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THE POWER 3/2 APPEARING IN THE ESTIMATE OF ANALYTIC CAPACITY

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We show that the power 3/2 appearing in the estimate of analytic capacity is best possible.

1. Introduction. For a compact set E in the complex plane $\mathbb{C}$, $H^\infty(E^c)$ denotes the Banach space of bounded analytic functions in $E^c = \mathbb{C}\cup\{\infty\} - E$ with supremum norm $\|\cdot\|_{H^\infty}$. The analytic capacity of E is defined by

$$\gamma(E) = \sup\{|f'(\infty)|\,;\ \|f\|_{H^\infty} \leq 1\,,\ f \in H^\infty(E^c)\}\,,$$

where $f'(\infty) = \lim_{z\to\infty} z(f(z)-f(\infty))$, i.e., $f'(\infty)$ is the $(1/z)$-coefficient of the Taylor expansion of $f(z)$ at infinity. It is easily seen that $\gamma(E) \leq |E|$, where $|E|$ is the (generalized) length of E; if E is a subset of the real line $\mathbb{R}$, then $|E|$ equals its 1-dimension Lebesgue measure (cf. Garnett [**4**, Chap. III]). Vitushkin [**12**] constructed an example Q_∞ such that $\gamma(Q_\infty) = 0$ and $|Q_\infty| > 0$ (cf. [**4**, p. 87]). Denjoy [**3**] showed that $\gamma(E) > 0$ if E is a subset of a rectifiable curve such that $|E| > 0$. But his proof has a serious gap, and his theorem was, for a while, called the Denjoy conjecture. As is easily seen, we may assume that E is a subset of a rectifiable graph. Let $\operatorname{pr} E$ denote the projection of E to $\mathbb{R}$. Since pr is a contraction [**6**, p. 377], it is natural to try the lower estimate of $\gamma(E)$ by $\gamma(\operatorname{pr} E)$. Pommerenke [**11**] showed that $\gamma(\operatorname{pr} E) = |\operatorname{pr} E|/4$. Hence this approach is equivalent to comparing $\gamma(E)$ with $|\operatorname{pr} E|$. To do this, the study of the Cauchy-Hilbert transform on C^1 graphs is necessary (Davie [**2**]). In 1977, Calderón [**1**] succeeded in proving its boundedness, and, using his theorem, Marshall [**8**] finally settled the Denjoy conjecture in the affirmative. After Marshall's theorem, we are concerned with studying further relations between $\gamma(E)$ and $|\operatorname{pr} E|$. Using an estimate of the Cauchy-Hilbert transform on Lipschitz graphs [**10**, p. 53], the author [**9**] showed that

$$\gamma(E) \geq C_0 |\operatorname{pr} E|^{3/2}$$

if E is a subset of a rectifiable graph Γ satisfying $|\Gamma| = 1$, where C_0 is an absolute constant. The main purpose of this paper is to show that the power $3/2$ is best possible. Our method gives a new approach to the computation of analytic capacity, and suggests that analytic capacity is related to the theory of fractals (Mandelbrot [7]).

For an integer $p \geq 2$, we put

$$B_p(x) = \frac{1}{2p}\{1-(-1)^k\} \qquad \left(\frac{k}{p} \leq x < \frac{k+1}{p}, \, 0 \leq k \leq p-1\right).$$

For an n-tuple $(p_1, \dots, p_n)$ of integers larger than or equal to 2, we put

$$A(x; p_1, \dots, p_n) = \sum_{j=1}^{n} B_{p_1 \cdots p_j}(x).$$

A set $\Gamma \subset \mathbb{C}$ is called a crank of degree n if it is expressed in the form

$$\Gamma = \Gamma(p_1, \dots, p_n) = \{x + iA(x; p_1, \dots, p_n); \, 0 \leq x < 1\}$$

for some n-tuple $(p_1, \dots, p_n)$ of integers larger than or equal to 2. (The class of cranks in this paper is smaller than a class defined in [**10**, Chap. III].) We shall show

THEOREM. *For any $n \geq 1$, there exists a crank Γ_n of degree n such that*

$$\frac{1}{C_1}\frac{1}{\sqrt{n}} \leq \gamma(\Gamma_n) \leq C_1\frac{1}{\sqrt{n}},$$

where C_1 is an absolute constant.

Once this theorem is established, we can deduce the exactness of the power $3/2$ as follows. Adding some segments (perpendicular to the x-axis) to Γ_n, we obtain an arc connecting 0 and 1. Then the length of this arc is less than or equal to $n+1$. Hence we can define a rectifiable graph Γ'_n so that $|\Gamma'_n| \leq 3n$, $|\operatorname{pr} E'_n| \geq 1/2$, where $E'_n = \Gamma_n \cap \Gamma'_n$. Then $\gamma(E'_n) \leq \gamma(\Gamma_n) \leq C_1/\sqrt{n}$. Contracting E'_n, Γ'_n, we define E''_n, Γ''_n so that $|\Gamma''_n| = 1$. Then

$$\begin{aligned}\gamma(E''_n) &= \gamma(E'_n)/|\Gamma'_n| \leq \sqrt{3}C_1|\Gamma'_n|^{-3/2} \\ &\leq 2^{3/2}\sqrt{3}C_1\{|\operatorname{pr} E'_n|/|\Gamma'_n|\}^{3/2} = 2^{3/2}\sqrt{3}C_1|\operatorname{pr} E''_n|^{3/2},\end{aligned}$$

which shows that the power $3/2$ cannot be replaced by any number less than $3/2$.

To prove our theorem, it is necessary to investigate cranks carefully. In §2, we shall give a formula ((1) in Proposition 1) to compute analytic capacity. Proposition 2 is a generalization of Garnett's example [**4**, p. 87], and will be used to prove our theorem. Using the method in the proof of the formula, we shall, in §3, give the proof of our theorem. In the last section, we shall give a new proof of Pommerenke's theorem **[11]** as another application of Proposition 1; our method shows how to construct the extremal functions.

2. A formula for the computation of $\gamma(\cdot)$**.** Let $L^2(\Gamma)$ denote the L^2 space of functions on a finite union Γ of smooth arcs with respect to the length element $|dz|$. The norm is denoted by $\|\cdot\|_{L^2(\Gamma)}$. The Cauchy-Hilbert transform $\mathscr{H}_\Gamma$ from $L^2(\Gamma)$ to itself is defined by

$$\mathscr{H}_\Gamma f(z) = \frac{1}{\pi}\,\mathrm{p.v.}\int_\Gamma \frac{f(w)}{w-z}|dw|,$$

where p.v. is the principal value. This is a bounded operator and the norm is denoted by $\|\mathscr{H}_\Gamma\|_{L^2(\Gamma),L^2(\Gamma)}$. An operator $\overline{\mathscr{H}}_\Gamma$ is defined by $\overline{\mathscr{H}}_\Gamma f = \overline{\mathscr{H}_\Gamma \overline{f}}$, and $\mathscr{I}_\Gamma$ is the identity operator. We show

PROPOSITION 1. *Let Γ be a finite union of smooth arcs. Then, for any $0 < \varepsilon < 1/\|\mathscr{H}_\Gamma\|_{L^2(\Gamma),L^2(\Gamma)}$,*

$$\gamma(\Gamma) = \frac{1}{\pi}\left\{|\Gamma| + \sum_{m=0}^{\infty}(1-\varepsilon^2)^m \sum_{l=0}^{\infty}\varepsilon^{2l+2}\frac{(l+1)(l+2)\cdots(l+m)}{m!}d_{2l+2}(\mathscr{H}_\Gamma)\right\}, \tag{1}$$

where

$$d_{2l}(\mathscr{H}_\Gamma) = \int_\Gamma (\mathscr{H}_\Gamma\overline{\mathscr{H}}_\Gamma)^l 1|dz| \qquad (l \geq 0,\ (\mathscr{H}_\Gamma\overline{\mathscr{H}}_\Gamma)^0 = \mathscr{I}_\Gamma)$$

and $(l+1)\cdots(l+m)/m! = 1$ if $m = 0$. (First $\sum_{l=0}^{\infty}$ is taken, and next $\sum_{m=0}^{\infty}$ is taken.) If $\|\mathscr{I}_\Gamma + \overline{\mathscr{H}}_\Gamma\mathscr{H}_\Gamma\|_{L^2(\Gamma),L^2(\Gamma)} < 2$, then

$$\gamma(\Gamma) = \frac{1}{\pi}\left\{|\Gamma| + \sum_{m=0}^{\infty}2^{-m-1}\sum_{l=0}^{\infty}\binom{m}{l}d_{2l+2}(\mathscr{H}_\Gamma)\right\}, \tag{2}$$

where $\binom{0}{0} = 1$. If $\lim_{l\to\infty} d_{2l}(\mathscr{H}_\Gamma) = 0$, then

$$\gamma(\Gamma) = \frac{1}{\pi}\sum_{l=0}^{\infty} d_{2l}(\mathscr{H}_\Gamma). \tag{3}$$

This is a version of Garabedian's theorem **[4**, p. 22] to $\mathscr{H}_\Gamma$. Equality (3) is applicable to give a new proof of Pommerenke's theorem. (See §4.) Notice that $\mathscr{I}_\mathbb{R} + \overline{\mathscr{H}}_\mathbb{R}\mathscr{H}_\mathbb{R} = 0$, where $\mathscr{H}_\mathbb{R}$ is the Hilbert transform on $\mathbb{R}$. Hence (2) is applicable to compact sets Γ on a Lipschitz graph which is a small perturbation of $\mathbb{R}$. For any $M > 0$, there exists a crank Γ' such that $d_2(\mathscr{H}_{\Gamma'}) \geq M$ [**10**, p. 84]. Then Cauchy-Schwarz' inequality yields that

$$d_{2^l}(\mathscr{H}_{\Gamma'}) \geq d_2(\mathscr{H}_{\Gamma'})^{2^{l-1}} \geq M^{2^{l-1}} \qquad (l \geq 1).$$

Hence (1) is necessary in this case.

Proof of Proposition 1. Let

$$\gamma^*(\mathscr{H}_\Gamma) = \inf\{\|1 + \mathscr{H}_\Gamma h\|^2_{L^2(\Gamma)} + \|h\|^2_{L^2(\Gamma)}\,;\, h \in L^2(\Gamma)\}. \tag{4}$$

We begin by showing that

$$\gamma(\Gamma) = \frac{1}{\pi}\gamma^*(\mathscr{H}_\Gamma). \tag{5}$$

For a compact set E bounded by a finite number of smooth Jordan curves, we have

$$\gamma(E) = \frac{1}{2\pi}\inf\left\{\int_{\partial E} |g(z)|^2\,|dz|\,;\, g(\infty) = 1\,,\ g \text{ is analytic in } E^c\right\} \tag{6}$$

[**4**, p. 22]. Hence a standard argument yields that (6) holds with E replaced by Γ; in this case, the boundary $\partial\Gamma$ has two sides. We define a smooth curve $\mathscr{L}$ tending to infinity so that $\Gamma \subset \mathscr{L}$ and that $\mathscr{L} = \mathbb{R}$ outside a large disk. Then $\mathscr{L}$ divides $\mathbb{C}$ into two domains $\Omega_\pm$. For an analytic function $g(z)$ in Γ^c such that $g(\infty) = 1$ and $\int_{\partial\Gamma} |g(z)|^2|dz| < \infty$, we can write

$$g(z) = 1 + \frac{1}{\pi}\int_\Gamma \frac{h(w)}{w - z}\,dw\,,$$

where the orientation of dw is chosen so that Ω_+ lies to the left. Let $g_\pm(z)$ be the nontangential limits of g at $z \in \Gamma$ with respect to $\Omega_\pm$, respectively. Then

$$\begin{aligned} g_+(z) &= 1 + \frac{1}{\pi}\,\mathrm{p.v.}\int_\Gamma \frac{h(w)}{w - z}\,dw + ih(z) \\ &= 1 + \mathscr{H}_\Gamma(h\psi)(z) + ih(z) \qquad (z \in \Gamma), \end{aligned}$$

where $\psi(z) = dz/|dz|$. Analogously,

$$g_-(z) = 1 + \mathscr{H}_\Gamma(h\psi)(z) - ih(z) \qquad (z \in \Gamma).$$

Thus

$$\begin{aligned} \int_{\partial\Gamma} |g(z)|^2|\,dz| &= \|g_+\|^2_{L^2(\Gamma)} + \|g_-\|^2_{L^2(\Gamma)} \\ &= \|1 + \mathscr{H}_\Gamma(h\psi) + ih\|^2_{L^2(\Gamma)} + \|1 + \mathscr{H}_\Gamma(h\psi) - ih\|^2_{L^2(\Gamma)} \\ &= 2\{\|1 + \mathscr{H}_\Gamma(h\psi)\|^2_{L^2(\Gamma)} + \|h\|^2_{L^2(\Gamma)}\} \\ &= 2\{\|1 + \mathscr{H}_\Gamma(h\psi)\|^2_{L^2(\Gamma)} + \|h\psi\|^2_{L^2(\Gamma)}\} \end{aligned}$$

because $|\psi(z)| = 1 \ \ (z \in \Gamma)$. This shows that the quantity in the right-hand side of (6) $(E = \Gamma)$ equals $\frac{1}{\pi}\gamma^*(\mathscr{H}_\Gamma)$, i.e., (5) holds.

We next compute $\gamma^*(\mathscr{H}_\Gamma)$. Fatou's lemma shows that there exists $h_\Gamma \in L^2(\Gamma)$ which attains the infimum in (4). A variational method yields that $(1 + \mathscr{H}_\Gamma h_\Gamma, \mathscr{H}_\Gamma h) + (h_\Gamma, h) = 0$ for all $h \in L^2(\Gamma)$, where $(\cdot, \cdot)$ is the (complex) inner product with respect to $|dz|$. Since the adjoint operator of $\mathscr{H}_\Gamma$ is $-\overline{\mathscr{H}}_\Gamma$, this shows that

$$(\mathscr{I}_\Gamma - \overline{\mathscr{H}}_\Gamma \mathscr{H}_\Gamma) h_\Gamma = \overline{\mathscr{H}}_\Gamma 1. \tag{7}$$

Suppose that $h'_\Gamma \in L^2(\Gamma)$ also attains the infimum in (4). Then h'_Γ satisfies (7), and hence

$$\begin{aligned} 0 &= ((\mathscr{I}_\Gamma - \overline{\mathscr{H}}_\Gamma \mathscr{H}_\Gamma)(h_\Gamma - h'_\Gamma), h_\Gamma - h'_\Gamma) \\ &= \|h_\Gamma - h'_\Gamma\|^2_{L^2(\Gamma)} + \|\mathscr{H}_\Gamma(h_\Gamma - h'_\Gamma)\|^2_{L^2(\Gamma)}. \end{aligned}$$

This shows that $h'_\Gamma = h_\Gamma$. Thus h_Γ is uniquely determined. By (7), we have

$$\begin{aligned} \gamma^*(\mathscr{H}_\Gamma) &= \|1 + \mathscr{H}_\Gamma h_\Gamma\|^2_{L^2(\Gamma)} + \|h_\Gamma\|^2_{L^2(\Gamma)} \\ &= (1 + \mathscr{H}_\Gamma h_\Gamma, 1) + ((\mathscr{I}_\Gamma - \overline{\mathscr{H}}_\Gamma \mathscr{H}_\Gamma) h_\Gamma - \overline{\mathscr{H}}_\Gamma 1, h_\Gamma) \\ &= \int_\Gamma \{1 + \mathscr{H}_\Gamma h_\Gamma\} |dz|. \end{aligned} \tag{8}$$

Let

$$T_\Gamma = (\mathscr{I}_\Gamma - \varepsilon^2 \overline{\mathscr{H}}_\Gamma \mathscr{H}_\Gamma)^{-1}.$$

Then we can write

$$T_\Gamma = \sum_{l=0}^{\infty} \varepsilon^{2l} (\overline{\mathscr{H}}_\Gamma \mathscr{H}_\Gamma)^l$$

because $0 < \varepsilon < 1/\|\mathscr{H}_\Gamma\|_{L^2(\Gamma), L^2(\Gamma)}$. We have, for any $h \in L^2(\Gamma)$,

$$\begin{aligned} \|T_\Gamma h\|^2_{L^2(\Gamma)} &\le \|T_\Gamma h\|^2_{L^2(\Gamma)} + \varepsilon^2 \|\mathscr{H}_\Gamma T_\Gamma h\|^2_{L^2(\Gamma)} \\ &= ((\mathscr{I}_\Gamma - \varepsilon^2 \overline{\mathscr{H}}_\Gamma \mathscr{H}_\Gamma) T_\Gamma h, T_\Gamma h) = (h, T_\Gamma h) \le \|h\|_{L^2(\Gamma)} \|T_\Gamma h\|_{L^2(\Gamma)}, \end{aligned}$$

which shows that $\|T_\Gamma\|_{L^2(\Gamma), L^2(\Gamma)} \le 1$. Equality (7) can be rewritten as

$$(\mathscr{I}_\Gamma - \varepsilon^2 \overline{\mathscr{H}}_\Gamma \mathscr{H}_\Gamma) h_\Gamma = (1 - \varepsilon^2) h_\Gamma + \varepsilon^2 \overline{\mathscr{H}}_\Gamma 1. \tag{9}$$

Observing this equality, we inductively define $(h_m)_{m=0}^\infty$ by $h_0 = 0$,

$$h_m = T_\Gamma\{(1 - \varepsilon^2) h_{m-1} + \varepsilon^2 \overline{\mathscr{H}}_\Gamma 1\} \qquad (m \ge 1).$$

Then

$$\|h_{m+1}-h_m\|_{L^2(\Gamma)} = (1-\varepsilon^2)\|T_\Gamma(h_m-h_{m-1})\|_{L^2(\Gamma)} \\ \leq (1-\varepsilon^2)\|h_m-h_{m-1}\|_{L^2(\Gamma)}.$$

Hence $\lim_{m\to\infty} h_m$ exists and satisfies (9), i.e., (7). Thus $h_\Gamma = \lim_{m\to\infty} h_m$. Since

$$h_{m+1}-h_m = (1-\varepsilon^2)T_\Gamma(h_m-h_{m-1}) = \cdots = (1-\varepsilon^2)^m T_\Gamma^m h_1 \\ = \varepsilon^2(1-\varepsilon^2)^m T_\Gamma^{m+1}\overline{\mathscr{H}}_\Gamma 1,$$

we have

$$h_\Gamma = \sum_{m=0}^{\infty}(h_{m+1}-h_m) = \varepsilon^2\sum_{m=0}^{\infty}(1-\varepsilon^2)^m T_\Gamma^{m+1}\overline{\mathscr{H}}_\Gamma 1.$$

Consequently, (8) yields that

$$\begin{aligned}\gamma^*(\mathscr{H}_\Gamma) &= |\Gamma| + \sum_{m=0}^{\infty}(1-\varepsilon^2)^m\int_\Gamma \varepsilon^2\mathscr{H}_\Gamma T_\Gamma^{m+1}\overline{\mathscr{H}}_\Gamma 1|dz| \\ &= |\Gamma| + \sum_{m=0}^{\infty}(1-\varepsilon^2)^m\int_\Gamma \varepsilon^2\mathscr{H}_\Gamma\left\{\sum_{l=0}^{\infty}\varepsilon^{2l}(\overline{\mathscr{H}}_\Gamma\mathscr{H}_\Gamma)^l\right\}^{m+1}\overline{\mathscr{H}}_\Gamma 1|dz| \\ &= |\Gamma| + \sum_{m=0}^{\infty}(1-\varepsilon^2)^m\sum_{l=0}^{\infty}\varepsilon^{2l+2}\frac{(l+1)\cdots(l+m)}{m!}\int_\Gamma(\mathscr{H}_\Gamma\overline{\mathscr{H}}_\Gamma)^{l+1}1|dz| \\ &= |\Gamma| + \sum_{m=0}^{\infty}(1-\varepsilon^2)^m\sum_{l=0}^{\infty}\varepsilon^{2l+2}\frac{(l+1)\cdots(l+m)}{m!}d_{2l+2}(\mathscr{H}_\Gamma).\end{aligned}$$

Using (5), we obtain (1).

We can write

$$\mathscr{I}_\Gamma - \overline{\mathscr{H}}_\Gamma\mathscr{H}_\Gamma = 2\{\mathscr{I}_\Gamma - \tfrac{1}{2}(\mathscr{I}_\Gamma + \overline{\mathscr{H}}_\Gamma\mathscr{H}_\Gamma)\}.$$

Hence, if $\|\mathscr{I}_\Gamma + \overline{\mathscr{H}}_\Gamma\mathscr{H}_\Gamma\|_{L^2(\Gamma),L^2(\Gamma)} < 2$, then

$$h_\Gamma = \sum_{m=0}^{\infty} 2^{-m}(\mathscr{I}_\Gamma + \overline{\mathscr{H}}_\Gamma\mathscr{H}_\Gamma)^m(\tfrac{1}{2}\overline{\mathscr{H}}_\Gamma)1.$$

Thus (5) and (8) yield (2).

Equality (7) shows that $\mathscr{H}_\Gamma h_\Gamma = \mathscr{H}_\Gamma\overline{\mathscr{H}}_\Gamma 1 + \mathscr{H}_\Gamma\overline{\mathscr{H}}_\Gamma\mathscr{H}_\Gamma h_\Gamma$, and hence, by (8),

$$\gamma^*(\mathscr{H}_\Gamma) = \int_\Gamma\{1 + \mathscr{H}_\Gamma\overline{\mathscr{H}}_\Gamma 1 + \mathscr{H}_\Gamma\overline{\mathscr{H}}_\Gamma\mathscr{H}_\Gamma h_\Gamma\}|dz|.$$

Repeating this argument, we have

$$\begin{aligned}\gamma^*(\mathscr{H}_\Gamma) &= \int_\Gamma \left\{ \sum_{l=0}^{L} (\mathscr{H}_\Gamma \overline{\mathscr{H}}_\Gamma)^l 1 + \mathscr{H}_\Gamma (\overline{\mathscr{H}}_\Gamma \mathscr{H}_\Gamma)^L h_\Gamma \right\} |dz| \\ &= \sum_{l=0}^{L} d_{2l}(\mathscr{H}_\Gamma) - \int_\Gamma \{(\mathscr{H}_\Gamma \overline{\mathscr{H}}_\Gamma)^L \mathscr{H}_\Gamma\} 1(z) h_\Gamma(z) |dz|.\end{aligned}$$

If $\lim_{L\to\infty} d_{2L}(\mathscr{H}_\Gamma) = 0$, then

$$\begin{aligned}&\lim_{L\to\infty} \left| \int_\Gamma \{(\mathscr{H}_\Gamma \overline{\mathscr{H}}_\Gamma)^L \mathscr{H}_\Gamma\} 1(z) h_\Gamma(z) |dz| \right| \\ &\quad \le \lim_{L\to\infty} \|(\mathscr{H}_\Gamma \overline{\mathscr{H}}_\Gamma)^L \mathscr{H}_\Gamma 1\|_{L^2(\Gamma)} \|h_\Gamma\|_{L^2(\Gamma)} \\ &\quad = \lim_{L\to\infty} d_{4L+2}(\mathscr{H}_\Gamma)^{1/2} \|h_\Gamma\|_{L^2(\Gamma)} = 0.\end{aligned}$$

Hence (5) gives (3). This completes the proof of our proposition.

We now give a remark. There exists an analytic function $g_\Gamma(z)$ in Γ^c such that $g_\Gamma(\infty) = 1$ and $\gamma(\Gamma) = (1/2\pi)\int_{\partial\Gamma} |g_\Gamma(z)|\,|dz|$ [**4**, p. 19]. This is called the Garabedian function of Γ. Equality (5) shows that

$$g_\Gamma(z) = \left\{ 1 + \frac{1}{\pi} \int_\Gamma \frac{h_\Gamma(w)}{w - z} |dw| \right\}^2 .$$

There exists $f_\Gamma \in H^\infty(\Gamma^c)$ such that $\|f_\Gamma\|_{H^\infty} = 1$ and $f_\Gamma'(\infty) = \gamma(\Gamma)$ [**4**, p. 18]. This is called the Ahlfors function of Γ. We have

$$f_\Gamma(z) = \frac{-\dfrac{1}{\pi}\left\{ \displaystyle\int_\Gamma \frac{|dw|}{w - z} + \int_\Gamma \frac{\overline{\mathscr{H}}_\Gamma \overline{h}_\Gamma(w)}{w - z} |dw| \right\}}{\left\{ 1 + \dfrac{1}{\pi} \displaystyle\int_\Gamma \frac{h_\Gamma(w)}{w - z} |dw| \right\}} \tag{10}$$

To see this, let $f(z)$ denote the function in the right-hand side. Since $g_\Gamma(z)$ does not take 0 in Γ^c, $f(z)$ is analytic in Γ^c [**4**, p. 21]. We have $f'(\infty) = \frac{1}{\pi}\gamma^*(\mathscr{H}_\Gamma) = \gamma(\Gamma)$ and

$$f_\pm(z) = -\frac{\mathscr{H}_\Gamma 1(z) \pm i\tilde{\psi}(z) + \mathscr{H}_\Gamma \overline{\mathscr{H}}_\Gamma \overline{h}_\Gamma(z) \pm i\overline{\mathscr{H}}_\Gamma \overline{h}_\Gamma(z) \tilde{\psi}(z)}{1 + \mathscr{H}_\Gamma h_\Gamma(z) \pm i h_\Gamma(z) \tilde{\psi}(z)},$$

where $\tilde{\psi}(z) = |dz|/dz$ and $f_\pm(z)$ are the nontangential limits of f at $z \in \Gamma$ with respect to $\Omega_\pm$, respectively. Equality (7) shows that

$$\begin{aligned}&\mathscr{H}_\Gamma 1 + i\tilde{\psi} + \mathscr{H}_\Gamma \overline{\mathscr{H}}_\Gamma \overline{h}_\Gamma + i(\overline{\mathscr{H}}_\Gamma \overline{h}_\Gamma)\tilde{\psi} \\ &\quad = \mathscr{H}_\Gamma 1 + i\tilde{\psi} + (\overline{h}_\Gamma - \mathscr{H}_\Gamma 1) + i(\overline{\mathscr{H}}_\Gamma \overline{h}_\Gamma)\tilde{\psi} \\ &\quad = i\tilde{\psi} + i(\overline{\mathscr{H}}_\Gamma \overline{h}_\Gamma)\tilde{\psi} + \overline{h}_\Gamma = i\tilde{\psi}\{\overline{1 + \mathscr{H}_\Gamma h_\Gamma + i h_\Gamma \tilde{\psi}}\},\end{aligned}$$

which yields that $|f_+(z)| = 1$ on Γ. Analogously, $|f_-(z)| = 1$ on Γ. Thus $\|f\|_{H^\infty} = 1$. This shows that $f = f_\Gamma$.

For the proof of our theorem, we note

PROPOSITION 2. *Let $0 < \delta_0 < 1$ and let $(q_n)_{n=1}^\infty$ be a sequence of integers larger than or equal to 2 such that*

$$\sum_{n=j}^{\infty} (q_j \cdots q_n)^{-1} \le \delta_0 \qquad (j \ge 1).$$

Then

$$\lim_{n\to\infty} \sup \gamma(\Gamma(p_1, \dots, p_n)) = 0,$$

where the supremum is taken over all n-tuples $(p_1, \cdots, p_n)$ satisfying $p_j \ge q_j$ $(1 \le j \le n)$.

This is a generalization of Garnett's example [**4**, p. 87], and used later. Notice that $\sum_{n=1}^\infty 2^{-n} = 1$. A sequence $(\Gamma(\mathbf{2}_n))_{n=1}^\infty$ ($\mathbf{2}_n$ is the n-tuple of 2) topologically converges to a segment $\{x + ix;\ 0 \le x < 1\}$, and these cranks behave like cranks of degree 1 with respect to this segment. Hence we have $\limsup_{n\to\infty} \gamma(\Gamma(\mathbf{2}_n)) > 0$. This shows that our proposition is sharp in a sense. Since a minor change of the argument in [**10**, p. 81] yields the required equality, we omit the proof (cf. Jones [**5**]).

3. Proof of Theorem. In this section, we give the proof of our theorem. Let L^q denote the L^q space of functions on $[0, 1)$ with respect to the 1-dimension Lebesgue measure $|\cdot|$ $(1 \le q < \infty)$. For a kernel $K = K(x, y)$ on $[0, 1)\times[0, 1)$, we simply write by the same notation K an operator defined by this kernel, and write by $\overline{K}$ an operator defined by $\overline{K(x, y)}$; $\|K\|_{L^q, L^{q'}}$ denotes the norm of K as an operator from L^q to $L^{q'}$. The identity operator is denoted by Id. A kernel K is anti-symmetric if $K(x, y) = -K(y, x)$ $(x \ne y)$. A kernel K is of type 0 if

$$\sup_{x, y\in[0, 1)} \left\{|K(x, y)| + \left|\frac{\partial}{\partial x}K(x, y)\right| + \left|\frac{\partial}{\partial y}K(x, y)\right|\right\} < \infty.$$

A kernel K is of type 1 if $\|K\|_{L^4, L^4} < \infty$ and if there exists a sequence $(K_j)_{j=1}^\infty$ of kernels of type 0 such that

$$\lim_{j\to\infty} \|K_j - K\|_{L^4, L^2} = 0, \quad \sup_{j\ge1} \|K_j\|_{L^4, L^4} < \infty.$$

Kernels used in this section are bounded as operators from L^q to itself for all $1 < q < \infty$. Let

$$\gamma^*(K) = \inf\{\|1 + Kh\|_{L^2}^2 + \|h\|_{L^2}^2 \, ; \, h \in L^2\},$$
$$d_{2l}(K) = \int_0^1 (K\overline{K})^l 1 \, dx \qquad (l \geq 0, \ (K\overline{K})^0 = \mathrm{Id}).$$

Recall the function $A(x; p_1, \dots, p_n)$ in the introduction. Let

$$H(x, y) = \mathscr{H}_{\mathbb{R}}(x, y) = \frac{1}{\pi}\frac{1}{y - x},$$

$$H[p_1, \dots, p_n](x, y) = \frac{1}{\pi}\frac{1}{(y - x) + i(A(y; p_1, \dots, p_n) - A(x; p_1, \dots, p_n))},$$

$$\Delta[p_1, \dots, p_n] = H[p_1, \dots, p_n] - H[p_1, \dots, p_{n-1}] \qquad (n \geq 1),$$

where $H[p_1, \dots, p_{n-1}] = H$ if $n = 1$. Then

$$H[p_1, \dots, p_n] = H + \sum_{j=1}^n \Delta[p_1, \dots, p_j].$$

Since all components/segments of $\Gamma(p_1, \cdots, p_n)$ are parallel to the x-axis, we can identify $\mathscr{H}_{\Gamma(p_1, \dots, p_n)}$, $L^2(\Gamma(p_1, \dots, p_n))$ with $H[p_1, \dots, p_n]$, L^2, respectively. We have $\|H[p_1, \dots, p_n]\|_{L^2, L^2} \leq C_2\sqrt{n}$ for some absolute constant C_2 [**10**, p. 84]. Hence Proposition 1 shows that

$$\begin{aligned} \text{(11)} \quad \gamma(\Gamma(p_1, \dots, p_n)) &= \frac{1}{\pi}\gamma^*(H[p_1, \dots, p_n]) \\ &= \frac{1}{\pi}\Bigg\{1 + \sum_{m=0}^\infty (1 - \varepsilon_n^2)^m \\ &\qquad \times \sum_{l=0}^\infty \varepsilon_n^{2l+2}\frac{(l+1)\cdots(l+m)}{m!} d_{2l+2}(H[p_1, \dots, p_n])\Bigg\}, \end{aligned}$$

where $\varepsilon_n = (2C_2\sqrt{n})^{-1}$. We shall inductively estimate

$$\lim_{p_1 \to \infty} \cdots \lim_{p_n \to \infty} \gamma^*(H[p_1, \dots, p_n]),$$

where $\lim_{p_n \to \infty}$ is taken first and $\lim_{p_1 \to \infty}$ is taken last. For $E \subset \mathbb{R}$, χ_E denotes its characteristic function, and, for $x \in \mathbb{R}$, $\imath(x)$ denotes its integral part. Here are some lemmas necessary for the estimate.

LEMMA 3. *For two kernels K and K',*

$$\gamma^*(K+K') \le 2(1+\|K'\|^2_{L^2,L^2})\gamma^*(K).$$

Proof. We have, for any $h \in L^2$,

$$\|1+(K+K')h\|^2_{L^2} + \|h\|^2_{L^2} \le 2(1+\|K'\|^2_{L^2,L^2})\{\|1+Kh\|^2_{L^2} + \|h\|^2_{L^2}\},$$

which yields the required inequality.

LEMMA 4. *Let K be an anti-symmetric kernel such that*

$$\lim_{l\to\infty} d_{2l}(K) = 0.$$

Then

$$\gamma^*(K) = \sum_{l=0}^{\infty} d_{2l}(K). \tag{12}$$

Since this is a version of (3) to K, we omit the proof.

LEMMA 5. *For an anti-symmetric kernel K, $0 < \varepsilon_0 \le (3\|K\|_{L^2,L^2})^{-1}$ and $w \in U = \{\zeta \in \mathbb{C}\,;\, |\zeta| < 2,\, |\arg\zeta| < \pi/4\}$,*

$$1 + \sum_{m=0}^{\infty}(1-\varepsilon_0^2)^m \sum_{l=0}^{\infty} w^{2l+2}\varepsilon_0^{2l+2}\frac{(l+1)\cdots(l+m)}{m!}d_{2l+2}(K) \tag{13}$$

$$(= \gamma^*(w\,;K),\ \textit{say})$$

exists and $\gamma^*(w\,;K)$ is analytic in U.

Proof. Let

$$T(w\,;K) = (\mathrm{Id} - w^2\varepsilon_0^2\overline{K}K)^{-1}.$$

Then

$$w^2\varepsilon_0^2\int_0^1 KT(w\,;K)^{m+1}\overline{K}1\,dx$$
$$= \sum_{l=0}^{\infty} w^{2l+2}\varepsilon_0^{2l+2}\frac{(l+1)\cdots(l+m)}{m!}d_{2l+2}(K)$$

because $2\varepsilon_0\|K\|_{L^2,L^2} < 1$. Evidently, this is analytic in U. Since K is anti-symmetric and $\mathrm{Re}\,w^2 > 0$ ($w \in U$, $\mathrm{Re}\,w^2$ is the real part of w^2), we have, in the same manner as in the proof of (1),

$$\|T(w\,;K)\|_{L^2,L^2} \le 1 \qquad (w \in U). \tag{14}$$

Thus the convergence of $\sum_{m=0}^{\infty}$ in (13) is uniform in U, which shows that $\gamma^*(w\,;K)$ exists and is analytic in U.

LEMMA 6. *For any* $l \geq 0$,

$$\lim_{p\to\infty} d_{2l}(\Delta[p]) \quad (= d_{2l}(\Delta[\infty]),\ \textit{say})$$

exists and

$$d_2(\Delta[\infty]) \leq -\frac{1}{25\pi^2}. \tag{15}$$

Proof. We put

$$\begin{aligned} R(s\,,t) &= \frac{1}{\pi}\left\{\frac{1}{t-s+1+i} - \frac{1}{t-s+1}\right\} \\ &\quad - \frac{2}{\pi}\sum_{m=1}^{\infty}\left\{\frac{t-s+1+i}{4m^2-(t-s+1+i)^2} - \frac{t-s+1}{4m^2-(t-s+1)^2}\right\} \end{aligned}$$

and show that

$$\lim_{p\to\infty} d_{2l}(\Delta[p]) = \frac{1}{2}\int_0^1 \{R^{2l}1 + \overline{R}^{2l}1\}\,ds. \tag{16}$$

Let

$$W_p = \bigcup_{m\,,\mathrm{odd}}\left[\frac{m}{p}\,,\frac{m+1}{p}\right),\quad W_p' = \bigcup_{m\,,\mathrm{even}}\left[\frac{m}{p}\,,\frac{m+1}{p}\right),$$

$$X_p = \bigcup_{m=\imath(\log p)}^{p-\imath(\log p)-1}\left[\frac{m}{p}\,,\frac{m+1}{p}\right),$$

$$s_x = px - \imath(px) \qquad (0 \leq x < 1\,,\ p \geq 2).$$

Notice that $|[0\,,1) - X_p| \leq 2\ \imath(\log p)/p$ and $\|\Delta[p]\|_{L^4\,,L^4} \leq 10$. Since $\Delta[p](x\,,y) = 0\ \ (x\,,y \in W_p\,;x\,,y\in W_p')$, we have

$$\begin{aligned} d_{2l}(\Delta[p]) &= \int_0^1 (\Delta[p]\overline{\Delta[p]})^{l-1}\Delta[p]\{\chi_{W_p}\overline{\Delta[p]}\chi_{W_p'} + \chi_{W_p'}\overline{\Delta[p]}\chi_{W_p}\}\,dx \\ &= \int_0^1 (\Delta[p]\overline{\Delta[p]})^{l-1}\Delta[p]\{\chi_{W_p\cap X_p}\overline{\Delta[p]}\chi_{W_p'} + \chi_{W_p'\cap X_p}\overline{\Delta[p]}\chi_{W_p}\}\,dx \\ &\quad + O\left(\left(\frac{\log p}{p}\right)^{1/4}\right). \end{aligned}$$

We now study $\overline{\Delta[p]}\chi_{W_p'}(x)\ \ (x \in W_p\cap X_p)$. Without loss of generality, we may assume that p is even. Since $x \in W_p \cap X_p$, $\imath(px)$ is even and

$\imath(\log p) \le \imath(px) \le p - \imath(\log p) - 1$. We may assume that $\imath(\log p) \le \imath(px) \le p/2$. We have

$$\begin{aligned}
\overline{\Delta[p]}\chi_{W'_p}(x) &= \frac{1}{\pi}\int_{W'_p}\left\{\frac{1}{y-x-i/p} - \frac{1}{y-x}\right\}dy \\
&= \frac{1}{\pi}\int_{W_p}\left\{\frac{1}{y-x+1/p-i/p} - \frac{1}{y-x+1/p}\right\}dy \\
&= \frac{1}{\pi}\sum_{m=0}^{(p/2)-1}\int_0^{1/p}\left\{\frac{1}{(2m/p+y)-(\imath(px)/p+x-\imath(px)/p)+1/p-i/p}\right. \\
&\qquad \left. - \frac{1}{(2m/p+y)-(\imath(px)/p+x-\imath(px)/p)+1/p}\right\}dy \\
&= \frac{1}{\pi}\sum_{m=0}^{\imath(px)} + \frac{1}{\pi}\sum_{m=\imath(px)+1}^{(p/2)-1} = L_1 + L_2,
\end{aligned}$$

$$\begin{aligned}
L_1 &= \frac{1}{\pi}\int_0^1\left\{\frac{1}{t-s_x+1-i} - \frac{1}{t-s_x+1}\right\}dt + \frac{1}{\pi}\sum_{0\le m\le \imath(px),\, m\ne \imath(px)/2} \\
&= \frac{1}{\pi}\int_0^1\left\{\frac{1}{t-s_x+1-i} - \frac{1}{t-s_x+1}\right\}dt \\
&\quad - \frac{2}{\pi}\sum_{m=1}^{\imath(px)/2}\int_0^1\left\{\frac{t-s_x+1-i}{4m^2-(t-s_x+1-i)^2}\right. \\
&\qquad \left. - \frac{t-s_x+1}{4m^2-(t-s_x+1)^2}\right\}dt \\
&= \overline{R}1(s_x) + O\left(\frac{1}{\log p}\right), \\
L_2 &= -\frac{i}{\pi}\sum_{m=\imath(px)+1}^{(p/2)-1}\int_0^1 \frac{1}{(2m-\imath(px))+(t-s_x+1-i)} \\
&\quad \times \frac{dt}{(2m-\imath(px))+(t-s_x+1)} = O\left(\frac{1}{\log p}\right),
\end{aligned}$$

which shows that $\overline{\Delta[p]}\chi_{W'_p}(x) = \overline{R}1(s_x) + O(1/\log p)\ \ (x \in W_p \cap X_p)$.

In the same manner, $\overline{\Delta[p]}\chi_{W_p}(x) = R1(s_x) + O(1/\log p)$ $(x \in W_p' \cap X_p)$. Thus

$$\begin{aligned} d_{2l}(\Delta[p]) &= \int_0^1 (\Delta[p]\overline{\Delta[p]})^{l-1}\Delta[p]\{\chi_{W_p\cap X_p}\overline{R}1(s_.) + \chi_{W_p'\cap X_p}R1(s_.)\}\,dx \\ &\quad + O\left(\frac{1}{\log p}\right) \\ &= \int_0^1 (\Delta[p]\overline{\Delta[p]})^{l-1}\Delta[p]\{\chi_{W_p}\overline{R}1(s_.) + \chi_{W_p'}R1(s_.)\}\,dx \\ &\quad + O\left(\frac{1}{\log p}\right) \\ &= \int_0^1 (\Delta[p]\overline{\Delta[p]})^{l-1}\{\chi_{W_p\cap X_p}\Delta[p](\chi_{W_p'}R1(s_.)) \\ &\qquad\qquad + \chi_{W_p'\cap X_p}\Delta[p](\chi_{W_p}\overline{R}1(s_.))\}\,dx \\ &\quad + O\left(\frac{1}{\log p}\right). \end{aligned}$$

Since $R1(s_x)$ is a periodic function with period $1/p$, we have, in the same manner as above,

$$\Delta[p](\chi_{W_p'}R1(s_.))(x) = R^2 1(s_x) + O\left(\frac{1}{\log p}\right) \qquad (x \in W_p \cap X_p),$$

$$\Delta[p](\chi_{W_p}\overline{R}1(s_.))(x) = \overline{R}^2 1(s_x) + O\left(\frac{1}{\log p}\right) \qquad (x \in W_p' \cap X_p).$$

Repeating this argument, we have

$$\begin{aligned} d_{2l}(\Delta[p]) &= \int_0^1 \{\chi_{W_p}(x)R^{2l}1(s_x) + \chi_{W_p'}(x)\overline{R}^{2l}1(s_x)\}\,dx + O\left(\frac{1}{\log p}\right) \\ &= \frac{1}{2}\int_0^1 \{R^{2l}1 + \overline{R}^{2l}1\}\,ds + O\left(\frac{1}{\log p}\right), \end{aligned}$$

which gives (16).

We have

$$\begin{aligned} R(s, t) &= \frac{1}{\pi}\sum_{m=-\infty}^{\infty}\left\{\frac{1}{2m+1+t-s+i} - \frac{1}{2m+1+t-s}\right\} \\ &= -\frac{1}{\pi}\sum_{m=-\infty}^{\infty}\frac{1}{(2m+1+t-s)\{(2m+1+t-s)^2+1\}} \\ &\quad - \frac{i}{\pi}\sum_{m=-\infty}^{\infty}\frac{1}{(2m+1+t-s)^2+1} \\ &= -R'(s, t) - iR''(s, t), \text{ say.} \end{aligned}$$

Then R' is anti-symmetric and R'' is symmetric, i.e., $R''(s, t) = R''(t, s)$. Thus

$$\begin{aligned} d_2(\Delta[\infty]) &= \mathrm{Re} \int_0^1 R^2 1\, ds \\ &= \mathrm{Re} \int_0^1 (-R'1 + iR''1)(R'1 + iR''1)\, ds \\ &= -\int_0^1 \{(R'1)^2 + (R''1)^2\}\, ds \\ &\le -\frac{1}{\pi^2} \int_0^1 \left\{ \int_0^1 \frac{dt}{(1+t-s)^2+1} \right\}^2 ds \le -\frac{1}{25\pi^2}. \end{aligned}$$

Thus (15) holds.

Lemma 7. *Let K be an anti-symmetric kernel of type 1, and let $(g_p)_{p=2}^{\infty}$, $(h_p)_{p=2}^{\infty}$ be two sequences in L^4 such that $\|g_p\|_{L^4} \le 1$, $\|h_p\|_{L^4} \le 1$. Then, for any $l \ge 0$,*

$$(17) \quad \lim_{p\to\infty} \left\{ \int_0^1 Kg_p \cdot (\Delta[p]\overline{\Delta[p]})^l \overline{K} h_p\, dx \right. \\ \left. -d_{2l}(\Delta[\infty]) \int_0^1 Kg_p \cdot \overline{K} h_p\, dx \right\} = 0,$$

$$(18) \quad \lim_{p\to\infty} \int_0^1 Kg_p \cdot \overline{\Delta[p]}(\Delta[p]\overline{\Delta[p]})^l K h_p\, dx = 0.$$

Equalities (17) *and* (18) *hold with Kg_p replaced by* 1.

Proof. First we assume that K is of type 0. Let

$$\Delta'[p](x, y) = \Delta[p](x, y)\chi_{[0, N/p)}(|y-x|) \qquad (p \ge 2),$$

where $N = \imath(\log p)$. Then $\|\Delta[p] - \Delta'[p]\|_{L^2, L^2} = O(1/\log p)$ (cf. Lemma 6), and hence

$$\int_0^1 Kg_p \cdot (\Delta[p]\overline{\Delta[p]})^l \overline{K} h_p\, dx \\ = \int_0^1 Kg_p \cdot (\Delta'[p]\overline{\Delta'[p]})^l \overline{K} h_p\, dx + O\left(\frac{1}{\log p}\right).$$

Notice that

$$(\Delta'[p]\overline{\Delta'[p]})^l(x, y) = 0 \qquad (|y-x| > 2lN/p),$$

and that $(\Delta'[p]\overline{\Delta'[p]})^l 1$ is a periodic function on $[2lN/p, 1-(2lN/p))$ with period $2/p$. Let

$$\eta_p^{(m)} = \frac{p}{2}\int_{2m/p}^{2(m+1)/p} (\Delta'[p]\overline{\Delta'[p]})^l 1\, dx - d_{2l}(\Delta[\infty]) \qquad (0 \le m \le \imath(p/2) - 1).$$

Then $\eta_p^{(m)} = \eta_p^{(lN)}$ $(lN \le m \le \imath(p/2) - lN - 1)$. Lemma 6 shows that

$$\begin{aligned}\limsup_{p\to\infty} |\eta_p^{(lN)}| &= \limsup_{p\to\infty} |d_{2l}(\Delta'[p]) - d_{2l}(\Delta[\infty])| \\ &= \limsup_{p\to\infty} |d_{2l}(\Delta[p]) - d_{2l}(\Delta[\infty])| = 0.\end{aligned}$$

Since K is of type 0, we have

$$\sup \frac{|Kh(y) - Kh(x)|}{|y-x|} \le \sup_{s,t\in[0,1)} \left|\frac{\partial}{\partial s} K(s, t)\right| < \infty,$$

where the supremum in the left-hand side is taken over all $x, y \in [0, 1)$ and all $h \in L^4$ satisfying $\|h\|_{L^4} \le 1$. Thus

$$\begin{aligned}&\int_0^1 Kg_p \cdot (\Delta[p]\overline{\Delta[p]})^l \overline{K} h_p\, dx \\ &= \int_{2lN/p}^{1-(2lN/p)} Kg_p(x) \\ &\quad \times \int_{|y-x|\le 2lN/p} (\Delta'[p]\overline{\Delta'[p]})^l(x, y)\overline{K}h_p(y)\, dy\, dx + o(1) \\ &= \int_{2lN/p}^{1-(2lN/p)} Kg_p \cdot \overline{K}h_p \cdot (\Delta'[p]\overline{\Delta'[p]})^l 1\, dx + o(1) \\ &= \frac{2}{p}\sum_{m=lN}^{\imath(p/2)-lN-1} Kg_p\left(\frac{2m}{p}\right)\overline{K}h_p\left(\frac{2m}{p}\right)\{\eta_p^{(m)} + d_{2l}(\Delta[\infty])\} + o(1) \\ &= d_{2l}(\Delta[\infty])\int_0^1 Kg_p \cdot \overline{K}h_p\, dx + O(\eta_p^{(lN)}) + o(1) \\ &= d_{2l}(\Delta[\infty])\int_0^1 Kg_p \cdot \overline{K}h_p\, dx + o(1),\end{aligned}$$

which shows that (17) holds. Let K be of type 1. Then there exists a sequence $(K_j)_{j=1}^{\infty}$ of kernels of type 0 such that

$$\|K - K_j\|_{L^4, L^2} \le \frac{1}{j} \qquad (j \ge 1), \quad \sup_{j\ge 1} \|K_j\|_{L^4, L^4} < \infty.$$

Then

$$\left| \int_0^1 K g_p \cdot (\Delta[p]\overline{\Delta[p]})^l \overline{K} h_p \, dx - \int_0^1 K_j g_p \cdot (\Delta[p]\overline{\Delta[p]})^l \overline{K}_j h_p \, dx \right| \le C_3/j,$$

$$\left| \int_0^1 K g_p \cdot \overline{K} h_p \, dx - \int_0^1 K_j g_p \cdot \overline{K}_j h_p \, dx \right| \le C_3/j \qquad (j \ge 1)$$

for some constant C_3 independent of p and j. Since (17) holds for all K_j $(j \ge 1)$, this shows that (17) holds.

Since $\overline{\Delta[p]}(\Delta[p]\overline{\Delta[p]})^l$ is anti-symmetric, we have

$$\int_0^1 \overline{\Delta[p]}(\Delta[p]\overline{\Delta[p]})^l 1 \, dx = 0.$$

Hence, in the same manner as above, we obtain (18). Analogously, we can replace Kg_p by 1.

LEMMA 8. *Let K be an anti-symmetric kernel of type* 1. *Then, for any $l \ge 0$,*

$$\lim_{p\to\infty} d_{2l}(\Delta[p] + K) \quad (= d_{2l}(\Delta[\infty] + K), \ \textit{say})$$

exists, and we can write

$$d_{2l}(\Delta[\infty] + K) = \sum_{k=0}^{l} c_{2k}^{(2l)} d_{2l-2k}(K) \tag{19}$$

so that $c_{2k}^{(m)}$ $(0 \le k \le \iota(m/2),\ m \ge 0)$ satisfy

$$c_0^{(m)} = 1, \quad c_{2k}^{(2k)} = d_{2k}(\Delta[\infty]) \qquad (m \ge 0,\ k \ge 0), \tag{20}$$

$$c_{2k}^{(m)} = \sum_{j=0}^{k} c_{2k-2j}^{(m-2j-1)} d_{2j}(\Delta[\infty]) \qquad \left(0 \le k \le \iota\left(\frac{m-1}{2}\right),\ m \ge 0\right). \tag{21}$$

Proof. We say that a $2l$-tuple $(\tau_1, \dots, \tau_{2l})$, $\tau_j = \pm 1$ is negligible if there exist two integers j_0, j_0' $(1 \le j_0 < j_0' \le 2l)$ such that $j_0' - j_0 - 1$ is odd, $\tau_j = -1$ $(j_0 \le j \le j_0')$ and $\tau_{j_0-1} = \tau_{j_0'+1} = 1$. (We put $\tau_0 = \tau_{2l+1} = 1$. Hence $\tau_{j_0-1} = 1$ if $j_0 = 1$, and $\tau_{j_0'+1} = 1$ if $j_0' = 2l$.) Let $\tau(\Delta[p]) = -1$ $(p \ge 2)$, $\tau(K) = 1$. Lemmas 6 and 7 show that $d_{2l}(\Delta[\infty] + K)$ exists and

$$\begin{aligned} d_{2l}(\Delta[\infty] + K) &= \lim_{p\to\infty} \sum_{(K_1, \dots, K_{2l}),\, K_j = \Delta[p],\, K} \int_0^1 K_1 \overline{K}_2 \cdots K_{2l-1} \overline{K}_{2l} 1\, dx \\ &= \lim_{p\to\infty} \sum_{(p)} \int_0^1 K_1 \overline{K}_2 \cdots K_{2l-1} \overline{K}_{2l} 1\, dx, \end{aligned}$$

where $\sum_{(p)}$ is the summation over all $2l$-tuples $(K_1, \dots, K_{2l})$, $K_j = \Delta[p], K$ such that $(\tau(K_1), \dots, \tau(K_{2l}))$ is not negligible. If $(\tau(K_1), \dots, \tau(K_{2l}))$ is not negligible, then K appears even times in $(K_1, \dots, K_{2l})$. We can choose $j_1 < j_2 < \cdots < j_{2\nu}$ so that $K_{j_\mu} = K$ $(1 \le \mu \le 2\nu)$, $K_j = \Delta[p]$ $(j \notin \{j_\mu\}_{\mu=1}^{2\nu})$. Then $j_1 - 1$, $j_{\mu+1} - j_\mu - 1$ $(1 \le \mu \le 2\nu - 1)$, $2l - j_{2\nu}$ are even. Notice that

$$d_{2j}(K) = \int_0^1 (\overline{K} K)^j 1\, dx \quad (j \ge 0).$$

Thus we can write

$$d_{2l}(\Delta[\infty] + K) = \sum_{k=0}^{l} c_{2k}^{(2l)} d_{2l-2k}(K).$$

Let κ_0 be an operator defined by $h \in L^2 \to (\int_0^1 h\, dx) \chi_{[0,1)}$. We put $Y_{p,-1}(t) = 1$,

$$Y_{p,m}(t) = \begin{cases} \int_0^1 K_{p,t}^{m/2} 1\, dx & (m \text{ is even}), \\ \int_0^1 (\kappa_0 + t\overline{\Delta[p]}) K_{p,t}^{(m-1)/2} 1\, dx & (m \text{ is odd}), \end{cases}$$

where $K_{p,t} = (\kappa_0 + t\Delta[p])(\kappa_0 + t\overline{\Delta[p]})$. Then $Y_{\infty,m}(t) = \lim_{p\to\infty} Y_{p,m}(t)$ exists, and $c_{2k}^{(2l)}$ equals the t^{2k}-coefficient of $Y_{\infty,2l}(t)$. Evidently, (20) holds. Since $\int_0^1 \Delta[p](\overline{\Delta[p]}\Delta[p])^j 1\, dx = 0 \ \ (j \geq 0)$, we have inductively

$$\begin{aligned}
Y_{p,2l}(t) &= Y_{p,2l-1}(t) + t\int_0^1 \Delta[p](\kappa_0 + t\overline{\Delta[p]})K_{p,t}^{l-1} 1\, dx \\
&= Y_{p,2l-1}(t) + t^2\int_0^1 \Delta[p]\overline{\Delta[p]}K_{p,t}^{l-1} 1\, dx \\
&= Y_{p,2l-1}(t) + t^2 d_2(\Delta[p])Y_{p,2l-3}(t) \\
&\quad + t^3\int_0^1 \Delta[p]\overline{\Delta[p]}\Delta[p](\kappa_0 + t\overline{\Delta[p]})K_{p,t}^{l-2} 1\, dx \\
&= Y_{p,2l-1}(t) + t^2 d_2(\Delta[p])Y_{p,2l-3}(t) \\
&\quad + t^4\int_0^1 (\Delta[p]\overline{\Delta[p]})^2 K_{p,t}^{l-2} 1\, dx \\
&= \cdots = \sum_{j=0}^{l} t^{2j} d_{2j}(\Delta[p])Y_{p,2l-2j-1}(t).
\end{aligned}$$

Letting p tend to infinity, we have

$$Y_{\infty,2l}(t) = \sum_{j=0}^{l} t^{2j} d_{2j}(\Delta[\infty])Y_{\infty,2l-2j-1}(t).$$

In the same manner,

$$Y_{\infty,2l+1}(t) = \sum_{j=0}^{l} t^{2j} d_{2j}(\Delta[\infty])Y_{\infty,2l-2j}(t).$$

Thus

$$Y_{\infty,m}(t) = \sum_{j=0}^{\iota(m/2)} t^{2j} d_{2j}(\Delta[\infty])Y_{\infty,m-2j-1}(t).$$

Comparing the t^{2k}-coefficients of both sides, we obtain (21).

Lemma 9. *Let K be an anti-symmetric kernel of type 1. Then, for any $0 < \delta \leq 1$,*

$$\lim_{p\to\infty} \gamma^*(\delta\Delta[p] + \delta K) \quad (= \gamma^*(\delta\Delta[\infty] + \delta K),\ \textit{say})$$

exists; we write $\gamma^*(\delta\Delta[\infty])$ if $K = 0$. Moreover,

$$\gamma^*(\delta\Delta[\infty] + \delta K) = \gamma^*(\delta\Delta[\infty])\gamma^*(\gamma^*(\delta\Delta[\infty])\delta K). \tag{22}$$

Proof. First we show that $\gamma^*(\delta\Delta[\infty] + \delta K)$ and $\gamma^*(\delta\Delta[\infty])$ exist. Define $\gamma^*(w; \Delta[p] + K)$, $T(w; \Delta[p] + K)$ $(w \in U)$ for $\varepsilon_0 =$

$(12 + 3\|K\|_{L^2, L^2})^{-1}$ in the same manner as in Lemma 5; we have $\varepsilon_0 \le (3\|\Delta[p] + K\|_{L^2, L^2})^{-1}$ because $\|\Delta[p]\|_{L^2, L^2} \le 4$. Lemma 8 shows that

$$\lim_{p\to\infty} w^2 \varepsilon_0^2 \int_0^1 (\Delta[p] + K) T(w\,; \Delta[p] + K)^{m+1} (\overline{\Delta[p] + K}) 1\, dx$$
$$= \sum_{l=0}^{\infty} w^{2l+2} \varepsilon_0^{2l+2} \frac{(l+1)\cdots(l+m)}{m!} d_{2l+2}(\Delta[\infty] + K) \qquad (m \ge 0).$$

Since (14) holds with K replaced by any $\Delta[p] + K$ $(p \ge 2)$, (13) exists with K replaced by $\Delta[\infty] + K$, i.e.,

$$\text{(23)} \qquad \lim_{p\to\infty} \gamma^*(w\,; \Delta[p] + K) \quad (= \gamma^*(w\,; \Delta[\infty] + K), \text{ say})$$

exists. Since

$$\gamma^*(\delta\,; \Delta[p] + K) = \gamma^*(\delta\Delta[p] + \delta K) \qquad (p \ge 2),$$

$\gamma^*(\delta\Delta[\infty] + \delta K)$ $(= \gamma^*(\delta\,; \Delta[\infty] + K))$ exists. Putting $K = 0$, we see that $\gamma^*(\delta\Delta[\infty])$ exists.

Next we show that $\gamma^*(w\,; \Delta[\infty] + K)$ and $\gamma^*(\gamma^*(w\,; \Delta[\infty])w\,; K)$ are analytic in a domain containing $(0, 1]$. The convergence of (23) is uniform in U. By Lemma 5, $\gamma^*(w\,; \Delta[p] + K)$ is analytic in U, and hence $\gamma^*(w\,; \Delta[\infty] + K)$ is analytic in U. The definition of $\gamma^*(\cdot)$ immediately shows that

$$\gamma^*(\operatorname{Re} w\,; \Delta[p]) = \gamma^*(\operatorname{Re} w \Delta[p]) \le 1 \qquad (w \in U).$$

Letting p tend to infinity, we have $\gamma^*(\operatorname{Re} w\,; \Delta[\infty]) \le 1$ $(w \in U)$. Since $\gamma^*(w\,; \Delta[\infty])$ is analytic in U, there exists $0 < \eta < \pi/8$ such that

$$|\gamma^*(w\,; \Delta[\infty])| \le \frac{4}{3}, \qquad |\arg \gamma^*(w\,; \Delta[\infty])| \le \frac{\pi}{8}$$

in $U_\eta = \{w \in \mathbb{C}; |w| < 4/3, |\arg w| < \eta\}$. Then $\gamma^*(w\,; \Delta[\infty])w \in U$ $(w \in U_\eta)$. Thus, by Lemma 5, $\gamma^*(\gamma^*(w\,; \Delta[\infty])w\,; K)$ is analytic in U_η.

By the theorem of identity, it is sufficient to show that (22) holds for $0 < \delta < (8 + 2\|K\|_{L^2, L^2})^{-1}$. Since

$$\lim_{l\to\infty} d_{2l}(\delta\Delta[p]) = \lim_{l\to\infty} d_{2l}(\delta\Delta[p] + \delta K) = 0,$$

(12) holds for $\delta\Delta[p]$, $\delta\Delta[p] + \delta K$ $(p \ge 2)$. Letting p tend to infinity, we have

$$\gamma^*(\delta\Delta[\infty]) = \sum_{l=0}^{\infty} d_{2l}(\delta\Delta[\infty]) = \sum_{l=0}^{\infty} \delta^{2l} d_{2l}(\Delta[\infty]),$$

$$\gamma^*(\delta\Delta[\infty] + \delta K) = \sum_{l=0}^{\infty} \delta^{2l} d_{2l}(\Delta[\infty] + K).$$

Let

$$\mu_m = \sum_{k=0}^{\infty} \delta^{2k} c_{2k}^{(m+2k)} \qquad (m \geq 0),$$

where $c_{2k}^{(m)}$ $(0 \leq k \leq \iota(m/2),\ m \geq 0)$ are numbers in Lemma 8. Then

$$\mu_0 = \sum_{k=0}^{\infty} \delta^{2k} c_{2k}^{(2k)} = \gamma^*(\delta\Delta[\infty]),$$

by (20). Equality (21) yields that

$$\begin{aligned}
\mu_m &= \sum_{k=0}^{\infty} \delta^{2k} \sum_{j=0}^{k} c_{2k-2j}^{(m+2k-2j-1)} d_{2j}(\Delta[\infty]) \\
&= \sum_{j=0}^{\infty} \delta^{2j} d_{2j}(\Delta[\infty]) \sum_{k=j}^{\infty} \delta^{2(k-j)} c_{2(k-j)}^{(m-1+2(k-j))} \\
&= \mu_{m-1}\mu_0 \qquad (m \geq 1),
\end{aligned}$$

which gives

$$\mu_m = \mu_0^{m+1} = \gamma^*(\delta\Delta[\infty])^{m+1} \qquad (m \geq 1).$$

Thus, by (21),

$$\begin{aligned}
\gamma^*(\delta\Delta[\infty] + \delta K) &= \sum_{l=0}^{\infty} \delta^{2l} d_{2l}(\Delta[\infty] + K) \\
&= \sum_{l=0}^{\infty} \delta^{2l} \sum_{k=0}^{l} c_{2k}^{(2l)} d_{2l-2k}(K) = \sum_{k=0}^{\infty} \sum_{l=k}^{\infty} \delta^{2l} c_{2k}^{(2l)} d_{2l-2k}(K) \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \delta^{2j+2k} c_{2k}^{(2j+2k)} d_{2j}(K) = \sum_{j=0}^{\infty} \mu_{2j} \delta^{2j} d_{2j}(K) \\
&= \sum_{j=0}^{\infty} \gamma^*(\delta\Delta[\infty])^{2j+1} \delta^{2j} d_{2j}(K) = \gamma^*(\delta\Delta[\infty])\gamma^*(\gamma^*(\delta\Delta[\infty])\delta K).
\end{aligned}$$

LEMMA 10. *We inductively define a sequence* $(\gamma_n^*)_{n=1}^{\infty}$ *of positive numbers by*

$$\gamma_1^* = \gamma^*(\Delta[\infty]), \quad \gamma_n^* = \gamma_{n-1}^* \gamma^*(\gamma_{n-1}^* \Delta[\infty]) \qquad (n \geq 2).$$

Then

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\cdots\lim_{p_n\to\infty}\gamma^*(H[p_1,\dots,p_n]-H)=\gamma_n^* \qquad (n\ge 1),$$

where $\lim_{p_n\to\infty}$ *is taken first and* $\lim_{p_1\to\infty}$ *is taken last.*

Proof. We define a sequence $(\lambda_n)_{n=1}^{\infty}$ of positive numbers by $\lambda_1=\gamma^*(\Delta[\infty])$, $\lambda_n=\gamma^*(\lambda_1\dots\lambda_{n-1}\Delta[\infty])$ $(n\ge 2)$. Then $\gamma_n^*=\lambda_1\cdots\lambda_n$ $(n\ge 1)$. Fixing an $(n-1)$-tuple $(p_1,\dots,p_{n-1})$ $(n\ge 2)$ of integers larger than or equal to 3, we study

$$\begin{aligned}&\lim_{p_n\to\infty}\gamma^*(H[p_1,\dots,p_n]-H)\\&\quad=\lim_{p_n\to\infty}\gamma^*(\Delta[p_1,\dots,p_n]+(H[p_1,\dots,p_{n-1}]-H)).\end{aligned}$$

Put $I_0=[0,1/(p_1\cdots p_{n-1}))$, $I_j=(I_0+j/(p_1\cdots p_{n-1}))$ $(0\le j\le (p_1\dots p_{n-1})-1)$. Then

$$(H[p_1,\dots,p_{n-1}]-H)(x,y)=0 \qquad (x,y\in I_j),$$

$$\begin{aligned}&|(H[p_1,\dots,p_{n-1}]-H)(x,y)|\\&\quad\le\frac{2}{\pi}p_1\cdots p_{n-1}+\frac{1}{\pi}\frac{1}{|y-x|} \qquad (x\in I_j,\ y\in I_k,\ j\ne k),\end{aligned}$$

which shows that $H[p_1,\dots,p_{n-1}]-H$ is of type 1. Let

$$\Delta'[p_1,\dots,p_n](x,y)=\Delta[p_1,\dots,p_n](x,y)\chi_{[0,\frac{N'}{p_1\cdots p_n})}(|y-x|),$$

$$\Delta'\left[\prod_{j=1}^{n}p_j\right](x,y)=\Delta\left[\prod_{j=1}^{n}p_j\right](x,y)\chi_{[0,\frac{N'}{p_1\cdots p_n})}(|y-x|)$$

$(N'=\iota(\log(p_1\dots p_n)))$. Then

$$\lim_{p_n\to\infty}\|\Delta[p_1,\dots,p_n]-\Delta'[p_1,\dots,p_n]\|_{L^4,L^4}=0,$$

$$\lim_{p_n\to\infty}\left\|\Delta\left[\prod_{j=1}^{n}p_j\right]-\Delta'\left[\prod_{j=1}^{n}p_j\right]\right\|_{L^4,L^4}=0$$

(cf. Lemmas 6 and 7). Since

$$\Delta'[p_1,\dots,p_n](x,y)=\Delta'\left[\prod_{j=1}^{n}p_j\right](x,y)$$

$$(x,y\in I_j,\ 0\le j\le (p_1\cdots p_{n-1})-1),$$

we have

$$\lim_{p_n\to\infty}\left\|\Delta[p_1,\dots,p_n]-\Delta\left[\prod_{j=1}^n p_j\right]\right\|_{L^4,L^2} = \lim_{p_n\to\infty}\left\|\Delta'[p_1,\dots,p_n]-\Delta'\left[\prod_{j=1}^n p_j\right]\right\|_{L^4,L^2}=0,$$

and hence, in the same manner as in the proof of the existence of (23),

$$\lim_{p_n\to\infty}\gamma^*(\Delta[p_1,\dots,p_n]+(H[p_1,\dots,p_{n-1}]-H)) = \lim_{p_n\to\infty}\gamma^*\left(\Delta\left[\prod_{j=1}^n p_j\right]+(H[p_1,\dots,p_{n-1}]-H)\right).$$

Using (22) with $\delta=1$, $K=H[p_1,\dots,p_{n-1}]-H$, we have

$$\lim_{p_n\to\infty}\gamma^*(H[p_1,\dots,p_n]-H)=\gamma^*(\Delta[\infty]+(H[p_1,\dots,p_{n-1}]-H)) = \lambda_1\gamma^*(\lambda_1(H[p_1,\dots,p_{n-1}]-H)).$$

In the same manner, using (22) with $\delta=\lambda_1$, $K=H[p_1,\dots,p_{n-2}]-H$, we have

$$\lim_{p_{n-1}\to\infty}\lim_{p_n\to\infty}\gamma^*(H[p_1,\dots,p_n]-H) = \lambda_1\gamma^*(\lambda_1\Delta[\infty])\gamma^*(\gamma^*(\lambda_1\Delta[\infty])\lambda_1(H[p_1,\dots,p_{n-2}]-H)) = \lambda_1\lambda_2\gamma^*(\lambda_1\lambda_2(H[p_1,\dots,p_{n-2}]-H)).$$

Repeating this argument,

$$\lim_{p_1\to\infty}\cdots\lim_{p_n\to\infty}\gamma^*(H[p_1,\dots,p_n]-H) = \lambda_1\cdots\lambda_{n-1}\lim_{p_1\to\infty}\gamma^*(\lambda_1\cdots\lambda_{n-1}\Delta[p_1])=\lambda_1\dots\lambda_n=\gamma_n^*.$$

This completes the proof of our lemma.

We now give the proof of our theorem. By Proposition 2, there exists a positive integer n_0 such that

$$\sup\gamma(\Gamma(p_1,\dots,p_n))\le 10^{-5}\qquad(n\ge n_0),\tag{24}$$

where the supremum is taken over all n-tuples $(p_1,\dots,p_n)$ of integers larger than or equal to 3. By Lemma 10, we can inductively choose a sequence $(p_n^0)_{n=1}^\infty$ of integers larger than or equal to 3 so that

$$\frac{1}{2}\gamma_n^*\le\gamma^*(H[p_1^0,\dots,p_n^0]-H)\le 2\gamma_n^*\qquad(n\ge1),$$

where $(\gamma_n^*)_{n=1}^{\infty}$ is the sequence in Lemma 10. We show that $\Gamma_n = \Gamma(p_1^0, \dots, p_n^0)$ $(n \geq 1)$ are required cranks. We may assume that $n \geq n_0$. Lemma 3 shows that

$$\begin{aligned}\frac{1}{4}\gamma^*(H[p_1^0, \dots, p_n^0] - H) &\leq \gamma^*(H[p_1^0, \dots, p_n^0]) \\ &\leq 4\gamma^*(H[p_1^0, \dots, p_n^0] - H),\end{aligned}$$

and hence

$$\frac{1}{8}\gamma_n^* \leq \gamma^*(H[p_1^0, \dots, p_n^0]) \leq 8\gamma_n^*.$$

Thus, by (11),

$$\frac{1}{8\pi}\gamma_n^* \leq \gamma(\Gamma_n) \leq \frac{8}{\pi}\gamma_n^*. \tag{25}$$

Using (24) and (25), we have $\gamma_n^* \leq 8\pi \cdot 10^{-5}$. Recall (15), and notice that $d_{2l}(\Delta[\infty]) \leq 4^l$ $(l \geq 1)$. Since $\lim_{l\to\infty} d_{2l}(\gamma_n^*\Delta[p]) = 0$, (12) holds for $\gamma_n^*\Delta[p]$. Letting p tend to infinity, we have

$$\begin{aligned}\gamma_{n+1}^* &= \gamma_n^*\gamma^*(\gamma_n^*\Delta[\infty]) = \gamma_n^* \sum_{l=0}^{\infty} d_{2l}(\gamma_n^*\Delta[\infty]) \\ &= \gamma_n^* \sum_{l=0}^{\infty} \gamma_n^{*^{2l}} d_{2l}(\Delta[\infty]) \leq \gamma_n^* - \frac{1}{25\pi^2}\gamma_n^{*^3} + \sum_{l=2}^{\infty} 4^l \gamma_n^{*^{2l+1}} \\ &\leq \gamma_n^* - 10^{-3}\gamma_n^{*^3},\end{aligned}$$

$$\gamma_{n+1}^* \geq \gamma_n^* - \sum_{l=1}^{\infty} 4^l \gamma_n^{*^{2l+1}} \geq \gamma_n^* - 10\gamma_n^{*^3}, \quad \text{i.e.,}$$

$$\gamma_n^* - 10\gamma_n^{*^3} \leq \gamma_{n+1}^* \leq \gamma_n^* - 10^{-3}\gamma_n^{*^3}.$$

Since this holds for all $n \geq n_0$, a simple induction yields that

$$\frac{1}{C_4}\frac{1}{\sqrt{n}} \leq \gamma_n^* \leq C_4\frac{1}{\sqrt{n}} \qquad (n \geq n_0)$$

for some absolute constant C_4. Using (25) again,

$$\frac{1}{8\pi C_4}\frac{1}{\sqrt{n}} \leq \gamma(\Gamma_n) \leq \frac{8}{\pi}C_4\frac{1}{\sqrt{n}} \qquad (n \geq n_0).$$

This completes the proof of our theorem.

REMARK 11. It is not known whether $\gamma(\cdot)$ is semi-additive [**4**, p. 11]. For $0 < \eta \leq 1$, we define $B_p^\eta(x)$ replacing $1/2p$ by $\eta/2p$ in the definition of $B_p(x)$. Then cranks $\Gamma^\eta(p_1, \dots, p_n)$ of degree n are analogously defined. We see that there exists a crank Γ_n^η of degree n such that $\gamma(\Gamma_n^\eta) \le C_\eta/\sqrt{n}$, where C_η is a constant depending only on η. Adding some segments (perpendicular to the x-axis) to Γ_n^η, we obtain an arc $\widetilde{\Gamma}_n^\eta$ connecting 0 and 1. Then the diameter of $\widetilde{\Gamma}_n^\eta$ is larger than or equal to 1. Since $\widetilde{\Gamma}_n^\eta$ is connected, $\gamma(\widetilde{\Gamma}_n^\eta) \ge 1/4$ [**4**, p. 9]. Hence, from the point of view of the above semi-additive problem, it seems interesting to compute $\gamma(\widetilde{\Gamma}_n^\eta - \Gamma_n^\eta)$.

4. Another application of Proposition 1. In this section, we show another application of our method. Let E be a compact set on $\mathbb{R}$. Pommerenke [**11**] showed that

$$\gamma(E) = |E|/4, \tag{26}$$

$$f_E(z) = \left\{1 - \exp\left(\frac{1}{2}\int_E \frac{dt}{t-z}\right)\right\} \Big/ \left\{1 + \exp\left(\frac{1}{2}\int_E \frac{dt}{t-z}\right)\right\}. \tag{27}$$

We deduce (26), (27) from (3), (10); our method explains a quarter and (27). Let $L^2(\mathbb{R})$ denote the L^2 space of functions on $\mathbb{R}$, and let M_E denote the multiplier: $h \in L^2(\mathbb{R}) \to \chi h \in L^2(\mathbb{R})$, where $\chi = \chi_E$. We inductively define a sequence $(H_E^{(m)})_{m=0}^\infty$ of operators from $L^2(\mathbb{R})$ to itself by $H_E^{(0)} = M_E$, $H_E^{(m)} = HM_E H_E^{(m-1)}$ $(m \ge 1)$. Notice that

$$\gamma(E) = \frac{1}{\pi}\gamma^*(M_E H M_E),$$
$$d_{2l}(M_E H M_E) = \int_E H_E^{(2l)}\chi\, dx \qquad (l \ge 0,\ \chi = \chi_E).$$

We also remark that

$$H(g \cdot Hh) + H(Hg \cdot h) = Hg \cdot Hh - gh \qquad (g, h \in L^2(\mathbb{R})). \tag{28}$$

We first show that, for any $m \ge 1$,

$$\chi H\chi \cdot H_E^{(m)}\chi = (m+1)\chi H_E^{(m+1)}\chi + m\chi H_E^{(m-1)}\chi. \tag{29}$$

Equality (28) shows that $2H(\chi H\chi) = (H\chi)^2 - \chi\chi$, which gives $\chi H\chi \cdot H_E^{(1)}\chi = 2\chi H_E^{(2)}\chi + \chi H_E^{(0)}\chi$. Suppose that (29) holds for m. Using (28) with $g = \chi$, $h = \chi H_E^{(m)}\chi$, we have

$$\begin{aligned}
\chi H\chi \cdot H_E^{(m+1)}\chi &= \chi H\chi \cdot H(\chi H_E^{(m)}\chi) \\
&= \chi H\{\chi H(\chi H_E^{(m)}\chi) + H\chi \cdot \chi H_E^{(m)}\chi\} + \chi\{\chi \cdot \chi H_E^{(m)}\chi\} \\
&= \chi H_E^{(m+2)}\chi + \chi H\{(m+1)\chi H_E^{(m+1)}\chi + m\chi H_E^{(m-1)}\chi\} + \chi H_E^{(m)}\chi \\
&= (m+2)\chi H_E^{(m+2)}\chi + (m+1)\chi H_E^{(m)}\chi, \quad \text{i.e.,}
\end{aligned}$$

(29) holds for $m+1$. Thus (29) holds for all $m \geq 1$.

We next show that

$$\int_E H_E^{(2l)}\chi\,dx = \frac{(-1)^l}{2l+1}|E| \qquad (l \geq 0). \tag{30}$$

We put $a_{2l} = \int_E H_E^{(2l)}\chi\,dx$ $(l \geq 0)$. Evidently, $a_0 = |E|$. Suppose that $a_{2l-2} = \{(-1)^{l-1}/(2l-1)\}|E|$. Equality (29) $(m = 2l-1)$ shows that

$$\begin{aligned}\int_E H\chi \cdot H_E^{(2l-1)}\chi\,dx &= 2l\int_E H_E^{(2l)}\chi\,dx + (2l-1)\int_E H_E^{(2l-2)}\chi\,dx \\ &= 2la_{2l} + (2l-1)a_{2l-2}.\end{aligned}$$

Since the adjoint operator of H equals $-H$, we have

$$\int_E H\chi \cdot H_E^{(2l-1)}\chi\,dx = -\int_E H\{\chi H_E^{(2l-1)}\chi\}\,dx = -a_{2l}.$$

Thus $-a_{2l} = 2la_{2l} + (2l-1)a_{2l-2}$, which yields that

$$a_{2l} = -\frac{2l-1}{2l+1}a_{2l-2} = \frac{(-1)^l}{2l+1}|E|.$$

Now the deduction of (26) is immediate. By (30),

$$\lim_{l\to\infty} d_{2l}(M_EHM_E) = \lim_{l\to\infty}\int_E H_E^{(2l)}\chi\,dx = 0.$$

Hence we can apply (3). Leibniz's formula and (30) yield that

$$\begin{aligned}\gamma(E) &= \frac{1}{\pi}\gamma^*(M_EHM_E) = \frac{1}{\pi}\sum_{l=0}^{\infty} d_{2l}(M_EHM_E) \\ &= \frac{1}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}|E| = \frac{1}{4}|E|.\end{aligned}$$

Last, we deduce (27) from (10). Equality (10) gives that

$$f_E(z) = -\frac{1}{\pi}\left\{\int_E \frac{ds}{s-z} + \int_E \frac{H_E^{(1)}h_E(s)}{s-z}\,ds\right\} \Big/ \left\{1 + \frac{1}{\pi}\int_E \frac{h_E(s)}{s-z}\,ds\right\},$$

where $h_E(s)$ is the function which attains $\gamma^*(M_EHM_E)$. We show that this equals the function in the right-hand side of (27). Let

$$u_0(z) = 1, \qquad u_m(z) = \frac{1}{\pi}\int_E \frac{H_E^{(m-1)}\chi(s)}{s-z}\,ds,$$

$$v_m(z) = \frac{1}{\pi}\int_E \frac{H\chi(s)H_E^{(m-2)}\chi(s)}{s-z}\,ds \qquad (m \geq 1),$$

where $H_E^{(-1)}\chi = \chi$. Let

$$P_t(z) = \sum_{m=0}^{\infty} t^m u_m(z) \qquad (t \in \mathbb{C},\ |t| < 1).$$

We begin by showing that

$$(31) \qquad (1+t^2)\frac{\partial}{\partial t}P_t(z) = u_1(z)P_t(z) \qquad (0 < t < 1).$$

Let $m \geq 1$. We have, on $\mathbb{R}$,

$$\begin{aligned}
&\lim_{\eta\downarrow 0}\{u_{m+1}(\cdot + i\eta) + v_{m+1}(\cdot + i\eta)\} \\
&\quad = H(\chi H_E^{(m)}\chi) + i\chi H_E^{(m)}\chi \\
&\qquad + H\{\chi H\chi \cdot H_E^{(m-1)}\chi\} + i\chi H\chi \cdot H_E^{(m-1)}\chi \\
&\quad = H\{\chi H(\chi H_E^{(m-1)}\chi) + H\chi \cdot \chi H_E^{(m-1)}\chi\} \\
&\qquad + i\{\chi H(\chi H_E^{(m-1)}\chi) + H\chi \cdot \chi H_E^{(m-1)}\chi\},
\end{aligned}$$

$$\begin{aligned}
&\lim_{\eta\downarrow 0} u_1(\cdot + i\eta)u_m(\cdot + i\eta) \\
&\quad = \{H\chi + i\chi\}\{H(\chi H_E^{(m-1)}\chi) + i\chi H_E^{(m-1)}\chi\} \\
&\quad = H\chi \cdot H(\chi H_E^{(m-1)}\chi) - \chi \cdot \chi H_E^{(m-1)}\chi \\
&\qquad + i\{\chi H(\chi H_E^{(m-1)}\chi) + H\chi \cdot \chi H_E^{(m-1)}\chi\}.
\end{aligned}$$

Hence (28) $(g = \chi,\ h = \chi H_E^{(m-1)}\chi)$ shows that

$$\lim_{\eta\downarrow 0}\{u_{m+1}(\cdot + i\eta) + v_{m+1}(\cdot + i\eta) - u_1(\cdot + i\eta)u_m(\cdot + i\eta)\} = 0$$

on $\mathbb{R}$. In particular, this holds on $\mathbb{R} - E$. Hence, by the theorem of identity, $u_{m+1}(z) + v_{m+1}(z) - u_1(z)u_m(z) = 0$. Equality (29) shows that $v_{m+1}(z) = mu_{m+1}(z) + (m-1)u_{m-1}(z)$. Thus

$$(m+1)u_{m+1}(z) + (m-1)u_{m-1}(z) - u_1(z)u_m(z) = 0 \qquad (m \geq 1),$$

which yields that

$$\sum_{m=0}^{\infty} mt^m u_m(z) + t^2 \sum_{m=0}^{\infty} mt^m u_m(z) = tu_1(z) \sum_{m=0}^{\infty} t^m u_m(z), \quad \text{i.e.,}$$

$$t\frac{\partial}{\partial t}P_t(z) + t^3\frac{\partial}{\partial t}P_t(z) = tu_1(z)P_t(z).$$

This is the required equality (31).

We can choose $x_0 \in \mathbb{R} - E$, $\eta > 0$ so that $P_t(x) > 0$, $u_1(x) > 0$ for all $x \in (x_0 - \eta, x_0 + \eta)$, $0 < t < 1$. Equality (31) shows that

$$\frac{1}{1+t^2} u_1(x) = \frac{\partial}{\partial t} P_t(x) \Big/ P_t(x) \qquad (x \in (x_0 - \eta, x_0 + \eta),\ 0 < t < 1),$$

which gives that

$$P_t(x) = \exp\left\{ \int_0^t \frac{ds}{1+s^2} u_1(x) \right\} \qquad (x \in (x_0 - \eta, x_0 + \eta),\ 0 < t < 1)$$

because $P_0 = 1$. By the theorem of identity,

$$P_t(z) = \exp\left\{ \int_0^t \frac{ds}{1+s^2} u_1(z) \right\} \qquad (0 < t < 1).$$

Since $P_t(z)$ and $\exp\{(\int_0^t (ds/(1+s^2))u_1(z)\}$ are analytic in the unit disk as functions of t, this equality holds for $-1 < t < 0$ also. Thus

$$\begin{aligned} 1 + \frac{1}{\pi} \int_E \frac{h_E(s)}{s-z}\, ds &= 1 + \frac{1}{\pi} \int_E \frac{1}{s-z} \sum_{l=1}^{\infty} H_E^{(2l-1)} \chi(s)\, ds \\ &= \lim_{t \uparrow 1} \sum_{l=0}^{\infty} t^{2l} u_{2l}(z) = \frac{1}{2} \lim_{t \uparrow 1} \{P_{-t}(z) + P_t(z)\} \\ &= \frac{1}{2} \left\{ \exp\left(-\frac{\pi}{4} u_1(z)\right) + \exp\left(\frac{\pi}{4} u_1(z)\right) \right\}, \\ -\frac{1}{\pi} \int_E \frac{ds}{s-z} - \frac{1}{\pi} \int_E \frac{H_E^{(1)} h_E(s)}{s-z}\, ds &= \frac{1}{2} \lim_{t \uparrow 1} \{P_{-t}(z) - P_t(z)\} \\ &= \frac{1}{2} \left\{ \exp\left(-\frac{\pi}{4} u_1(z)\right) - \exp\left(\frac{\pi}{4} u_1(z)\right) \right\}, \end{aligned}$$

which gives (27).

References

[1] A. P. Calderón, *Cauchy integrals on Lipschitz curves and related operators*, Proc. Nat. Acad. Sci. USA, **74** (1977), 1324–1327.

[2] A. M. Davie, *Analytic capacity and approximation problems*, Trans. Amer. Math. Soc., **171** (1972), 409–444.

[3] A. Denjoy, *Sur les fonctions analytiques uniformes à singularités discontinues*, C. R. Acad. Sci. Paris, **149** (1909), 258–260.

[4] J. Garnett, *Analytic Capacity and Measure*, Lecture Notes in Mathematics, Vol. 297, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[5] P. W. Jones, *Square functions, Cauchy integrals, analytic capacity, and harmonic measure*, preprint.

[6] N. S. Landkof, *Foundations of Modern Potential Theory*, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[7] B. B. Mandelbrot, *The Fractal Geometry and Nature*, Freeman, San Francisco, 1982.
[8] D. E. Marshall, *Removable sets for bounded analytic functions*, in *Linear and Complex Analysis Problem Book* (Edited by V. P. Havin, S. V. Hruščëv and N. K. Nikol′skiĭ), Lecture Notes in Mathematics, Vol. 1043, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1984, 485–490.
[9] T. Murai, *Comparison between analytic capacity and the Buffon needle probability*, Trans. Amer. Math. Soc., **304** (1987), 501–514.
[10] ——, *A Real Variable Method for the Cauchy Transform, and Analytic Capacity*, Lecture Notes in Mathematics, Vol. 1307, Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo, 1988.
[11] Ch. Pommerenke, *Über die analytische Kapazität*, Arch. Math., **11** (1960), 270–277.
[12] A. G. Vitushkin, *Example of a set of positive length but of zero analytic capacity*, (Russian) Dokl. Akad. Nauk SSSR, **127** (1959), 246–249.

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