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MASS OF RAYS ON COMPLETE OPEN SURFACES

K. SHIOHAMA, T. SHIOYA, AND M. TANAKA

The total curvature of a complete open surface describes certain properties of the Riemannian structure which defines it. We study relationships between the total curvature and the mass of rays on a finitely connected complete open surface and obtain some integral formulas.

0. Introduction. Throughout this paper let M be a connected, finitely connected, oriented, complete and noncompact Riemannian 2-manifold without boundary. The total curvature $c(M)$ of M is defined to be an improper integral over M of Gaussian curvature G with respect to the area element dM of M. A well-known theorem due to Cohn-Vossen [**1**] states that if M admits total curvature, then $2\pi\chi(M) - c(M) \geq 0$, where $\chi(M)$ is the Euler characteristic of M. Clearly $c(M)$ depends on the choice of Riemannian metric. This phenomenon gives rise to the idea that the value $2\pi\chi(M) - c(M)$ should describe certain properties of Riemannian metric which defines it.

A ray (respectively, a straight line) on M is by definition a unit speed geodesic parametrized on $[0, \infty)$ (respectively, on $\mathbb{R}$) every subarc of which realizes distance between its terminal points. For a point $p \in M$ let $S_p(1)$ be the unit circle centered at the origin of the tangent space M_p to M at p. Let $A(p)$ be the set of all unit vectors tangent to rays emanating from p. $A(p)$ is closed in $S_p(1)$. Let $\mathfrak{M}$ be the natural measure on $S_p(1)$ induced from the Riemannian metric. A relation between the mass of rays and the total curvature was first investigated by Maeda in [**6**], [**7**]. He proved that if M is homeomorphic to R^2 and if $G \geq 0$, then $\mathfrak{M} \circ A \geq 2\pi - c(M)$, and in particular $\inf_M \mathfrak{M} \circ A = 2\pi - c(M)$. These results were extended by Shiga in [**10**], [**11**] to Riemannian planes whose Gaussian curvatures change sign, and later by Oguchi [**9**] to finitely connected M with one endpoint. In connection with an isoperimetric problem discussed by Fiala [**3**] and Hartman [**4**], the first-named author proved in [**14**] that if M has one end and if $2\pi\chi(M) - c(M) < 2\pi$, then for every monotone increasing sequence $\{K_j\}$ of compact sets with $\bigcup K_j = M$,

$$\lim_{j\to\infty} \frac{\int_{K_j} \mathfrak{M} \circ A \, dM}{\int_{K_j} dM} = 2\pi\chi(M) - c(M).$$

The proof of this equation essentially depends on the fact that M *admits no straight lines*. This property is guaranteed by the assumptions on the total curvature and the uniqueness of endpoint of M.

It should also be noted that all results mentioned above are obtained under the assumption that M has one endpoint. In the case where M has more than one endpoint (and this is the case where we are interested in this paper), it will be natural to consider that each endpoint shares the value $2\pi\chi(M) - c(M)$ in the following sense. Let M have k endpoints and let $K \subset M$ be a compact set with the property that $M \setminus \mathrm{Int}(K)$ consists of k tubes $U_1, \ldots, U_k$ such that each U_i is homeomorphic to $S^1 \times [0, \infty)$ and that each ∂U_i is a piecewise smooth simply closed curve. Then the Gauss-Bonnet theorem states that $c(K) + \sum_{i=1}^{k} \kappa(\partial U_i) = 2\pi\chi(M)$, where $c(K) = \int_K G \, dM$ and $\kappa(\partial U_i)$ denotes the curvature integral over the boundary curve ∂U_i. For each $i = 1, \ldots, k$ the value

$$s_i(M) := \kappa(\partial U_i) - c(U_i)$$

is nonnegative and independent of the choice of tube. Moreover

$$\sum_{i=1}^{k} s_i(M) = 2\pi\chi(M) - c(M).$$

For details see [**15**]. Thus one observes that each endpoint corresponding to U_i shares the value $2\pi\chi(M) - c(M)$.

With these notations our main results will be stated as follows.

THEOREM A. *Assume that M admits total curvature and has k endpoints. If $s_i(M) \le 2\pi$ holds for each $i = 1, \ldots, k$, then for every monotone increasing sequence $\{K_j\}$ of compact sets with $\bigcup K_j = M$,*

$$\begin{aligned} \operatorname*{Min}_{1\le i\le k} s_i(M) &\le \liminf_{j\to\infty} \frac{\int_{K_j} \mathfrak{M} \circ A \, dM}{\int_{K_j} dM} \\ &\le \limsup_{j\to\infty} \frac{\int_{K_j} \mathfrak{M} \circ A \, dM}{\int_{K_j} dM} \le \operatorname*{Max}_{1\le i\le k} s_i(M). \end{aligned}$$

THEOREM B. *Assume that M admits total curvature and has k endpoints. Let $\mathfrak{C}$ be a simply closed smooth curve in M and let $B(t) := \{x \in M ; d(x, \mathfrak{C}) \le t\}$ and $S(t) := \{x \in M ; d(x, \mathfrak{C}) = t\}$,* *where d is the distance function induced from Riemannian metric. If $s_i(M) \leq 2\pi$ holds for each $i = 1, \ldots, k$, then*

$$\lim_{t\to\infty} \frac{\int_{B(t)} \mathfrak{M} \circ A\, dM}{\int_{B(t)} dM} = \begin{cases} \dfrac{\sum_{i=1}^{k} s_i^2(M)}{2\pi\chi(M) - c(M)} & \text{if } 2\pi\chi(M) - c(M) > 0, \\ 0 & \text{if } 2\pi\chi(M) - c(M) = 0. \end{cases}$$

REMARK 1. Shiohama first proved an inequality in Theorem B under the stronger assumption that $s_i(M) < 2\pi$. But subsequent improvement on the asymptotic behavior of $\mathfrak{M} \circ A$ was obtained by Shioya and Tanaka. It turns out that the existence of straight lines on M is no objection at all. Tanaka's proof for the asymptotic behavior of $\mathfrak{M} \circ A$ by assuming $s_i(M) = 2\pi$ will be provided in Lemma 1.1. Shioya has extended this result to the case where $+\infty \geq s_i(M) \geq 2\pi$. This result will be published independently because the proof is fascinating and of independent interest in itself.

REMARK 2. Theorem B does not hold for any monotone increasing sequence $\{K_j\}$ of compact sets with $\bigcup K_j = M$. For example, consider a surface M of revolution in $\mathbb{R}^3$: Let $f: \mathbb{R} \to (0, \infty)$ be a positive smooth function satisfying $f(t) = 1$ for $t \leq -1$, $f(t) = (t \cdot \tan\theta + 1)$ for $t \geq 1$, where θ is a constant in $(0, \pi/2)$. M is defined as

$$M = \{(x, y, z) \in \mathbb{R}^3;\, y^2 + z^2 = f(x)^2,\, x \in \mathbb{R}\}.$$

Then $s_1(M)$ and $s_2(M)$ are 0 and $2\pi\sin\theta$ and $2\pi\chi(M) - c(M) = 2\pi\sin\theta$. For any given $\varepsilon > 0$ there exists a positive number t_ε such that if $p \in M$ satisfies $x(p) < -t_\varepsilon$, then $\mathfrak{M} \circ A(p) < \varepsilon$, and such that if $x(p) > t_\varepsilon$, then $\mathfrak{M} \circ A(p) \in (s_2(M) - \varepsilon, s_2(M) + \varepsilon)$. For an arbitrary fixed number $\alpha > 0$ choose a monotone increasing sequence $\{K_j^\alpha\}$ of compact sets of M with $\bigcup K_j^\alpha = M$ such that

$$\mathrm{Area}\{p \in K_j^\alpha;\, x(p) > 0\} / \mathrm{Area}\{p \in K_j^\alpha;\, x(p) < 0\} = \alpha.$$

Then, computation will show that

$$\lim_{j\to\infty} \frac{\int_{K_j^\alpha} \mathfrak{M} \circ A\, dM}{\int_{K_j^\alpha} dM} = \frac{s_1(M) + \alpha s_2(M)}{\alpha + 1} = \frac{(2\pi\chi(M) - c(M))\alpha}{\alpha + 1}.$$

Since $\alpha > 0$ is arbitrary, this example will suggest the validity of Theorem A.

1. Preliminaries. Let $K \subset M$ be a compact set with the property that $M \setminus \mathrm{Int}(K)$ consists of k tubes $U_1, \dots, U_k$ such that each ∂U_i is a piecewise smooth closed curve. For a point $p \in M \setminus \mathrm{Int}(K)$ taken sufficiently away from K, $A(p)$ is divided into two subsets $A_K(p)$ and $A'_K(p)$ as follows: For $u \in A(p)$ set $\gamma_u(t) := \exp_p tu$, $t \geq 0$.

$$A_K(p) := \{u \in A(p);\ \gamma_u([0, \infty)) \cap K \neq \varnothing\},$$
$$A'_K(p) := \{u \in A(p);\ \gamma_u([0, \infty)) \cap \mathrm{Int}(K) = \varnothing\}.$$

Both $A_K(p)$ and $A'_K(p)$ are closed in $S_p(1)$. It follows from minimizing property of rays emanating from p that $A_K(p) \cap A'_K(p)$ consists of at most two elements. Therefore

$$\mathfrak{M} \circ A(p) = \mathfrak{M} \circ A_K(p) + \mathfrak{M} \circ A'_K(p).$$

It was proved in §§2 and 3 in **[14]** that if $0 \leq s_i(M) < 2\pi$, then for any given $\varepsilon > 0$ there exists an $R(\varepsilon)$ such that for every $p \in U_i$ with $d(p, K) > R(\varepsilon)$

$$(*) \qquad s_i(M) - \varepsilon \leq \mathfrak{M} \circ A'_K(p) \leq s_i(M) + \varepsilon.$$

A crucial step of the proof of Theorems A and B is to obtain the asymptotic behavior of $\mathfrak{M} \circ A$. What is left for this purpose is to prove for all $i = 1, \dots, k$ and for all $p \in U_i$ with $d(p, K) > R(\varepsilon)$,

$$(**) \qquad \mathfrak{M} \circ A_K(p) < \varepsilon$$

and the following

LEMMA 1.1 (*Tanaka*). *Assume that $s_i(M) = 2\pi$. Then there exists a compact set K with the property that for any $\varepsilon > 0$ there exists an $R_i(\varepsilon) > 0$ such that if $p \in U_i$ satisfies $d(p, K) > R_i(\varepsilon)$, then*

$$\mathfrak{M} \circ A'_K(p) > 2\pi - \varepsilon.$$

Making use of a slightly extended version of an idea developed in the proof of Theorem C in **[12]**, $(**)$ is proved for a more general closed subinterval $S_p(D(p))$ of $S_p(1)$ which contains $A_K(p)$. For $p \in U_i$ and for $u, v \in A_K(p)$ let $D_{u,v}(p)$ be the disk domain in U_i bounded by the subarcs of γ_u and γ_v between $p = \gamma_u(0) = \gamma_v(0)$ and their first intersections with K and a subarc of ∂U_i between them. Let $D(p)$ be the maximal disk domain among $\{D_{u,v}(p) \colon u, v \in A_K(p)\}$ and $S_p(D(p)) \subset S_p(1)$ the set of all unit vectors at p tangent to $D(p)$. Define an angle

$$\theta_K(p) := \mathfrak{M}(S_p(D(p))).$$

Then the proof of $(**)$ is a direct consequence of the following.

LEMMA 1.2 (*Shioya*). *Let $K \subset M$ be as above and assume that $s_i(M) \leq +\infty$ holds for all $i = 1, \dots, k$. For any $\varepsilon > 0$ there exists an $R(\varepsilon) > 0$ such that if $p \in M \backslash K$ satisfies $d(p, K) > R(\varepsilon)$, then*

$$\theta_K(p) < \varepsilon.$$

2. Proof of Theorems A and B by assuming Lemmas 1.1 and 1.2. First of all consider the case where the total area of M is bounded. Then a slight modification of Lemma 3.1 in **[14]** implies that there exist k distinct Busemann functions on M, each of which corresponds to an endpoint of M. A Busemann function is differentiable except a set of measure zero since it is Lipschitz continuous. This fact means that there exists a measure zero set E on M such that $A(p)$ for every $p \in M \backslash E$ consists of exactly k elements. Furthermore one has $2\pi\chi(M) - c(M) = 0$ if the total area of M is bounded (see Theorem 12 in **[5]** and Corollary of Theorem A in **[13]**). Therefore the proof of theorems in this case is complete.

Assume that the total area of M is unbounded. Let

$$R(\varepsilon) := \operatorname*{Max}_{1 \leq i \leq k} R_i(\varepsilon).$$

Let a be the area of closed $R(\varepsilon)$-ball around K and b the integral of $\mathfrak{M} \circ A$ over this closed ball. It follows from $(*)$, Lemmas 1.1 and 1.2 that for all sufficiently large j

$$\frac{b + \left(\operatorname{Min}_{1 \leq i \leq k} s_i(M) - \varepsilon\right) \left\{\int_{K_j} dM - a\right\}}{\int_{K_j} dM} \leq \frac{\int_{K_j} \mathfrak{M} \circ A \, dM}{\int_{K_j} dM} \leq \frac{b + \left(\operatorname{Max}_{1 \leq i \leq k} s_i(M) + \varepsilon\right) \left\{\int_{K_j} dM - a\right\}}{\int_{K_j} dM}.$$

The proof of Theorem A is complete since ε is any and the total area of M is unbounded.

For the proof of Theorem B one applies the Fiala-Hartman type isoperimetric inequality which was refined by Shiohama in **[12]** and **[13]**. Fix a compact set K containing $\mathfrak{C}$ as in Lemmas 1.1 and 1.2. For every $i = 1, \dots, k$ and for sufficiently large $t > 0$ let $L_i(t)$ and $A_i(t)$ be the length of $S(t) \cap U_i$ and the area of $B(t) \cap U_i$. Because M admits total curvature $S(t) \cap U_i$ is homeomorphic to a circle for all large t (see Theorem B in **[13]**), and is piecewise smooth for almost all t. Note that $A_i(t) - A_i(t') = \int_{t'}^{t} L_i(u) \, du$. For every $i = 1, \dots, k$

$$\lim_{t \to \infty} \frac{L_i(t)}{t} = \lim_{t \to \infty} \frac{2A_i(t)}{t^2} = s_i(M).$$

By choosing $R(\varepsilon)$ sufficiently large so as to fulfil

$$s_i(M) - \varepsilon < \frac{L_i(t)}{t} < s_i(M) + \varepsilon$$

for all $i = 1, \dots, k$ and for all $t > R(\varepsilon)$, one obtains

$$\frac{b + \sum_{i=1}^{k}(s_i(M) - 2\varepsilon)(s_i(M) - \varepsilon)^{(t^2 - R(\varepsilon)^2)/2}}{\sum_{i=1}^{k}(s_i(M) + \varepsilon)^{(t^2 - R(\varepsilon)^2)/2} + a} \leq \frac{\int_{B(t)} \mathfrak{M} \circ A \, dM}{\int_{B(t)} dM}$$
$$\leq \frac{b + \sum_{i=1}^{k}(s_i(M) + 2\varepsilon)(s_i(M) + \varepsilon)^{(t^2 - R(\varepsilon)^2)/2}}{\sum_{i=1}^{k}(s_i(M) - \varepsilon)^{(t^2 - R(\varepsilon)^2)/2} + a}.$$

This completes the proof of Theorem B.

3. Proof of Lemmas. A general formula for the mass of rays emanating from a point $p \in M$ is obtained by using an idea developed by Shiga in [10]. This is stated as

$$(***) \qquad \mathfrak{M} \circ A(p) = 2\pi\chi(M) - c(M \backslash F_p),$$

where $F_p := \{\exp_p tu;\, u \in A(p),\, t \geq 0\}$. This formula plays an essential role for the proof of Lemma 1.1.

For the proof of $(***)$ fix a point $p \in M$ and let $T > 0$ be a sufficiently large number such that $S(p, T) := \{x \in M;\, d(p, x) = T\}$ consists of k piecewise smooth closed curves $C_1, \dots, C_k$ in $U_1, \dots, U_k$ and such that the break points $x_{i,1}, \dots, x_{i,m(i)}$ of C_i are joined to p by exactly two distinct minimizing geodesics $\alpha_{i,1}^-$, $\alpha_{i,1}^+, \dots, \alpha_{i,m(i)}^-$, $\alpha_{i,m(i)}^+$ with $\alpha_{i,m}^-(0) = \alpha_{i,m}^+(0) = p$, $\alpha_{i,m}^-(T) = \alpha_{i,m}^+(T) = x_{i,m}$ and $x_{i,m}$ is not conjugate to p along $\alpha_{i,m}^-$ and $\alpha_{i,m}^+$. This is possible whenever T is taken to be a sufficiently large non-exceptional value (see [4], [13]). Let $F_{i,m}$ $(i = 1, \dots, k$, $1 \leq m \leq m(i))$ be a disk domain surrounded by $\alpha_{i,m}^+([0, T])$, the smooth subarc of $S(p, T)$ with terminal points $x_{i,m}$ and $x_{i,m+1}$ and $\alpha_{i,m+1}^-([0, T])$, and $\theta_{i,m}$ the angle between $-\dot{\alpha}_{i,m}^-(T)$ and $-\dot{\alpha}_{i,m}^+(T)$. If $\kappa_{i,m}$ is the curvature integral of the subarc on $\partial F_{i,m} \cap S(p, T)$, then

$$c(F_{i,m}) = \mathfrak{M}(S_p(F_{i,m})) - \kappa_{i,m}.$$

If $B(p, T)$ is the closed T-ball around p, then

$$c(B(p, T)) + \sum_{i=1}^{k}\sum_{m=1}^{m(i)} \kappa_{i,m} - \sum_{i=1}^{k}\sum_{m=1}^{m(i)} \theta_{i,m} = 2\pi\chi(M).$$

It follows from construction that $\bigcup_i \bigcup_m S_p(F_{i,m})$ is monotone decreasing with T and converges to $A(p)$ as $T \to \infty$. The proof of $(***)$ is complete since $\lim_{T\to\infty} \sum_{i=1}^{k} \sum_{m=1}^{m(i)} \theta_{i,m} = 0$ (see Theorem C, [12]) and $\lim_{T\to\infty} c(B(p, T)\setminus \bigcup_i \bigcup_m F_{i,m}) = c(M\setminus F_p)$.

Proof of Lemma 1.1. For a compact set C such that $M\setminus C$ consists of k tubes, we choose a K containing C such that every minimizing geodesic joining points in C does not meet ∂K. Let M_i be a complete open 2-manifold having one end with the properties that there exists an isometric embedding $\imath$ of $K \cup U_i$ into M_i and that $M_i\setminus \imath(K \cup U_i)$ consists of $k-1$ disks. From construction it follows that $2\pi\chi(M_i) - c(M_i) = s_i(M)$ and $\chi(M_i) = \chi(M) + (k-1)$. Without loss of generality one may identify points in U_i with those images in M_i as well as other objects. For $p \in U_i$ let $A_i(p)$, $A_{K,i}(p)$ and $A'_{K,i}(p)$ be the set of all unit vectors tangent to rays on M_i from p with the same properties as defined in M. Then $A'_{K,i}(p) = A'_K(p)$ follows from the choice of K. There is no strict relationship between $A_{K,i}(p)$ and $A_K(p)$. But both of them will be estimated in Lemma 1.2. Since $\mathfrak{M} \circ A(p) = (\mathfrak{M} \circ A_K(p) - \mathfrak{M} \circ A_{K,i}(p)) + \mathfrak{M} \circ A_i(p)$ and the first term in the right-hand side turns out to be small by Lemma 1.2, one only needs to show that $\mathfrak{M} \circ A_i(p) > 2\pi - \varepsilon$ if p is taken sufficiently away from K in M_i.

From now on one identifies M_i with M. For any $\varepsilon > 0$ let $K_\varepsilon \subset M$ be a compact set containing K such that

$$\int_{M\setminus K_\varepsilon} |G|\, dM < \varepsilon.$$

By means of $(***)$ it suffices for the proof of Lemma 1.1 to show $c(M\setminus F_p) < c(M) + 5\varepsilon$ for $p \in M$ with $d(p, K) > R(\varepsilon)$. It follows from finite connectivity of M that there are at most finitely many non-overlapping sectors $V_1(p), \dots, V_l(p)$ in M with the following properties: (1) $V_j(p) \cap K_\varepsilon \neq \varnothing$, (2) $\partial V_j(p)$ consists of two rays emanating from p, (3) $V_j(p)$ is homeomorphic to a closed half-plane, and (4) every ray emanating from p is contained in some $V_j(p)$ if it intersects K_ε. $V_j(p)$ has the property that if $V'_j(p) \subset V_j(p)$ is a subsector such that there is no ray emanating from p and passing through a point on $\mathrm{Int}(V'_j(p))$, then $c(V'_j(p)) = \mathfrak{M}(S_p(V'_j(p)))$. Let $\{p_n\}$ be a divergent sequence of points in $M\setminus K_\varepsilon$ such that $\{V_j(p_n)\}$ for each $j = 1, \dots, l$ has a limit V_j as $n \to \infty$. This V_j is a strip if it has a nonempty interior. If $V'_j \subset V_j$ is a substrip such that there exists no straight line contained entirely in $\mathrm{Int}(V'_j)$, then $c(V'_j) = 0$.

Set $V = V_1 \cup \cdots \cup V_l$. $c(M \backslash F_{p_n}) \leq c(K_\varepsilon) - c(K_\varepsilon \cap F_{p_n}) + \varepsilon$ and $\{c(K_\varepsilon \cap F_{p_n})\}_n$ tends to $c(K_\varepsilon \cap V)$ as $n \to \infty$. Thus for all sufficiently large numbers n, $c(M \backslash F_{p_n}) \leq c(M \backslash V) + 4\varepsilon$. Since V_j is a strip, a result of Cohn-Vossen (see Satz 3, [2]) implies that $c(V_j) \leq 0$ for all $j = 1, \ldots, l$. This implies that $c(M \backslash V_j) \leq 2\pi\chi(M \backslash V_j) - 4\pi$. But since $\chi(M \backslash V_j) = \chi(M) + 1$ the above inequality reduces to $c(M \backslash V_j) \leq 2\pi\chi(M) - 2\pi$. It follows from the assumption for $c(M)$ that $c(M \backslash V_j) \leq c(M)$, and in particular $c(V_j) = 0$ for all $j = 1, \ldots, l$. Therefore $c(M \backslash F_{p_n}) \leq c(M \backslash V) + 4\varepsilon \leq c(M) + 5\varepsilon$. This together with (***) proves Lemma 1.1.

Proof of Lemma 1.2. A contradiction will be derived by supposing that there exists a divergent sequence $\{p_n\}$ of points such that $\theta_K(p_n) \geq \varepsilon_0$ holds for all n and for some $\varepsilon_0 > 0$. Without loss of generality we may consider that $\{p_n\}$ is contained in a tube U.

To derive a contradiction consider the universal Riemannian covering $\widetilde{U}$ of U whose covering projection is denoted by π. Let $\tau: [0, \infty) \to M$ be a ray emanating from a point on ∂U such that $\tau([0, \infty))$ is contained entirely in U. Cut open U along $\tau([0, \infty))$ and let $\widetilde{U}_{-1}$, $\widetilde{U}_0$, $\widetilde{U}_1, \ldots$ be the fundamental domains of U lying in this order in $\widetilde{U}$. Let $\tilde{\tau}_i: [0, \infty) \to \widetilde{U}$ be the lifted ray of τ such that its image lies in $\partial\widetilde{U}_{i-1} \cap \partial\widetilde{U}_i$ and $\widetilde{W} := \widetilde{U}_0 \cup \widetilde{U}_1 \cup \widetilde{U}_2$. Then $\partial\widetilde{W}$ consists of two rays $\tilde{\tau}_0([0, \infty))$, $\tilde{\tau}_3([0, \infty))$ and a subarc of $\partial\widetilde{U}$ whose terminal points are $\tilde{\tau}_0(0)$ and $\tilde{\tau}_3(0)$.

The intersection of the two minimizing segments on $\partial D(p_n)$ with ∂U will be denoted by x_n and y_n. Set $D_n = D(p_n)$ and let $\tilde{p}_n := \pi^{-1}(p_n) \cap \widetilde{U}_1$ and $\widetilde{D}_n \subset \widetilde{U}$ the lift up of D_n satisfying $\tilde{p}_n \in \partial\widetilde{D}_n$. Let $\tilde{x}_n := \pi^{-1}(x_n) \cap \partial\widetilde{D}_n$ and $\tilde{y}_n := \pi^{-1}(y_n) \cap \partial\widetilde{D}_n$. It follows from minimizing property of rays that the lifted minimizing geodesics joining $\tilde{p}_n$ to $\tilde{x}_n$ and $\tilde{p}_n$ to $\tilde{y}_n$ intersect $\pi^{-1}(\tau)$ at most at one point. This fact means that these geodesics are in $\widetilde{W}$, and in particular, $\tilde{x}_n$ and $\tilde{y}_n$ are on $\partial\widetilde{W} \cap \partial\widetilde{U}$. By choosing a subsequence, if necessary, one may consider that $\{\tilde{x}_n\}$, $\{\tilde{y}_n\}$ and $\{\widetilde{D}_n\}$ converge to $\tilde{x}$, $\tilde{y}$ and to an unbounded domain $\widetilde{D}$ in $\widetilde{W}$. Two cases occur in the convergence of $\{\widetilde{D}_n\}$. In the first case, assume that $\{\tilde{p}_n\}$ is contained in the closure of $\widetilde{D}$. Then one may consider that $\{\widetilde{D}_n\}$ is monotone increasing and $\bigcup \widetilde{D}_n = \widetilde{D}$. A slight modification of Theorem C in [**12**] implies that $\{\theta_K(p_n)\}$ converges to 0, a contradiction. In the second case, assume that $\{\tilde{p}_n\}$ is not contained in the closure of $\widetilde{D}$. Without loss of generality one may consider that the lifted minimizing geodesic joining

$\tilde{p}_n$ to $\tilde{x}_n$ intersects $\partial\widetilde{D}$ at a point $\tilde{r}_n$. Set $\widetilde{E}_n := \widetilde{D}_n \backslash \widetilde{D}$ and let $\alpha_n \in (0, \pi)$ be the angle at $\tilde{r}_n$ of the corner of $\widetilde{D}_n \cap \widetilde{D}$. By construction, $\{\tilde{r}_n\}$ contains a divergent subsequence. Then Cohn-Vossen's argument (see §5, **[2]**) implies that $\{\alpha_n\}$ has a limit 0. Let $K_\varepsilon \subset M$ be a compact set so as to satisfy

$$\int_{M \backslash K_\varepsilon} G_+ \, dM < \varepsilon.$$

Then the area of $\pi^{-1}(K_\varepsilon \cap U) \cap \widetilde{E_n}$ tends to zero as $n \to \infty$ and the curvature integral over $\widetilde{E}_n \backslash \pi^{-1}(K_\varepsilon \cap U)$ is bounded above by ε. These facts together with the Gauss-Bonnet theorem for $\widetilde{E_n}$ imply that $\{\theta_K(p_n)\}$ contains a subsequence converging to 0 as $n \to \infty$, a contradiction. This completes the proof of Lemma 1.2.

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KYUSHU UNIVERSITY
FUKUOKA, 812—JAPAN

AND

TOKAI UNIVERSITY
HIRATSUKA, 259-12—JAPAN