PACIFIC JOURNAL OF MATHEMATICS
Vol. 143, No. 2, 1990

ON THE GENERALIZED DIFFERENCE POLYNOMIALS

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We study some factorization properties of a family of polynomials which includes the generalized difference polynomials. We deduce new irreducibility criteria for polynomials in two variables with coefficients in an algebraically closed field. We also obtain new proofs for the irreducibility criteria of Ehrenfeucht and Angermüller.

Let k be a commutative algebraically closed field. A polynomial in two variables $P(X, Y) \in k[X, Y]$ is called a *difference polynomial* if $P(X, Y) = f(X) - g(Y)$, where $f, g \in k[X]\backslash k$.

A. Ehrenfeucht [**6**] and H. Tverberg [**11**] studied a case of irreducibility of the difference polynomials and A. Schinzel [**10**] established conditions for the factorization of the difference polynomials. L. A. Rubel and S. S. Abhyankar [**2**], L. A. Rubel, A. Schinzel and H. Tverberg [**8**] and G. Angermüller [**3**] studied some factorization and irreducibility conditions of the larger class of the *generalized difference polynomials*

$$(*) \qquad Q(X, Y) = cY^n + \sum_{i=1}^{n} P_i(X) Y^{n-i},$$

where $c \in k\backslash\{0\}$, $n \in \mathbb{N}^*$, $\deg P_n(X) = m \geq 1$ and $\deg P_i(X) < mi/n$ for every i, $1 \leq i \leq n-1$.

In this paper we study factorization properties of polynomials of the form

$$(**) \qquad F(X, Y) = cY^n + \sum_{i=1}^{n} P_i(X) Y^{n-i},$$

where $c \in k\backslash\{0\}$, $P_i(X) \in k[X]$, $n \geq 1$.

The family of the polynomials $(**)$ includes the class of the generalized difference polynomials $(*)$. We introduce a rational number $p_Y(F)$ associated with a polynomial $F(X, Y) \in k[X, Y]$ that satisfies $(**)$. We shall establish some properties of $p_Y(F)$ using a Newton polygon argument. We deduce irreducibility criteria for the polynomials $(**)$. We also obtain new proofs of the criteria of Ehrenfeucht and Angermüller.

DEFINITION. Let X, Y be two indeterminates over k and let

$$F(X, Y) = cY^n + \sum_{i=1}^{n} P_i(X)Y^{n-i} \in k[X, Y],$$

where $c \in k\setminus\{0\}$, $n \geq 1$, $P_i(X) \in k[X]$.

We call the *degree index* of the polynomial $F(X, Y)$ the rational number

$$p_Y(F) = \max\left\{\frac{\deg P_i}{i};\ 1 \leq i \leq n\right\}.$$

REMARKS. (i) $p_Y(F) = 0$ if and only if $F(X, Y) \in k[Y]$.

(ii) If

$$p_Y(F) = \frac{\deg P_n}{n} \quad \text{and} \quad p_Y(F) > \max\left\{\frac{\deg P_i}{i};\ 1 \leq i \leq n-1\right\}$$

then $F(X, Y)$ is a generalized difference polynomial.

THEOREM 1. *Let*

$$F(X, Y) = cY^n + \sum_{i=1}^{n} P_i(X)Y^{n-1} \in k[X, Y],$$

where $c \in k\setminus\{0\}$, $P_i(X) \in k[X]$, $n \geq 1$. *If* $F = F_1F_2$, *with* $F_1, F_2 \in k[X, Y]\setminus k$, *then* $p = \max(p_1, p_2)$, *where* $p = p_Y(F)$, $p_1 = p_Y(F_1)$, $p_2 = p_Y(F_2)$.

Proof. We shall prove that p can be obtained as a suitable number associated with the polynomial $G(X, Y) := Y^nF(X^{-1}, Y^{-1}) \in k((X))[Y]$. ($G$ is a polynomial in Y with the coefficients meromorphic formal power series in X.)

Let

$$G(X, Y) = \sum_{i=0}^{n} H_i(X)Y^i \in k((X))[Y],$$

where $H_i(X) \in k((X))$. Let $r_i = \operatorname{ord}_X H_i(X)$ and let

$$e(G) = \max\left(\frac{r_0 - r_1}{1}, \frac{r_0 - r_2}{2}, \ldots, \frac{r_0 - r_n}{n}\right).$$

But

$$G(X, Y) = P_n\left(\frac{1}{X}\right)Y^n + P_{n-1}\left(\frac{1}{X}\right)Y^{n-1} + \cdots + P_1\left(\frac{1}{X}\right)Y + c.$$

Hence $r_0 = 0$ and $r_i = \operatorname{ord}_X P_i(\frac{1}{X}) = -\deg(P_i)$ for $i = 1, 2, \dots, n$. Therefore $r_0 - r_1 = 0 - (-\deg(P_i)) = \deg(P_i)$ and it follows that

$$e(G) = p_Y(F). \tag{1}$$

If the characteristic of the field k is zero or is positive and does not divide the degree n of the polynomial $G(X, Y)$ then $e(G)$ is the smallest exponent of a Puiseux series

$$y(X) = \sum_i c_i X^i \in \bigcup_{m=1}^{\infty} k((X^{1/m})) \quad \text{such that } G(X, y) = 0.$$

(Such a series exists because k is algebraically closed and the characteristic of k does not divide n.)

If the characteristic of k is positive and divides n then a root $y(X)$ of the equation $G(X, y) = 0$ is not necessary a Puiseux series, as it was remarked by C. Chevalley in **[4]**, p. 64. In this case a root of the equation $G(X, y) = 0$ is a general power series $y(X) = \sum_{i \in S(f)} c_i X^i \in k((T^{\mathbb{Q}}))$, where the support $S(f)$ is a well ordered subset of $\mathbb{Q}$. Let i_0 be the smallest of the exponents of $y(X)$. Then i_0 can be obtained with a Newton polygon argument (cf. **[7]** pp. 42–48) as

$$i_0 = \max\left(\frac{r_0 - r_1}{1}, \frac{r_0 - r_2}{2}, \dots, \frac{r_0 - r_n}{n}\right).$$

Indeed, it suffices to remark that the determination of i_0 above does not depend on the characteristic. Therefore $i_0 = e(G)$.

Let $G_1(X, Y), G_2(X, Y) \in k((X))[Y]$ corresponding to the polynomials F_1 and F_2 respectively. From a result of G. Dumas relative to the Newton polygon of the product of two polynomials (**[5]**, pp. 216–217) it follows that $e(G) = \max(e(G_1), e(G_2))$. From (1) we deduce that $p = \max(p_1, p_2)$.

PROPOSITION 2. *Let*

$$F(X, Y) = cY^n + \sum_{i=1}^{n} P_i(X) Y^{n-i} \in k[X, Y],$$

where $c \in k^*$, $n \geq 1$ *and let* $m = \deg P_n(X)$. *Let us suppose that* $p_Y(F) = m/n$. *If* $F = F_1 F_2$, *with* $F_1, F_2 \in k[X, Y] \backslash k$ *then* $p_Y(F) = p_Y(F_1) = p_Y(F_2)$.

Proof. Let

$$F_i(X, Y) = c_i Y^{n_i} + \sum_{j=1}^{n_i} P_{ij}(X) Y^{n_i - j},$$

where $c_i \in k^*$, $P_{ij}(X) \in k[X]$ and let $m_i = \deg P_{in_i}$ $(i = 1, 2)$. Then $n_1 + n_2 = n$, $m_1 + m_2 = m$.

From Theorem 1 it follows that

$$\frac{m_1}{n_1} \leq \frac{m}{n} \quad \text{and} \quad \frac{m_2}{n_2} \leq \frac{m}{n}.$$

But

$$\frac{m}{n} = \frac{m_1 + m_2}{n_1 + n_2}.$$

We deduce that $m_2 n_1 \geq m_1 n_2 \geq m_2 n_1$, and hence $m_2 n_1 = m_1 n_2$. Therefore

$$\frac{m}{m_2} = \frac{m_1 + m_2}{m_2} = \frac{n_1 + n_2}{n_2} = \frac{n}{n_2}.$$

It follows that $m/n = m_2/n_2$, and hence $p_Y(F_2) = p_Y(F)$. In the same way we deduce that $p_Y(F_1) = p_Y(F)$.

COROLLARY 3. *Let*

$$F(X, Y) = cY^n + \sum_{i=1}^{n} P_i(X) Y^{n-i} \in k[X, Y],$$

where $c \in k^*$, $P_i(X) \in k[X]$, $n \geq 1$ *and let* $m = \deg P_n(X)$. *If* $p_Y(F) = m/n$ *and* $(m, n) = 1$ *then the polynomial* $F(X, Y)$ *is irreducible in* $k[X, Y]$.

Proof. Let us suppose that there are $F_1, F_2 \in k[X, Y] \backslash k$ such that $F = F_1 F_2$. We suppose F_1, F_2 expressed as in the proof of the former proposition.

Because $p_Y(F) = p_Y(F_1) = p_Y(F_2) = m/n$ there is $i \in \{1, 2, \dots, n_1\}$ such that

$$\frac{\deg(P_{1i})}{i} = \frac{m}{n}.$$

Therefore $im = n \cdot \deg(P_{1i})$. Since $(m, n) = 1$ there is $s \in \mathbb{N}^*$ such that $i = sn$. It follows that $i = n_1 = n$ and $\deg(P_{1i}) = \deg(P_{1n_1}) = \deg(P_{1n}) = m$. Therefore $F_2(X, Y) \in k[X]$ and we deduce that $P_n(X) = P_{1n}(X) F_2(X, Y)$. We conclude that $F_2(X, Y) \in k$, a contradiction. It follows that $F(X, Y)$ is irreducible in $k[X, Y]$.

REMARKS. (i) If the characteristic of k does not divide n one can prove Corollary 3 using the Newton-Puiseux expansion theorem [**1**], 5.14.

(ii) The class of the polynomials

$$F(X, Y) = cY^n + \sum_{i=1}^{n} P_i(X)Y^{n-i} \in k[X, Y], \quad c \in k^*,$$
$$P_i(X) \in k[X], \quad n \geq 1$$

such that

$$p_Y(F) = \frac{\deg(P_n)}{n}$$

includes the family of the generalized difference polynomials. Therefore Corollary 3 establishes an irreducibility criterion for the generalized difference polynomials.

LEMMA 4. *Let*

$$F(X, Y) = cY^n + \sum_{i=1}^{n} P_i(X)Y^{n-1} \in k[X, y], \quad c \in k^*, \quad P_i(X) \in k[X]$$

and let $f \in [X]\backslash k$, $g \in k[Y]\backslash k$. *Then*

$$p_Y(F(f, g)) = \frac{\deg(f)}{\deg(g)} \cdot p_Y(F).$$

Proof. Let $u = \deg(f), v = \deg(g)$ and $H(X, Y) = F(f(X), g(Y))$. Then

$$H(X, Y) = c[g(Y)]^n + P_1(f(X))[g(Y)]^{n-1} + \cdots + P_{n-1}(f(X))g(Y) + P_n(f(X)).$$

Because

$$\frac{\deg[P_i(f(X))]}{iv} \geq \frac{\deg[P_i(f(X))]}{s} \quad \text{for } iv \leq s \leq nv$$

it follows that

$$p_Y(G) = \max\left\{\frac{\deg P_i(f)}{iv}; 1 \leq i \leq n\right\}.$$

But $\deg P_i(f) = u \cdot \deg(P_i)$. Therefore

$$p_Y(H) = \frac{u}{v} \cdot p_Y(F) = \frac{\deg(f)}{\deg(g)} \cdot p_Y(F).$$

COROLLARY 5. *Let*

$$F(X, Y) = cY^n + \sum_{i=1}^{n} P_i(X)Y^{n-i} \in k[X, Y], \quad c \in k^*,$$
$$P_i(X) \in k[X], \quad n \geq 1$$

such that $m = \deg(P_n) \geq 1$ *and* $p_Y(F) = m/n$. *If* $f \in k[X]\backslash k$ *and* $g \in k[X]\backslash k$ *are such that* $(m \cdot \deg(f), n \cdot \deg(g)) = 1$ *then the polynomial* $F(f, g)$ *is irreducible in* $k[X, Y]$.

Proof. Let $H(X, Y) = F(f, g)$. From the above lemma it follows that $p_Y(G) = m \cdot \deg(f)/n \cdot \deg(g)$. From Corollary 3 it follows that $F(f, g)$ is irreducible in $k[X, Y]$.

REMARK. Corollary 5 was obtained by G. Angermüller in [**3**] with different methods in the special case $F(X, Y)$ is a generalized difference polynomial.

THEOREM 6. *Let*

$$F(X, Y) = cY^n + \sum_{i=1}^{n} P_i(X)Y^{n-i} \in k[X, Y], \quad c \in k^*,$$
$$P_i(X) \in k[X], \quad n \geq 1$$

and let $a = \deg_X F(X, Y)$. *If* $p_Y(F) = a/b$, $(a, b) = 1$, *then the polynomial* $F(X, Y)$ *is irreducible in* $k[X, Y]$ *or it has a factor from* $k[Y]$.

Proof. Let us suppose that there are $F_1, F_2 \in k[X, Y]\backslash k$ such that $F = F_1F_2$. Let

$$F_i(X, Y) = c_iY^{n_i} + \sum_{j=1}^{n_i} P_{ij}(X)Y^{n_i-j} \in k[X, Y], \quad c_i \in k^*,$$
$$P_{ij}(X) \in k[X] \qquad (i = 1, 2).$$

From Theorem 1 it follows that we may suppose that $p_Y(F_1) = p_Y(F)$. Hence there is $j \in \{1, 2, \ldots, n_1\}$ such that $\deg(P_{1j})/j = a/b$, i.e. $ai = b \cdot \deg(P_{1j})$.

Since $(a, b) = 1$ it follows a divides $\deg(P_{1j})$. But $0 \leq \deg(P_{1j}) \leq \deg_X(F) = a$. Therefore $\deg(P_{1j}) = 0$ or a.

If $\deg(P_{1j}) = 0$ then $p_Y(F_1) = 0$; hence $p_Y(F) = 0$ and it follows that $F(X, Y) \in k[Y]$.

If $\deg(P_{1j}) = a$ then $\deg_X(F_2) = 0$. Therefore F_2 is a polynomial from $k[Y]$.

If follows that $F(X, Y)$ is irreducible or has a factor from $k[Y]$.

REMARKS. (i) If the polynomial $F(X, Y)$ has a factor from $k[Y]$ then this factor is the greatest common divisor of the polynomials

$Q_i \in k[Y]$ such that

$$F(X, Y) = \sum_{i=0}^{a} Q_i(Y) X^{a-i}.$$

(ii) If $p_Y(F) = m/n$, where $m = \deg(P_n)$, then $m/n \geq \deg(P_i)/i$ for every $i = 1, 2, \dots, n$. Hence $m \geq (n/i) \cdot \deg P_i \geq \deg P_i$ and it follows that $m = a = \deg_X(F)$. Therefore the class of the generalized difference polynomials is contained in the family of the polynomials satisfying the assumptions from Theorem 6.

COROLLARY 7. *Let*

$$F(X, Y) = cY^n + \sum_{i=1}^{n} P_i(X) Y^{n-i} \in k[X, Y],$$

where $c \in k^*$, $P_i(X) \in k[X]$, $n \geq 1$, *let* $a = \deg_X F(X, Y)$ *and* $f \in k[X] \backslash k$, $g \in k[X] \backslash k$. *If* $p_Y(F) = a/b$ *and* $(a \cdot \deg(f), b \cdot \deg(g)) = 1$ *then* $F(f, g)$ *is irreducible in* $k[X, Y]$ *or it has a factor from* $k[Y]$.

Proof. Let $u = \deg(f)$, $v = \deg(g)$ and $H(X, Y) = F(f(X), g(Y))$. From Lemma 4 it follows that $p_Y(H) = ua/vb$. Since $\deg_X(H) = \deg(f) \cdot \deg_X(F) = ua$ and $(ua, vb) = 1$ we conclude by Theorem 6.

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Received July 28, 1988.

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