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A PRETENDER TO THE TITLE "CANONICAL MOEBIUS STRIP"

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The Moebius Strip that results from identifying two opposite sides of a rectangle is embedded analytically and isometrically in Euclidean 3-space, as part of the rectifying developable of the algebraic curve given in paramatric form by

$$x = \sin t, \qquad y = (1-\cos t)^3, \qquad z = \sin(1-\cos t)$$

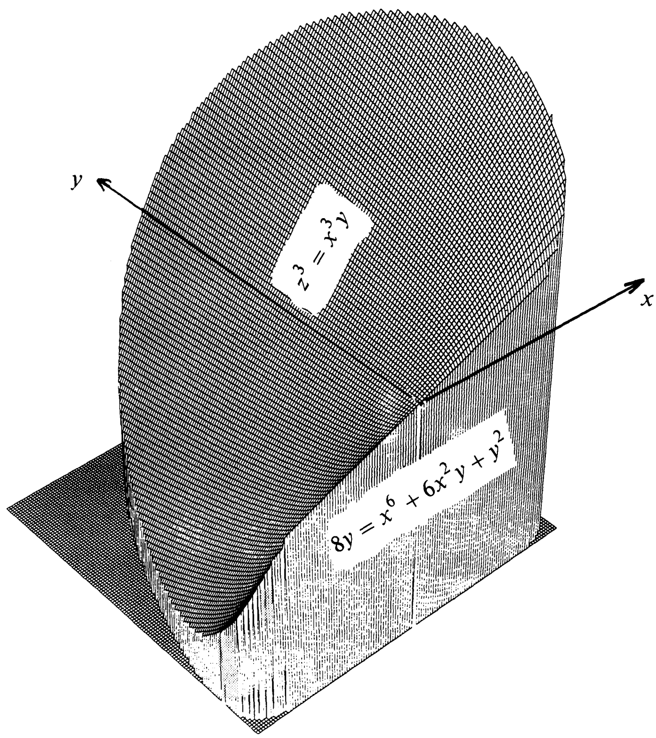
or equivalently, by

$$y^2 + 6x^2y - 8y + x^6 = x^3y - z^3 = 0.$$

1. A Moebius strip is the topological space obtained from a closed rectangle by identifying two opposite sides "with a twist", that is, so that each vertex is identified with its diagonal opposite. There are many ways to embed this space in Euclidean 3-space. One embedding that comes to mind naturally is obtained by choosing an interval on the positive half of the x-axis, rotating it around the z-axis at some fixed rate, and, at the same time, rotating it at half that rate around its perpendicular bisector in the x-y-plane. If the parameter t is defined as the angle through which the first rotation has gone, and s is length measured along the rotating interval, the equations of the surface are easily seen to be analytic functions of the parameters.

In the real world, Moebius Strips are made out of paper: a paper rectangle whose length is sufficiently large compared to its width can be made into a Moebius Strip, without doing violence to the paper (such as tearing, stretching or creasing it), by bending it smoothly, and pasting together two edges in the appropriate manner. Is the first embedding an acceptable mathematical model for the "real" strip? The answer is "no". Our qualifications of what may be done to the paper amount to requiring the embedding to be an isometry of the metric space defined by identifying two opposites of a rectangle appropriately. An isometry would preserve the Gaussian curvature, which is 0 for the rectangle, while the surface obtained in the analytical embedding above is easily seen to have everywhere negative curvature.

Recently Carmen Chicone [1] has shown that there exist analytic embeddings of the Moebius Strip as a regular flat surface in 3-space. A flat surface, i.e. one of zero Gaussian curvature, is locally isometric to a portion of the plane. Chicone's Moebius Strips can indeed be cut and rolled out isometrically on a plane, but the shapes they take on have not been shown to be rectangles. To illustrate the problem, consider embedding a cylindrical band in a circular cone. The embedded band can be made isometric to a section of a planar annulus with its two straight edges identified, but not to a rectangle. Had the "center curve" of Chicone's surface been a geodesic, a narrow band around it would have been the required surface, isometric to a rectangle; but this has not been shown to be the case.



FIGURE

2. A flat surface with a curve that is a geodesic on it can be interpreted as the rectifying developable (see e.g. [2] p. 70) of the curve, provided the curve is regular, and has no straight parts; the surface is thus determined by the curve. Furthermore, if the curve is analytic and regular, so is the surface. This suggests the following plan for constructing an isometric analytic embedding of a rectangular Moebius Strip: find a simple closed regular analytic curve in 3-space, whose principal normal reverses its direction when the curve is traversed one time. A sufficiently narrow band of the rectifying developable will be the required solution.

To make the curve closed, its cartesian coordinates will be assumed to be periodic functions of the parameter t. The simplest such functions are trigonometric polynomials. The form of the polynomials is found by some heuristics based on empirical evidence. Consider a Moebius Strip made by glueing together the overlapping ends of a paper rectangle. The glued paper strip will arrange itself so that the glued area, being stiffer than the rest of the paper strip, will contain the point where the mean curvature of the strip changes sign. The shape of the strip suggests that it has an axis of symmetry passing through that point. This axis cuts across the band along the glued part, and pierces the band orthogonally at the point that had been the center of the rectangle before it was twisted. Note that any surface that is carried into itself by a rotation of 180 degrees around an axis that is tangent to it at one point, and perpendicular to it at another point, is nonorientable: if a normal to the surface had been chosen everywhere, the rotation would reverse the normal at the former point, and would leave the normal at the latter point unchanged. Assume that the surface, and hence also the curve, have such an axis of symmetry; let the first point be the origin, where $x = y = z = t = 0$, and let the axis of symmetry be the y-axis, with its positive direction pointing towards the second point. The symmetry of the curve under a rotation of 180 degrees around the y-axis is attained by choosing for y an even function of t, and for x and z, odd functions; the former will be a sum of cosine terms, and the latter, sums of sine terms. For x a single term turns out to be sufficient. Put $x = A\sin t$, $y = B_0 + B_1\cos t + B_2\cos 2t + \cdots$, $z = C_1\sin t + C_2\sin 2t, \ldots$. The choice of origin leads to the condition that the coefficients of y add up to 0. For the positive direction of the x-axis, the tangent of the curve at the origin can be chosen, leading to $C_1 + 2C_2 + \cdots = 0$. The assumption that at the origin the normal to the strip, which is the principal normal to the curve, reverses direction, leads to $B_1 + 4B_2 + \cdots = 0$. All these conditions can be fulfilled by y and z being just second degree trigonometric polynomials. By rewriting the appropriate polynomials in factored form, the curve

$$x = \sin t, \qquad y = (1 - \cos t)^2, \qquad z = \sin t(1 - \cos t)$$

is obtained. This is a simple regular analytic closed curve, that is invariant under a 180 degree rotation around the y-axis, has curvature

0 at the origin, and its rectifying plane at the point where $t = \pi$, $x = 0$, $y = 4$, $z = 0$ is orthogonal to the y-axis. However, this curve has nonzero torsion at the origin, where the curvature is 0. The direction of the ruling on a rectifying developable is given, in terms of the curvature k, the torsion τ, the tangent unit vector $\mathbf{T}$ and the unit binormal $\mathbf{B}$, by $k\mathbf{B} + \tau\mathbf{T}$. Therefore, when a point along the curve given above, approaches the origin, the direction of the rulings approaches the tangent to the curve. At the origin there is thus no way to broaden the curve into a strip: the rectifying developable is singular there. To prevent this from happening, the torsion must be made to vanish wherever the curvature does, and at least at the same rate. With this in mind, after some trial and error, one finds that redefining $y(t)$ as $(1 - \cos t)^3$ will do the job: it will yield the required behavior at the origin, while at $t = \pi$ the rectifying plane will still be orthogonal to the y-axis. As is easily seen, the curve given by

$$x = \sin t, \qquad y = (1 - \cos t)^3, \qquad z = \sin t(1 - \cos t)$$

is not only analytic but algebraic: it is the intersection of the surfaces given by

$$y^2 + 6x^2y - 8y + x^6 = 0 \quad \text{and} \quad x^3 = yz^3.$$

3. The "velocity" $v(t)$, the length of the vector (x', y', z'), is the square root of a polynomial of the sixth degree in $\cos t$. It is positive for all t, since $x' = \cos t$ and $y' = 3\sin t(1 - \cos t)^2$ never vanish together, and $v(t)$ has no singularities. The vector (x'', y'', z'') does vanish at 0. Its cross-product with (x', y', z') has the y-component $-\sin t(1 + 2\cos^2 t)$; therefore $w(t)$, the squared length of the cross product of (x', y', z') and (x'', y'', z''), vanishes only at the origin where it has a zero of order 2. The curvature of the curve, that is given by $k(t) = \sqrt{w(t)}/v^3(t)$, is thus seen to have its only zero at the origin; it is $|t|$ times an analytic function, and is not analytic itself. Note, however, that the second order zero of k^2 comes from a factor $1 - \cos t$, which has the analytic square-root $\sqrt{2}\sin\frac{1}{2}t$. One can therefore define an analytic "signed curvature" $q(t) = k(t)\text{sign}(\sin\frac{1}{2}t)$. The fact that q has the double period 4π corresponds to the fact that the principal normal to the curve, which is also the normal to its rectifying developable, flips 180 degrees when t passes through 0. The torsion $\tau(t)$, depends on the determinant $\Delta(t)$ of the first three derivatives of (x, y, z). The leading terms near $t = 0$ are as follows:

	x	y	z
First derivatives:	1	$\frac{3}{4}t^5$	$\frac{3}{2}t^2$;
Second derivatives:	0	$\frac{15}{4}st^4$	$3t$;
Third derivatives:	-1	$15t^3$	$3.$

The lowest power of t appearing in the summands of this determinant is 4, and the corresponding terms are $\frac{45}{4}t^4$, and $-45t^4$. Therefore $\Delta(t)$ has a fourth-order zero at the origin. The torsion $\tau(t) = \Delta(t)/w(t)$ has, consequently, a second-order zero at the origin (it may have other zeros, but since $w(t)$ vanishes only at the origin they need not concern us), and the ratio of the torsion of $q(t)$ has a first-order zero there. In parametric form, the Moebius Strip can be written as

$$(x(t, s)y(t, s), z(t, s)) = (x(t), y(t), z(t)) + s\left(\mathbf{B} + \frac{\tau(t)}{k(t)}\mathbf{T}\right),$$

where t is arbitrary, and s, the parameter that represents geodesic distance from the curve, is restricted to a sufficiently small interval, symmetric around 0. In this form the parametrization is not analytic at $t = 0$. The singularity at 0 is removed, however, when s is replaced by $s\,\mathrm{sign}(\sin\frac{1}{2}t)$, yielding

$$\begin{aligned}(x(t, s), y(t, s)z(t, s)) &= (x(t), y(t), z(t)) \\ &\quad + s\left(\mathrm{sign}(\sin\tfrac{1}{2}t)\mathbf{B} + \frac{\tau(t)}{q(t)}\mathbf{T}\right).\end{aligned}$$

The direction of the binormal is that of the cross-product above, whose leading term at the origin is $(0, -3t, 0)$. Therefore $\mathbf{B}$ flips around when t passes 0, and $\mathrm{sign}(\sin\frac{1}{2}t)\mathbf{B}$ is analytic there. Since τ and q have zeros of order two and one respectively, the quotient τ/q vanishes at the origin, and the ruling of the rectifying developable to the curve at the origin lies on the y-axis, as required.

The fact that the curve is algebraic, and that the added rulings are algebraic as well, implies, by the Seidenberg-Tarski Theorem [**3**], that the strip is itself part of an algebraic surface.

I am indebted to Professor Brumfiel for pointing out this last implication. Thanks also to Moe Hirsch, who made me aware of the manuscript of Carmen Chicone.

P.S After this paper had been submitted, Gil Bor called my attention to a paper by Wunderlich [**4**], that also describes an algebraic isometrically embedded Moebius Strip. The initial approach in Wunderlich's paper is similar to the present one; however, mainly by the use of trigonometric polynomials, the path to the result is considerably shorter and simpler here. See [5] for an exposition of related work.

References

[1] Carmen Chicone, untitled preprint, University of Missouri, 1984.

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