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# AMENABILITY OF DISCRETE CONVOLUTION ALGEBRAS, THE COMMUTATIVE CASE

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**A Banach algebra $\mathfrak{A}$ is called amenable if all bounded derivations into dual Banach $\mathfrak{A}$-modules are inner. Let $S$ be a semigroup and let $l^1(S)$ be the corresponding discrete convolution algebra. This paper is on the theme: "On the hypothesis that $l^1(S)$ is amenable, what conclusions can be drawn about the (algebraic) structure of $S$?" We give a complete characterization of commutative semigroups carrying amenable semigroup algebras. If $S$ is commutative, then $l^1(S)$ is amenable if and only if $S$ is a finite semilattice of groups, that is, there is a finite semilattice $Y$ and disjoint commutative groups $G_\alpha$ $(\alpha \in Y)$ such that $S = \bigcup_{\alpha\in Y} G_\alpha$ and $G_\alpha G_\beta \subseteq G_{\alpha\beta}$ $(\alpha, \beta \in Y)$.**

The theme above has previously been studied in [3] and [4]. In both papers it is apparent that the condition of amenability imposes strong algebraic constraints on the semigroup. In [3] a rather complete description of inverse semigroups carrying amenable semigroup algebras is given. Of particular interest for this paper is that a semilattice carries an amenable semigroup algebra if and only if it is finite [3, Theorem 10]. In [4] it is proved that, if a one-sided cancellative semigroup carries an amenable semigroup algebra, then it is a group. The result of this paper, that for a commutative semigroup $S$, the semigroup algebra $l^1(S)$ is amenable if and only if $S$ is a finite lattice of groups, is proved by looking at the gross structure of $S$ by means of the "principle of maximal homomorphic image of a given type". Using the fact that homomorphic images of $S$ carry amenable semigroup algebras when $S$ does, we establish the necessity of the characterization by showing that each archimedean component of $S$ is a group. This is obtained by applying the results from [3] and [4], mentioned above, to the maximal semilattice, the maximal cancellative, and the maximal separative homomorphic images of $S$. The sufficiency of the characterization is easily verified. Alternatively, it follows from [3, Theorem 8].

**1. Preliminaries.** We shall need some elementary semigroup theory. We prefer to keep our exposition self-contained, so although most of what follows can be found in standard texts on the subject, we shall, with a few exceptions, give proofs in some detail. For a further discussion the reader is referred to [**1**]. Throughout $S$ will denote a commutative semigroup, with the binary operation written multiplicatively.

1.1. DEFINITIONS. Consider the following conditions on $S$:
(A) Each element of $S$ is an idempotent.
(B) For all $s, t \in S$ there is $n \in \mathbf{N}$ such that

$$s^n \in tS \quad \text{and} \quad t^n \in sS.$$

(C) $s^2 = t^2 = st \Rightarrow s = t \quad (s, t \in S)$.
If $S$ satisfies (A) we call $S$ a *semilattice.*
If $S$ satisfies (B) we call $S$ *archimedean.*
If $S$ satisfies (C) we call $S$ *separative.*
An *ideal* in $S$ is a subset $I$ such that $SI \subseteqq I$. A *prime ideal* in $S$ is an ideal, whose complement is a subsemigroup of $S$.
A *congruence* on $S$ is an equivalence relation which is compatible with the semigroup operation.
A congruence $\sim$ on $S$ will be called separative (cancellative, archimedean, etc.) if the semigroup $S/\sim$ is separative (cancellative, archimedean, etc.).

1.2. DEFINITION. (Principle of maximal homomorphic image of a given type). Let $\mathfrak{C}$ be a class of congruences on $S$, closed under intersections. Put $\rho_0 = \bigcap\{\rho | \rho \in \mathfrak{C}\}$. Then $S/\rho_0$ is the *maximal "type class $\mathfrak{C}$" homomorphic image of* $S$.
See also [**1**, p. 18] and [7, §1].

EXAMPLE. Let $\rho_0 = \bigcap\{\rho | s^2 \rho s \quad (s \in S)\}$. Then $S/\rho_0$ is the maximal semilattice homomorphic image of $S$.

1.3. DEFINITION. Let $s \in S$ and choose $m \in \mathbf{N}$ smallest possible so that $s^m = s^{m+r}$ for some $r \in \mathbf{N}$. Then order$(s) = m$ and the smallest possible $r$ is called period$(s)$. If no such $m \in \mathbf{N}$ can be found we put order$(s) = \infty$.

1.4. DEFINITION. Let $S$ be a semigroup and suppose that there is a semilattice $Y$ and disjoint subsemigroups $S_\alpha$ $(\alpha \in Y)$ of $S$ such that $S = \bigcup_{\alpha \in Y} S_\alpha$ and $S_\alpha S_\beta \subseteqq S_{\alpha\beta}$ $(\alpha, \beta \in Y)$. Then $S$ is called a *semilattice of the subsemigroups* $S_\alpha$ $(\alpha \in Y)$.

The following lemma is the main structure theorem for commutative semigroups.

1.5. LEMMA. *Let $S$ be a commutative semigroup and let $Y$ be the maximal semilattice homomorphic image of $S$. Then there are disjoint archimedean subsemigroups $S_\alpha$ $(\alpha \in Y)$ of $S$ such that $S$ is a semilattice of the semigroups $S_\alpha$ $(\alpha \in Y)$. This decomposition of $S$ into archimedean subsemigroups is unique up to isomorphism of $Y$, and $S$ is separative if and only if each archimedean component $S_\alpha$ is cancellative.*

*Proof.* See [**1**, §4.3].

1.6. LEMMA. *On $S$ define the relations:*

$$s\,c\,t \Leftrightarrow \exists u \in S\ su = tu$$

*and*

$$s\,\sigma\,t \Leftrightarrow \exists n_0 \in \mathbf{N} \forall n \geq n_0\ s^n = t^n.$$

*Then $c$ and $\sigma$ are congruences and $S/c$ is the maximal cancellative homomorphic image of $S$ and $S/\sigma$ is the maximal separative homomorphic image of $S$.*

*Proof.* It is clear that both relations are congruences. Now suppose $\rho$ is a cancellative congruence; that is, $su\,\rho\,tu \Rightarrow s\,\rho\,t$ $(s, t, u \in S)$. Then clearly $s\,c\,t \Rightarrow s\,\rho\,t$ $(s, t \in S)$ so that $c \subseteqq \rho$. Since $c$ is cancellative we are done with the statements about $c$.

Now suppose that $s^2\,\sigma\,t^2\,\sigma\,st$; that is, there is $n_0 \in \mathbf{N}$ so that $s^{2n} = t^{2n} = s^n t^n$ for $n \geq n_0$. Then $s^{4n_0+1}t = ss^{2n_0} \cdot t^{2n_0} \cdot t = s^{2n_0+1}t^{2n_0+1} = s^{4n_0+2}$ so that for $n \geq 8n_0 + 2$ we have $s^n = t^n$. Hence $s\,\sigma\,t$, proving that $\sigma$ is separative. Let $\rho$ be a separative congruence. If $s\,\sigma\,t$, then there is $k \in \mathbf{N}$ so that $st^k = t^{k+1}$. In particular $st^k\,\rho\,t^{k+1}$. This gives

$$(st^{k-1})^2 = st^{k-2}st^k\,\rho\,st^{k-2}t^{k+1} = st^{k-1}t^k\,\rho\,t^{k+1}t^{k-1} = (t^k)^2.$$

With $x = st^{k-1}$ and $y = t^k$ we have $x^2\,\rho\,y^2\,\rho\,xy$ so that $x\,\rho\,y$, that is, $st^{k-1}\,\rho\,t^k$. Repeating as necessary, we get $st\,\rho\,t^2\,\rho\,s^2$, where the second relation follows from symmetry. Thus $s\,\rho\,t$, proving that $\sigma \subseteqq \rho$. □

1.7. LEMMA. $s^2\,\sigma\,s \Leftrightarrow$ order$(s) < \infty$ *and* period$(s) = 1$. *If $e$, $f$ are idempotents in $S$, then $e\,\sigma\,f \Leftrightarrow e = f$.*

*Proof.* Suppose $s^2\,\sigma\,s$. Then there is $n_0 \in \mathbf{N}$ so that $s^{2n} = s^n$ for $n \geq n_0$. If $r$ is the period of $s$ we have $2n \equiv n \pmod r$ for $n \geq n_0$ so that $r = 1$. The rest is obvious. □

1.8. LEMMA. *$S/\sigma$ is a group if and only if $S$ is archimedean with unique idempotent.*

*Proof*. First suppose that $S/\sigma$ is a group. From Lemma 1.7 it follows that $S$ has a unique idempotent. Let $s, t \in S$. Since $S/\sigma$ is a group there are $u$, $v \in S$ so that $su\,\sigma\,t$ and $tv\,\sigma\,s$. By definition of $\sigma$, $s$ divides a power of $t$ and $t$ divides a power of $s$, that is, $S$ is archimedean. Conversely, let $s \in S$ and let $e$ denote the unique idempotent in $S$. Since $S$ is archimedean there are $t$, $u \in S$ so that $st = e$ and $ue = s^{n_0}$ for some $n_0$. We have $(es)^{n_0+p} = e^{n_0+p}s^{n_0}s^p = e^{n_0+p}ues^p = ues^p = s^{n_0+p}$ $(p \in \mathbf{N})$ so that $es\,\sigma\,s$. Clearly $st\,\sigma\,e$, so $S/\sigma$ is a group. □

**2. The main theorem.** For the remainder of this paper we shall assume that $S$ is a commutative semigroup such that $l^1(S)$ is amenable. We shall make frequent use of the fact that, if $T$ is a homomorphic image of $S$, then $l^1(T)$ is amenable, and if $I$ is an ideal in $S$ which is generated by an idempotent, then $l^1(I)$, being a closed $l^1(S)$-ideal which is unital as a Banach algebra, is amenable [**6**, Proposition 5.1]. Thus, if $S = \bigcup_{\alpha\in Y} S_\alpha$ is the decomposition of $S$ into its archimedean components, then the semilattice $Y$ is finite, since $l^1(Y)$ is amenable ([**3**, Theorem 10]). We give $Y$ the usual semilattice ordering $\alpha \le \beta \Leftrightarrow \alpha\beta = \alpha$ $(\alpha, \beta \in Y)$. Since $Y$ is finite, $Y$ has a minimal element, namely the product of all elements in $Y$.

It is convenient to start with the case where $S$ is separative; that is, we are assuming that each archimedean component is cancellative.

2.1. LEMMA. *Let $S$ and $Y$ be as above and let $\alpha_0$ be the minimal element of $Y$. Then $S_{\alpha_0}$ is a group.*

*Proof*. By [**4**, Theorem 2.3] $S/c$ is a group. Let $s \in S_{\alpha_0}$ Then there is $t \in S$ so that for all $u \in S$ $stu\,c\,u$, that is, for all $u \in S$ there is $v \in S$ so that $stuv = uv$. Since $\alpha_0$ is minimal, $st \in S_{\alpha_0}$ and $uv \in S_{\alpha_0}$, so, using the cancellation law in $S_{\alpha_0}$, we see that $st$ is a neutral element in $S_{\alpha_0}$. Consequently $l^1(S_{\alpha_0})$ can be identified canonically with an ideal generated by an idempotent in $l^1(S)$. It follows that $l^1(S_{\alpha_0})$ is amenable and therefore $S_{\alpha_0}$ is a group, again by [**4**, Theorem 2.3]. □

2.2. LEMMA. *Let $l^1(S)$ be amenable and suppose that $S$ is separative. Then $S$ is a finite semilattice of groups.*

*Proof.* Let $S = \bigcup_{\alpha\in Y} S_\alpha$ be the decomposition of $S$ into its archimedean components. Let $\beta \in Y$, and define $T = \bigcup_{\alpha\geq\beta} S_\alpha$. Then $T$ is a subsemigroup of $S$ and $S \setminus T$ is a (prime) ideal in $S$. Hence the canonical Banach space direct sum $l^1(S) = l^1(T) \oplus l^1(S\setminus T)$ is a semidirect product, so that $l^1(T)$ is amenable. Since $\beta$ is minimal in $\{\alpha \in Y | \alpha \geq \beta\}$, Lemma 2.1 implies that $S_\beta$ is a group. But $\beta$ was arbitrary in $Y$. □

We now turn to the general case.

2.3. LEMMA. *Suppose $l^1(S)$ is amenable. Then $S$ is a finite semilattice of its archimedean components, $S = \bigcup_{\alpha\in Y} S_\alpha$. Each $S_\alpha$ has a unique idempotent $e_\alpha$, and $e_\alpha S_\alpha$ is a group, isomorphic to the maximal separative homomorphic image of $S_\alpha$.*

*Proof.* By Lemma 2.2 $S/\sigma$ is a finite semilattice of groups, $S/\sigma = \bigcup_{\alpha\in Y} G_\alpha$. Let $S_\alpha$ be the preimage of $G_\alpha$ by the canonical map $S \to S/\sigma$. With slight abuse of notation we have $S_\alpha/\sigma = G_\alpha$, so that $S_\alpha$ is archimedean with unique idempotent, $e_\alpha$ say, by Lemma 1.8. It follows that $S = \bigcup_{\alpha\in Y} S_\alpha$ is the decomposition of $S$ into its archimedean components. Now let $s \in S_\alpha$. Since $G_\alpha$ is a group, there is $t \in S_\alpha$ so that $st\,\sigma\, e_\alpha$, i.e. $(st)^n = e_\alpha$ for some $n \in \mathbf{N}$. Hence $e_\alpha s^{n-1} t^n$ is an inverse to $e_\alpha s$. Clearly the canonical map from $e_\alpha S_\alpha$ to $G_\alpha$ is surjective. Assume that $e_\alpha s\,\sigma\, e_\alpha$ for some $s \in S_\alpha$. Since $e_\alpha S_\alpha$ is a group it follows from Lemma 1.7 that $e_\alpha s = e_\alpha$, proving injectivity of the canonical map. □

We shall finish the proof of the main theorem by proving that $e_\alpha S_\alpha = S_\alpha$ for each $\alpha \in Y$. This is done by exploiting that $l^1(S)$, being amenable, has a bounded approximate identity. First we need a definition.

2.4. DEFINITION. Let $s \in S$. Then we define

$$[ss^{-1}] = \{u \in S | us = s\}.$$

Since $l^1(S)$ has a bounded approximate identity $[ss^{-1}] \neq \varnothing$ for all $s \in S$ [**4**, Theorem 1.1].

2.5. LEMMA. *Let $S = \bigcup_{\alpha\in Y} S_\alpha$ be the decomposition of $S$ into its archimedean components, as in Lemma 2.3, and let $s \in S_\alpha$. If $[ss^{-1}] \cap S_\alpha \neq \varnothing$, then $s \in e_\alpha S_\alpha$. If $\alpha$ is maximal in $Y$, then $S_\alpha$ is a group.*

*Proof*. Let $u \in [ss^{-1}] \cap S_\alpha$. Then $us \,\sigma\, e_\alpha s$. Since $S_\alpha/\sigma$ is a group we have $u \,\sigma\, e_\alpha$, i.e. $u^n = e_\alpha$ for some $n \in \mathbf{N}$. Hence $s = u^n s = e_\alpha s$. In general, if $s \in S_\alpha$ and $u \in [ss^{-1}] \cap S_\beta$, then $s = us \in S_\alpha \cap S_{\beta\alpha}$, so $\beta \geq \alpha$. Thus, when $\alpha$ is maximal in $Y$ we have that $[ss^{-1}] \subseteqq S_\alpha$ for all $s \in S_\alpha$. It follows that $e_\alpha S_\alpha = S_\alpha$, so that $S_\alpha$ is a group by Lemma 2.3. □

2.6. Lemma. *Let* $s = \bigcup_{\alpha \in Y} S_\alpha$ *be as in Lemma* 2.3. *Then* $[ss^{-1}] \cap \{e_\alpha | \alpha \in Y\} \neq \emptyset$ *for all* $s \in S$. *In particular* $l^1(S)$ *is unital.*

*Proof*. First note that, if $u \in [ss^{-1}]$, then $[uu^{-1}] \subseteqq [ss^{-1}]$. Let $s \in S$ and let $S_{\alpha_0}$ be the archimedean component of $s$. Put $u_0 = s$ and choose successively $u_k \in [u_{k-1}u_{k-1}^{-1}]$. Let $S_{\alpha_k}$ be the archimedean component of $u_k$. As noted in the proof of Lemma 2.5 we have $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq \cdots$. Since $\operatorname{card} Y < \infty$, we eventually have $S_{\alpha_k} = S_{\alpha_{k+1}}$, whence $[u_k u_k^{-1}] \cap S_{\alpha_k} \neq \emptyset$, so that $e_{\alpha_k} \in [u_k u_k^{-1}]$ by Lemma 2.5. As observed in the beginning of the proof $e_{\alpha_k} \in [ss^{-1}]$. From [**5**, Theorem 7.5] it follows that $l^1(S)$ has a unit. □

We are now able to prove:

2.7. Theorem. *Let* $S$ *be a commutative semigroup. Then* $l^1(S)$ *is amenable if and only if* $S$ *is a finite semilattice of commutative groups.*

*Proof*. The sufficiency has been noted in the introduction. Hence we assume that $l^1(S)$ is amenable. Let $s = \bigcup_{\alpha \in Y} S_\alpha$ be the decomposition as in Lemma 2.3. By Lemma 2.5 the theorem is true if $\operatorname{card} Y = 1$. We proceed by induction on $n = \operatorname{card} Y$. Assume that $n \geq 2$ and that the theorem is true for semigroups which are semilattices of archimedean semigroups with cardinality of the semilattice strictly less than $n$. Let $\alpha_0$ be the minimal element in $Y$. Let $\beta \in Y \setminus \{\alpha_0\}$, and define $T_\beta = \bigcup_{\alpha \geq \beta} S_\alpha$. As in the proof of Lemma 2.2, we see that $l^1(T_\beta)$ is amenable. Thus, by the induction hypothesis, we have that $S_\alpha$ is a group for $\alpha \in Y \setminus \{\alpha_0\}$. We finish the induction step by proving that $S_{\alpha_0} = e_{\alpha_0} S_{\alpha_0}$. To this end, define a congruence $\sim$ on $S$ by

$$s \sim t \Leftrightarrow Ss = St \qquad (s, t \in S).$$

Note that, if $s \sim t$, then $s \in St$, since $[ss^{-1}] \neq \emptyset$. Using that $S_\alpha$ is a group for $\alpha \neq \alpha_0$, we see that $S/{\sim} \cong \bigcup_{\alpha \neq \alpha_0} \{e_\alpha\} \cup S_{\alpha_0}/{\sim}$. Hence $l^1(S_{\alpha_0}/{\sim})$ is (isomorphic to) a closed ideal of finite codimension in the amenable Banach algebra $l^1(S/\sim)$, and therefore $l^1(S_{\alpha_0}/\sim)$ is itself amenable [2, Theorem 4.1]. From Lemma 2.5 we get that $S_{\alpha_0}/\sim$ is a group. In particular we have for all $s \in S_{g a_0}$ that $s \sim e_{\alpha_0} s$, so, by the note above, $S_{\alpha_0} \subseteqq e_{\alpha_0} S_{\alpha_0}$. The induction step is hereby completed. □

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