K is an upper semicontinuous set function whose values $K(z)$ are compact subsets of $\mathbb{C}^m$. Set

$$X = \{(z, w) \colon z \in D, w \in K(z)\},$$
$$W = D \times \mathbb{C}^m, \qquad U = (D \times \mathbb{C}^m) \setminus X.$$

K is an analytic multifunction if X is a $(k-1)$-maximum set or equivalently, by what we mentioned above, if U is $(m-1)$-pseudoconvex in $D \times \mathbb{C}^m$ (see **[S2]**). The construction for Corollary 6.3 then gives:

6.4. COROLLARY. *An analytic multifunction K on $D \subset \mathbb{C}^k$ as described above can be written as the intersection of a decreasing sequence of analytic multifunctions K_n whose graphs X_n are unions of complex manifolds of dimension k.*

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