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ON THE FIX-POINTS OF COMPOSITE FUNCTIONS

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Gross has conjectured that a composite transcendental entire function has infinitely many fix-points. We show that the conjecture is true if one of the two components has finite order.

1. Introduction and results. Let f and g be two nonlinear entire functions, at least one of them transcendental. Gross [**4**] has conjectured that the composite function $f \circ g$ has infinitely many fix-points.

Gross and Osgood [**5**] have proved that the conjecture is true, if one of the functions f and g is of finite order while the other one is of finite lower order. The conjecture has also been proved under various other conditions on f and g (cf. [**6**], [**9**], [**13**], [**14**]).

We shall prove

THEOREM 1. *Let f and g be nonlinear entire functions, at least one of them transcendental. If one of the functions f and g is of finite order, then $f \circ g$ has infinitely many fix-points.*

As a consequence of Theorem 1 we obtain

THEOREM 2. *Let f and g be nonlinear entire functions, at least one of them transcendental. If*

$$\limsup_{r\to\infty} \frac{\log\log\log M(r, f\circ g)}{\log r} < \infty,$$

then $f \circ g$ has infinitely many fix-points.

These two theorems contain and generalize many of the results referred to above.

2. Lemmas. Our proofs will be based partially on Nevanlinna theory (for notations see [7]), but mainly on Wiman-Valiron theory. We denote the maximum term of an entire function h by $\mu(r, h)$ and the central index by $N = N(r, h)$. By F we denote an exceptional set of finite logarithmic measure, not necessarily the same at each occurrence. For the convenience of the reader we state the results of

Wiman-Valiron theory that we need. In fact Hayman [8] has obtained much more precise estimations, but the following results suffice for our purposes.

LEMMA 1([8], *see also* [12]). *Let* h *be entire,* $k > 0$, $\gamma > 1/2$, $0 < \eta < 1$ *and* $\varepsilon > 0$. *Assume that* $|z_0| = r$, $|h(z_0)| \geq \eta M(r, h)$ *and* $|\tau| \leq kN^{-\gamma}$. *Then*

$$h(z_0 e^{\tau}) \sim h(z_0)e^{N\tau} \qquad (r \notin F), \tag{2.1}$$

$$h'(z_0 e^{\tau}) \sim \frac{N}{z_0 e^{\tau}} h(z_0)e^{N\tau} \qquad (r \notin F), \tag{2.2}$$

$$\log \mu(r, h) \sim \log M(r, h) \sim \log M(r, h') \qquad (r \notin F), \tag{2.3}$$

$$N \leq (\log \mu(r, h))^{1+\varepsilon} \qquad (r \notin F), \tag{2.4}$$

$$\log \mu(r, h) \leq N \log r + O(1). \tag{2.5}$$

LEMMA 2. *Let* h *be entire,* $K > 0$, $0 < \eta < 1$ *and* $\varepsilon > 0$. *If* $|\sigma_1| < K$, $|h(z_0)| \geq \eta M(r, h)$ *and if* $|z_0| = r \notin F$ *is large enough, then there exists* τ_1 *such that* $|N\tau_1 - \sigma_1| < \varepsilon$ *and* $h(z_0 e^{\tau_1}) = h(z_0)e^{\sigma_1}$. *If* $\varepsilon < 2\pi$ *and if* $r \notin F$ *is large enough, then* τ_1 *is unique.*

Proof. Put $w_1 = h(z_0)e^{\sigma_1}$ and consider $f_1(\tau) = h(z_0 e^{\tau})$ and $f_2(\tau) = h(z_0)e^{N\tau} = w_1 \exp(N\tau - \sigma_1)$. If $|N\tau - \sigma_1| = \varepsilon$, then

$$f_1(\tau) \sim h(z_0)e^{N\tau} = f_2(\tau)$$

by (2.1) and therefore

$$|(f_1(\tau) - w_1) - (f_2(\tau) - w_1)| = |f_1(\tau) - f_2(\tau)| = o(|f_2(\tau)|). \tag{2.6}$$

On the other hand, we have for $|N\tau - \sigma_1| = \varepsilon$

$$\begin{aligned} |f_2(\tau) - w_1| &= |w_1(\exp(N\tau - \sigma_1) - 1)| \\ &\geq \delta_1 |w_1| \geq \delta_2 |f_2(\tau)| \end{aligned} \tag{2.7}$$

for some $\delta_1 \geq \delta_2 > 0$, if $0 < \varepsilon < 2\pi$. The conclusion follows from (2.6) and (2.7) by Rouché's theorem.

Clunie [3] has given the following application.

LEMMA 3. *If* f *and* g *are entire, then*

$$M(r, f \circ g) = M((1 + o(1))M(r, g), f) \qquad (r \notin F). \tag{2.8}$$

Next we note that if $f \circ g$ has only a finite number of fix-points, then

$$f(g(z)) = P(z)e^{\alpha(z)} + z, \tag{2.9}$$

where α is an entire function and P is a polynomial. A consequence of Lemma 3 is

LEMMA 4. *If* (2.9) *holds, then*

$$M(r, \alpha) \sim \log M((1+o(1))M(r, g), f) \qquad (r \notin F). \tag{2.10}$$

The following lemma is implicit in the work of Gross and Osgood **[5]**.

LEMMA 5. *If* (2.9) *holds, then*

$$T(r, g) = o(T(r, \alpha')) \qquad (r \notin E), \tag{2.11}$$

where E has finite linear measure.

In fact, if $T(r, \alpha') \leq KT(r, g)$ for a constant K on a set of infinite measure, then a modification of a theorem of Steinmetz **[11]** (cf. **[5]**) yields that f satisfies a certain differential equation. As shown in **[5]**, this leads to a contradiction.

We remark that for our purposes the weaker inequality

$$T(r, g) = O(T(r, \alpha')) \qquad (r \notin E) \tag{2.12}$$

will be sufficient. This inequality is easier to obtain than (2.11), in fact the method used in **[2]** for the Riccati equation applies also to the linear equation

$$\begin{aligned}\frac{d}{dz}(f(g(z))) = \left(\frac{P'(z)}{P(z)} + \alpha'(z)\right) f(g(z)) \\ - \left(\frac{P'(z)}{P(z)} + \alpha'(z)\right) z + 1,\end{aligned}$$

which is a consequence of (2.9).

We also need

LEMMA 6 **[1]**. *Let $h(x)$ and $k(x)$ be non-negative, non-decreasing and convex for $x \geq 0$. Let $K > 1$ and suppose that $h(x) \leq k(x)$ for all $x \geq 0$. Then $h'(x) \leq Kk'(x)$ on a set of lower density at least $(K-1)/K$.*

A consequence is

LEMMA 7. *Let* α *and* g *be entire functions,* $c > 0$, $K > 1$ *and assume that*

$$\log M(r, \alpha) < c \log M(r, g) \qquad (r \notin F).$$

Then

$$N(r, \alpha) \leq KcN(r, g)$$

on a set of positive lower logarithmic density.

Proof. Let $\varepsilon > 0$ and put $x = \log r$, $h(x) = \max\{0, \log \mu(r, \alpha)\}$, $k(x) = \max\{h(x), (c+\varepsilon)\log \mu(r, g)\}$. The conclusion follows from (2.3) and Lemma 6, since

$$\frac{d\mu(r, h)}{d \log r} = N(r, h) \tag{2.13}$$

for an entire function h, except for the discontinuities of $N(r, h)$.

3. Proof of Theorems.

Proof of Theorem 1. Since $f \circ g$ has infinitely many fix-points if and only if $g \circ f$ does [**6**, p. 214, proof of Theorem 2], we may assume that the order of f is finite. The conclusion follows from the result of Gross and Osgood [**5**] mentioned in the introduction, if the lower order of g is finite. Hence we may assume that the lower order of g is infinite. What we need, however, is only that g has non-zero lower order.

Suppose that $f \circ g$ has only a finite number of fix-points, so that (2.9) holds. Lemma 4 shows that $\log M(r, \alpha) = O(\log M(r, g))$ for $r \notin F$ and Lemma 7 implies that there exists a positive constant c such that

$$N(r, \alpha) \leq cN(r, g) \qquad (r \in H) \tag{3.1}$$

where H has positive lower logarithmic density. It follows easily from a classical lemma due to Borel [7, Lemma 2.4] that for $\beta > 0$

$$\log M(r, g) \leq T(r, g)^{1+\beta} \qquad (r \notin E), \tag{3.2}$$

where E has finite linear measure. Combining (2.3), (2.4), (2.5), (2.12) and (3.2), we get for $\varepsilon > 0$ and $r \notin F$

$$\begin{aligned} N(r, g) &\leq [\log \mu(r, g)]^{1+\varepsilon} \leq [\log M(r, g)]^{1+\varepsilon} \\ &\leq T(r, g)^{1+2\varepsilon} \leq T(r, \alpha')^{1+3\varepsilon} \leq [\log M(r, \alpha')]^{1+3\varepsilon} \\ &\leq [\log \mu(r, \alpha)]^{1+4\varepsilon} \leq [N(r, \alpha) \log r]^{1+5\varepsilon}. \end{aligned} \tag{3.3}$$

From the assumption that the lower order of g is positive (or infinite) we can deduce that

$$\log r \le N(r, g)^{\varepsilon}, \tag{3.4}$$

if r is large enough. If $1/2 < \gamma < 1$ and if $\varepsilon > 0$ is suitably chosen, then (3.3) and (3.4) imply that

$$N(r, g)^{\gamma} \le N(r, \alpha) \qquad (r \notin F). \tag{3.5}$$

Now choose z_0 such that $|f(g(z_0))| = M(r, f \circ g)$, where $r = |z_0|$. It follows from Lemma 3 that

$$|g(z_0)| = (1 - o(1))M(r, g) \qquad (r \notin F) \tag{3.6}$$

and that

$$M(r, e^{\alpha}) = \exp((1 - o(1))M(r, \alpha)) \qquad (r \notin F). \tag{3.7}$$

If we put $m(r, P) = \min\{|P(z)|; |z| = r\}$, where P is the polynomial from the representation (2.9), then

(3.8)
$$\begin{aligned} M(r, e^{\alpha}) &= M\left(r, \frac{Pe^{\alpha} + z - z}{P}\right) \le \frac{M(r, Pe^{\alpha} + z) + r}{m(r, P)} \\ &= \frac{|P(z_0)e^{\alpha(z_0)} + z_0| + r}{m(r, P)} \le \frac{M(r, P)}{m(r, P)}|e^{\alpha(z_0)}| + \frac{2r}{m(r, P)} \\ &= (1 + o(1))\exp(\operatorname{Re}\alpha(z_0)). \end{aligned}$$

Combining (3.7) and (3.8) we get

$$|\alpha(z_0)| \ge \operatorname{Re}\alpha(z_0) \ge (1 - o(1))M(r, \alpha) \qquad (r \notin F). \tag{3.9}$$

Lemma 2 implies that there exists τ_1 satisfying $|\tau_1 N(r, g) - 2\pi i| = o(1)$ such that $g(z_0 e^{\tau_1}) = g(z_0)$, provided $r \notin F$. Let $z_1 = z_0 e^{\tau_1}$ and

$$l(z) = \frac{f'(g(z))g'(z) - 1}{f(g(z)) - z}.$$

Then

$$\frac{l(z_1)}{l(z_0)} \sim \frac{g'(z_1)}{g'(z_0)} \sim \frac{g(z_1)}{g(z_0)} = 1 \tag{3.10}$$

by (3.6) and Lemma 1. On the other hand we have

$$l(z) = \frac{P'(z)}{P(z)} + \alpha'(z)$$

by (2.9). Since

$$|\tau_1| \le \frac{2\pi + o(1)}{N(r, g)} \le \frac{2\pi c + o(1)}{N(r, \alpha)} \qquad (r \in H\backslash F)$$

by (3.1), we have

$$\frac{l(z_1)}{l(z_0)} \sim \frac{\alpha'(z_1)}{\alpha'(z_0)} \sim \frac{\alpha(z_1)}{\alpha(z_0)} \sim \exp(\tau_1 N(r, \alpha)) \qquad (r \in H\backslash F) \tag{3.11}$$

by (3.9) and Lemma 1. It follows from (3.10) and (3.11) that $\tau_1 N(r, \alpha) = 2\pi i k + o(1)$ for some integer $k = k(r)$, provided $r \in H\backslash F$. Hence we have

$$\frac{N(r, \alpha)}{N(r, g)} \sim k(r) \in \mathbf{Z} \qquad (r \in H\backslash F) \tag{3.12}$$

where $k(r) \le c$ by (3.1).

Now let $\tau_2 = i\pi/N(r, \alpha)$ and $z_2 = z_0 e^{\tau_2}$. Lemma 1 and (3.9) imply that $\alpha(z_2) \sim (-\alpha(z_0))$ and $\operatorname{Re}\alpha(z_2) \sim (-M(r, \alpha))$ for $r \notin F$. It follows from (3.5) that $|\tau_2| \le \pi N(r, g)^{-\gamma}$ for $r \notin F$. Hence we have

$$|g(z_2)| \sim |g(z_0)\exp(N(r, g)\tau_2)| \sim |g(z_0)| \sim M(r, g) \qquad (r \notin F)$$

by Lemma 1. Lemma 2 implies that there exists τ_3 satisfying $|\tau_3 N(r, g) - 2\pi i| = o(1)$ such that $g(z_2 e^{\tau_3}) = g(z_2)$. Let $z_3 = z_2 e^{\tau_3}$. To estimate $\alpha(z_3)$ we note that

$$|\tau_3| \le \frac{2\pi c + o(1)}{N(r, \alpha)} \qquad (r \in H\backslash F)$$

by (3.1). Hence Lemma 1 and (3.12) imply that

$$\begin{aligned} \alpha(z_3) &\sim \alpha(z_2)\exp(N(r, \alpha)\tau_3) \\ &\sim \alpha(z_2)\exp((k(r) + o(1))(2\pi i + o(1))) \\ &\sim \alpha(z_2) \qquad (r \in H\backslash F). \end{aligned}$$

Since $g(z_2) = g(z_3)$ we have

$$z_2 + P(z_2)e^{\alpha(z_2)} = f(g(z_2)) = f(g(z_3)) = z_3 + P(z_3)e^{\alpha(z_3)}.$$

It follows that

$$\begin{aligned} |z_3 - z_2| &\le |P(z_2)e^{\alpha(z_2)}| + |P(z_3)e^{\alpha(z_3)}| \\ &\le r^K \exp(-(1 - o(1))M(r, \alpha)) \end{aligned}$$

for some constant K and $r \in H\backslash F$. On the other hand we have

$$|z_3 - z_2| = |z_2(e^{\tau_3} - 1)| \sim r|\tau_3| \sim \frac{2\pi r}{N(r, g)},$$

so that

$$N(r, g) \geq (1 - o(1))2\pi r^{1-K} \exp((1 - o(1))M(r, \alpha)) \geq \exp \frac{M(r, \alpha)}{2}$$

for sufficiently large $r \in H \backslash F$. By (3.3) we have

$$N(r, g) \leq [\log \mu(r, \alpha)]^{1+4\varepsilon} \leq [\log M(r, \alpha)]^{1+4\varepsilon} \qquad (r \notin F).$$

Altogether we find for $\varepsilon = 1/4$ that

$$\exp \frac{M(r, \alpha)}{2} \leq [\log M(r, \alpha)]^2 \qquad (r \in H \backslash F).$$

This is an obvious contradiction and the theorem is proved.

Proof of Theorem 2. Assume that $f \circ g$ has only a finite number of fix-points so that (2.9) holds. It is easy to show that

$$\rho(\alpha) = \limsup_{r \to \infty} \frac{\log \log \log M(r, e^{\alpha})}{\log r},$$

where $\rho(\alpha)$ denotes the order of α. In fact this is a special case of a theorem of Schönhage [**10**, Satz 6]. It follows from (2.9) and the hypothesis that $\rho(\alpha) < \infty$. Moreover, we have $\rho(\alpha') = \rho(\alpha)$ and (2.11) or (2.12) imply that $\rho(g) \leq \rho(\alpha')$. Hence we have $\rho(g) < \infty$, and the conclusion follows from Theorem 1.

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References

[1] W. Bergweiler, *An inequality for real functions with applications to function theory*, Bull. London Math. Soc., **21** (1989), 171–175.

[2] W. Bergweiler, G. Jank and L. Volkmann, *Über faktorisierbare Lösungen Riccatischer Differentialgleichungen*, Resultate Math., **10** (1986), 40–53.

[3] J. Clunie, *The Composition of Entire and Meromorphic Functions*, Math. Essays Dedicated to A. J. Macintyre (1970), 75–92, Ohio Univ. Press.

[4] F. Gross, *Factorization of Meromorphic Functions*, U.S. Government Printing Office, Washington, D.C., 1972.

[5] F. Gross and C. F. Osgood, *On fixed points of composite entire functions*, J. London Math. Soc., (2) **28** (1983), 57–61.

[6] F. Gross and C. C. Yang, *The fix-points and factorization of meromorphic functions*, Trans. Amer. Math. Soc., **168** (1972), 211–219.

[7] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs (1964), Clarendon Press, Oxford.

[8] ——, *The local growth of power series: A survey of the Wiman-Valiron method*, Canad. Math. Bull., (3) **17** (1974), 317–358.

[9] P. C. Rosenbloom, *The fix-points of Entire Functions*, Medd. Lunds Univ. Mat. Sem. Suppl.-Bd. M. Riesz, (1952), 187–192.

[10] A. Schönhage, *Über das Wachstum zusammengesetzter Funktionen*, Math. Z., **73** (1960), 22–44.

[11] N. Steinmetz, *Über die faktorisierbaren Lösungen gewöhnlicher Differentialgleichungen*, Math. Z., **170** (1980), 169–180.

[12] G. Valiron, *Lectures on the General Theory of Integral Functions*, Edouard Privat, Toulouse, 1923.

[13] C. C. Yang, *Further results on the fix-points of composite transcendental functions*, J. Math. Anal. Appl., **90** (1982), 259–269.

[14] ——, *On the fix-points of composite transcendental entire functions*, J. Math. Anal. Appl., **108** (1985), 366–370.

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