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SIMPLE PERIODIC MODULES OF TWISTED CHEVALLEY GROUPS

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Consider a finite twisted Chevalley group constructed over a field of prime characteristic and its representations over an algebraically closed field of the same characteristic. In this paper we classify all those irreducible representations that are periodic, i.e., that have a periodic projective resolution. There is always the Steinberg module that is both simple and projective. We show that there are further periodic simple modules only for groups of types $^2A_2$ and $^2B_2$.

0. Introduction. Let K be an algebraically closed field of characteristic $p > 0$ and Γ a finite group with $p \mid |\Gamma|$. A finite dimensional $K\Gamma$-module M is called periodic, if it has a periodic minimal projective resolution. Of course projective modules are also periodic, but the non-projective ones are characterized by their complexity $c_\Gamma(M)$ being one. Here we understand complexity in the sense of Alperin [**1**].

In this article we look at the case where Γ is an "almost simple" twisted group of Lie type, defined over a finite field of the same characteristic p. We will give a classification of all periodic simple modules for these groups and the finite simple groups related to them.

It is a well-known fact that all finite groups of Lie type have a unique simple projective module, the Steinberg module St. We will prove in this paper that, if Γ is not of type $^2A_2$ or $^2B_2$, then the only periodic simple $K\Gamma$-module is St. With the classification for $^2A_2$ in [**6**], [**7**], and for $^2B_2$ in part 3 of this paper, we will achieve a full classification of simple periodic modules for twisted groups.

The corresponding classification for non-twisted Chevalley groups was recently accomplished in [**9**], [**10**]. Together with the initial classification for the type A_1 in [**15**], the final result is that for finite groups of Lie type there are simple periodic modules, other than St, only for groups of type A_1, $^2A_2$ and $^2B_2$.

To state our results more precisely, we have to introduce some notations which we take over from [**5**]. For conceptual reasons we prefer working with universal groups:

So let K be an algebraic closure of the prime field $\mathbb{F}_p$, let G be an almost simple, simply connected and connected affine algebraic group over K, defined and split over $\mathbb{F}_p$. Let F be a Frobenius endomorphism of G, as in [5], T an F-stable maximal torus of G with character group $X(T)$ and $R \subset X(T)$ be the (indecomposable) root system of G. We choose a basis $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_l\}$ of R and assume F to induce a non-trivial automorphism τ of the Dynkin diagram of G. So either τ is of order 3 and R of type D_4 or τ is of order 2 and R of type A_l $(l > 1)$, D_l $(l > 3)$, E_6, or B_2, G_2, F_4 (in which cases p must be $2, 3, 2$ respectively).

Attached to F are natural numbers $i, e \in \mathbb{N}$ such that $F^i(t) = t^{p^e}$ for all $t \in T$ and the number $q := p^{e/i}$ is uniquely determined by G and F. The group G^F of F-fixed points is either a finite (universal) Steinberg group (usually denoted by ${}^3D_4(q^3)_{\text{s.c.}}$, ${}^2A_l(q^2)_{\text{s.c.}}$, ${}^2D_l(q^2)_{\text{s.c.}}$ or ${}^2E_6(q^2)_{\text{s.c.}}$) or a Ree or Suzuki group (usually denoted by ${}^2B_2(q^2)$, ${}^2F_4(q^2)$, ${}^2G_2(q^2)$ where $q^2 = 2^{2m+1}$ resp. $q^2 = 3^{2m+1}$ for some m).

Excluding the cases (a) ${}^2A_2(4)_{\text{s.c.}}$, (b) ${}^2B_2(2)_{\text{s.c.}}$, (c) ${}^2G_2(3)_{\text{s.c.}}$ and (d) ${}^2F_4(2)_{\text{s.c.}}$, their quotient by the center is simple (see [19], Theorem 34, pg. 188) and is usually denoted by ${}^3D_4(q^3)$, ${}^2A_l(q^2)$ etc. (or by ${}^3D_4(q), {}^2A_l(q)$, according to the taste of the author.)

The groups in (a) and (b) are solvable, whereas in (c) and (d) their commutator subgroups have index 3 and 2 and are simple non-abelian finite groups ($\cong \mathrm{PSl}_2(8)$ in (c), $\cong$ Tits' simple group in (d)) ([19], pg. 188).

In this framework the main theorem of this article reads as follows:

0.1. THEOREM. *Let Γ be G^F, $G^F/Z(G^F)$ or $(G^F)'$, then all periodic simple $K\Gamma$-modules are projective if and only if R is not of type A_2 or B_2, in which case they are isomorphic to $\mathrm{St}|_{G^F}$ or to a Clifford component of it, if $(G^F)' < G^F$.*

In §1 we will reduce the problem for the Steinberg groups to the case $G^F \cong {}^2A_3(q^2)_{\text{s.c.}} \cong \mathrm{SU}_4(q^2)$, which will be dealt with in §2. Factoring out the center, which does not affect the Sylow-p-structure, leads to the result for the simple groups. Section 3 is devoted to the Ree and Suzuki groups.

1. The reduction to SU_4. In this section we exclude the cases B_2, G_2 and F_4 but keep the previous notations. So all roots in R have the same length. We assume moreover T to be defined and split over $\mathbb{F}_p$. So the usual Frobenius endomorphism $x \mapsto x^p$ on K gives rise to an endomorphism F_0 of G, acting on T via $F_0(t) = t^p$ for all $t \in T$. There is an automorphism of G that has the same order as τ, commutes with F, stabilizes T and induces on $\Pi \subseteq X(T)$ the same map as τ. If this automorphism is also denoted by τ, then we can write $F = \tau \circ F_0^n = F_0^n \circ \tau$ for some integer $n > 0$; in this case $q = p^n$. We denote the fundamental dominant weights of R by λ_α and set $\rho := \sum \lambda_\alpha$ where the sum is over all simple roots. Let $X(T)^+$ be the set of dominant weights and set for all $n > 0$:

$$(1.1) \qquad X_n(T) := \left\{\lambda = \sum r_\alpha \lambda_\alpha \in X(T) | 0 \leq r_\alpha < p^n \text{ for all } \alpha \in \Pi\right\}.$$

For any $\lambda \in X(T)^+$ let $V(\lambda)$ be the Weyl module for G with highest weight λ and $L(\lambda)$ its unique simple quotient. One has $V((p^n - 1)\rho) = L((p^n - 1)\rho)$ for all n; this module is often called the n th Steinberg module and denoted by St_n.

For any G-module M and any $r \in \mathbb{N}$ let $M^{(r)}$ denote the G-module got from M by twisting the G-action with F_0^r. If $\mu = \sum p^i \mu_i$ where i ranges from 0 to $n-1$ and $\mu_i \in X_1(T)$, then

$$(1.2) \qquad L(\mu) \cong L(\mu_0) \otimes L(\mu_1)^{(1)} \otimes \cdots \otimes L(\mu_{n-1})^{(n-1)}$$

by Steinberg's tensor product theorem **[18]**.

According to another theorem of Steinberg **[18]** the $L(\lambda)$ with $\lambda \in X_n(T)$ remain irreducible when regarded as a representation of G^F, and each simple KG^F-module is isomorphic to exactly one $L(\lambda)$ with $\lambda \in X_n(T)$.

The n th Steinberg module $\mathrm{St}_n = L((p^n - 1)\rho)$ is known to be projective as a KG^F-module. So in the case of the Steinberg groups our main theorem amounts to:

(1.3) THEOREM. *Suppose that G is a Steinberg group not of type A_2. Then $L(\lambda)$ is not a periodic KG^F-module for any $\lambda \in X_n(T)$, $\lambda \neq (p^n - 1)\rho$.*

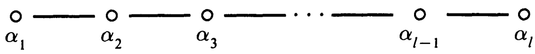
Using the corresponding results for non-twisted groups of type A_l, $l > 1$ (proved in **[9]**) we reduce the proof of this theorem to the case of ${}^2A_3$ (i.e. to the groups $\mathrm{SU}_4(q^2)$) that will be dealt with in §2.

(1.4) PROPOSITION. *If Theorem 1.3 holds in the case ${}^2A_3$, then it holds in all cases.*

Proof. Suppose that $L(\lambda)$ is periodic. Then for any subgroup H of G^F any direct summand M of $L(\lambda)$ as a KH-module is periodic. So we prove the proposition by constructing in each case some H and M such that M is not periodic for KH.

Here and in the following we will abbreviate a weight $\lambda = \sum r_i \lambda_{\alpha_i}$ by $(r_1, r_2, \dots, r_l)$, where $\Pi = \{\alpha_1, \dots, \alpha_l\}$.

Case 1. Type A_l, $l > 3$. Here the Dynkin diagram is:



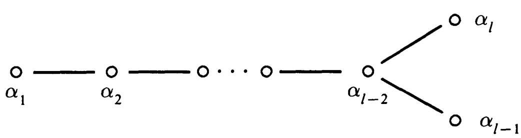
We consider $G^F \cong \mathrm{SU}_{l+1}(q^2)$ with respect to the hermitian form given by the unit matrix.

Then $c_1: g \mapsto \left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $c_2: g \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & g \end{smallmatrix}\right)$ define two canonical embeddings of $\mathrm{SU}_l(q^2) \to \mathrm{SU}_{l+1}(q^2)$. If L_J denotes the Levi subgroup for $J \subseteq \Pi$ in G, then the image $\operatorname{Im} c_1$ is contained in the derived group $L'_{\Pi\setminus\{\alpha_l\}} := L_1$ and $\operatorname{Im} c_2 \subseteq L'_{\Pi\setminus\{\alpha_1\}} := L_2$; hence we can apply a theorem of Smith **[17]**:

If $\lambda = (r_1, \dots, r_l) \in X_n(T)$ then $L(\lambda)|_{L_i} = L(\lambda_i) \oplus M_i$ where $L(\lambda_i)$ is an irreducible module for L_i with highest weight λ_i and M_i is some other L_i module. We get $\lambda_1 = (r_1, r_2, \dots, r_{l-1})$ and $\lambda_2 = (r_2, r_3, \dots, r_l)$. See also **[16]** for a proof of these facts.

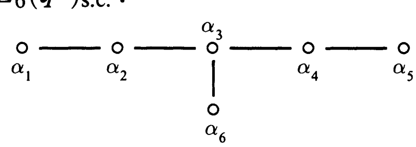
Furthermore L_i is simply connected of type A_{l-1}. If $\lambda \neq (p^n - 1)\rho$, then we may assume without loss of generality that $\lambda_1 \neq (p^n - 1)(\rho - \lambda_{\alpha_l})$. Then $L(\lambda_1)$ is a simple periodic and non-projective $\mathrm{SU}_l(q^2)$ module in contradiction to the induction hypothesis.

Case 2. Type ${}^2D_l(q^2)_{\text{s.c.}}$



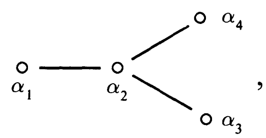
Let L_i be the derived group of the Levi subgroup corresponding to J_i, where $J_i = \{\alpha_i, \alpha_{i+1}\}$ for $i < l-2$ and $J_{l-2} = \{\alpha_{l-2}, \alpha_{l-1}, \alpha_l\}$. Each L_i is F-stable and simply connected; one has $L_i^F \cong A_2(q)_{\text{s.c.}}$ for $i < l-2$ and $(L_{l-2})^F \cong {}^2A_3(q^2)_{\text{s.c.}}$. If again $\lambda = (r_1, r_2, \dots, r_l)$, then by Smith's theorem $L(\lambda)|_{L_i^F} = L(\lambda_i) \oplus M_i$ as above with $\lambda_i = (r_i, r_{i+1})$ for $1 \le i \le l-3$, and $\lambda_{l-2} = (r_{l-2}, r_{l-1}, r_l)$. Again, since $L(\lambda) \neq \mathrm{St}_n$, one of these $L(\lambda_i)$'s must be non-projective and hence non-periodic for L_i^F, by the corresponding result for $A_2(q)$ in **[9]** and the hypothesis.

Case 3. Type ${}^2E_6(q^2)_{\text{s.c.}}$:



The same kind of argument applies here with $J_1 = \Pi\backslash\{\alpha_6\}$, $J_2 = \{\alpha_3, \alpha_6\}$ and $(L_{J_1})^F \cong {}^2A_5(q^2)$, $(L_{J_2})^F \cong A_2(q)$, using Case 1 and the result in **[9]**.

Case 4. Type ${}^3D_4(q^3)_{\text{s.c.}}$:


$$R = \{\pm(\varepsilon_i \pm \varepsilon_j),\ i \neq j\}.$$

The Weyl group $W(G, T)$ acts on $\mathbb{Z}^4$ as permutations of the $\pm\varepsilon_i$'s with an even number of sign changes.

If $\alpha_1 := \varepsilon_1 - \varepsilon_2$, $\alpha_2 := \varepsilon_2 - \varepsilon_3$, $\alpha_3 := \varepsilon_3 - \varepsilon_4$, $\alpha_4 := \varepsilon_3 + \varepsilon_4$, then the τ-fixed points α_2, $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = \varepsilon_1+\varepsilon_3 =: \beta$, $\alpha_1+2\alpha_2+\alpha_3+\alpha_4 = \varepsilon_1+\varepsilon_2$, form an A_2-subsystem of R. Moreover W maps

$$(\alpha_2, \beta) \mapsto (\alpha_2, \alpha_1) \quad \text{under} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & -3 & -2 & 4 \end{pmatrix}$$

$$(\alpha_1, \alpha_2) \mapsto (\alpha_2, \alpha_3) \quad \text{under} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$(\alpha_1, \alpha_2) \mapsto (\alpha_2, \alpha_4) \quad \text{under} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & -4 & -1 \end{pmatrix}$$

If U_α denotes the root subgroup of G, then U_{α_2}, U_β are F-stable and $U_{\alpha_2}^F$, $U_\beta^F \cong \mathbb{F}_q^+$. Let H be the subgroup $\langle U_{\pm\alpha_2}, U_{\pm\beta}\rangle \leq G$; then H is of type A_2 and H^F of type $A_2(q)$. For $i = 1, 3, 4$ let $J_i = \{\alpha_i, \alpha_2\}$; hence $L'_{J_i} =: L_i$ is of type A_2 but not F-stable. Let now $L(\lambda) \neq \mathrm{St}_n$, with $\lambda = (r_1, r_2, r_3, r_4)$.

Then $L(\lambda)|_{L_i} = L(\lambda_i) \oplus M_i$, where $\lambda_i = (r_2, r_1)$ resp. (r_1, r_2).

For each i there is some $w_i \in W$ such that $w_i(H) = w_i H w_i^{-1}$ is contained in L_i. Since $L(\lambda)|_{H^F}$ is periodic, also the restrictions $L(\lambda)|_{w_i(H^F)}$ must be periodic. This can be seen for instance looking at the rank variety in the sense of **[3]**, **[4]** and the fact that rank varieties of conjugate groups are isomorphic. Since $w_i(H)$ is contained in L_i, also $L(\lambda_i)|_{w_i(H^F)}$ is periodic. But $w_i(H^F) \cong A_2(q)$ and $r_i < q$, so $L(\lambda_i)$ is also irreducible for $w_i(H^F)$, which forces r_i and r_2 to be $q-1$ by the result in **[9]**. This gives a contradiction. □

2. The Case SU_4. Keep all assumptions and notations from §1 and assume that R is of type A_3.

(2.1) LEMMA. *Let* $\mu = \sum_{l=0}^{n-1} p^l \mu_l \in X_n(T)$ *with all* $\mu_l \in X_1(T)$. *If* $L(\mu)$ *is a periodic* KG^F*-module, then each* μ_l *has one of the following forms*:

(a) $(p-1, p-1, p-1) = (p-1)\rho$,
(b) $(p-1, p-1, i)$ *with* $0 \le i < p-1$,
(b′) $(i, p-1, p-1)$ *with* $0 \le i < p-1$,
(c) $(p-1, i, j)$ *with* $0 \le i, j, i+j = p-2$,
(c′) $(i, j, p-1)$ *with* $0 \le i, j, i+j = p-2$,
(d) $(p-1, i, p-1)$ *with* $0 \le i < p-1$,
(e) $(i, p-1, j)$ *with* $0 \le i, j < p-1$,
(f) $(p-2-i, i, p-2-i)$ *with* $0 \le i < p-1$.

Proof. Suppose $\mu = (m_1, m_2, m_3)$. The same argument as in the proof of (1.4), Case 1 shows that $L((m_1, m_2))$ and $L((m_2, m_3))$ are periodic as $KSU_3(q^2)$-modules. The classification of these modules in [7] yields now the claim. □

(2.2) LEMMA. *Let* $\lambda \in X_1(T)$ *be of type* (d), (e), (f) *in Lemma* (2.1). *Then* $\dim L(\lambda)$ *is not divisible by* p^3.

Proof. The formulae for the characters of the $L(\lambda)$ in **[12]** §7 imply:

$$\dim L(p-1, i, p-1) = \dim V(p-1, i, p-1) - \dim V(p-(i+2), i, p-(i+2)) \quad \text{for } 0 \le i < p-1;$$

$$\dim L(i, p-1, j) = \dim V(i, p-1, j) - \dim V(p-(j+2), p-1, p-(i+2)) + \dim V(p-(i+2), i+j+1-p, p-(j+2)) \quad \text{for } 0 \le i, j < p-1, i+j > p-2;$$

$$\dim L(i, p-1, j) = \dim V(i, p-1, j) \quad \text{for } 0 \le i, j, i+j = p-2;$$

$$\dim L(i, p-1, j) = \dim V(i, p-1, j) - \dim V(j, p-(i+j+3), i) \quad \text{for } 0 \le i, j, i+j < p-2;$$

$$\dim L(p-2-i, i, p-2-i) = \dim V(p-2-i, i, p-2-i) \quad \text{for } 0 \le i < p-1.$$

By Weyl's dimension formula

$$\dim V(r, s, t) = (r+1)(s+1)(t+1)(r+s+2)(s+t+2)(r+s+t+3)/12.$$

In each case one sees immediately that p^3 does not divide $\dim V(\mu)$ for any μ. In the case $\lambda = (p-1, i, p-1)$ one gets

$$\begin{aligned} \dim L(\lambda) &\equiv 2\dim V(\lambda) \not\equiv 0 \pmod{p^3} \quad \text{for } p \geq 5, \\ \dim L(\lambda) &\equiv 2\dim V(\lambda) \not\equiv 0 \pmod{p^2} \quad \text{for } p = 3, \\ \dim L(\lambda) &= 14 \not\equiv 0 \pmod{8} \quad \text{for } p = 2. \end{aligned}$$

The other cases are similar and are left to the reader. □

Recall the definition of the rank variety $V_E M$ of a KE-module M for any elementary abelian p-group E as in [3] or [4]. So it is a homogeneous subvariety of the vector space K^m where m is the rank of E. It is called linear, if it is a linear subspace of K^m.

(2.3) LEMMA. *Let E be an elementary abelian p-subgroup of $G^F = \mathrm{SU}_4(q^2)$ and let $\mu \in X_1(T)$ be of type* (b) *or* (c) *in Lemma* (2.1). *Then*

$$V_E L(\mu) = V_E L((p-1, 0, p-2)).$$

Proof. Set $C^* = \{(r_1, r_2, r_3) \in X(T) | -1 \leq r_1, r_2, r_3, r_1+r_2+r_3 \leq p-3\}$. Then C^* is a fundamental domain for the "dot" operation (i.e., $w \cdot \mu = w(\mu+\rho) - \rho$) of W_p on X, where W_p is the affine Weyl group (as in [14], II, 6.1). So there is a unique $\mu' \in W_p \cdot \mu \cap C^*$ and a unique $\lambda' \in W_p \cdot \lambda \cap C^*$ where $\lambda = (p-1, 0, p-2)$; in fact $\lambda' = (-1, -1, 0)$.

We want to use the translation functors $T_{\lambda'}^{\mu'}$ and $T_{\mu'}^{\lambda'}$, cf. [14], II, 7.6. More precisely, we shall show that $T_{\lambda'}^{\mu'} L(\lambda) \cong L(\mu)$ and $T_{\mu'}^{\lambda'} L(\mu) \cong L(\lambda)$. As $T_{\lambda'}^{\mu'} L(\lambda)$ is a direct summand of $L(\lambda) \otimes M$ for a suitable finite dimensional G-module M, and $T_{\mu'}^{\lambda'} L(\mu)$ a direct summand of $L(\mu) \otimes M^*$, elementary properties of the rank variety (cf. [3]) yield $V_E L(\lambda) = V_E L(\mu)$.

If μ is of type (c) in Lemma (2.1), then λ and μ belong to the same facet for W_p. In this case $T_{\lambda'}^{\mu'} L(\lambda) \cong L(\mu)$ and $T_{\mu'}^{\lambda'} L(\mu) \cong L(\lambda)$ are special cases of [14], II, 7.15.

Suppose now that μ is of type b) in Lemma (2.1), i.e. $\mu = (p-1, p-1, i)$ for some i, $0 \leq i < p-1$. Then $\mu' = (-1, i, p-(i+2))$. One has $V(\lambda) = L(\lambda)$ and $V(\mu) = L(\mu)$ by [11], p. 120 or by [12].

We can compute the formal characters of $T_{\lambda'}^{\mu'} L(\lambda)$ and $T_{\mu'}^{\lambda'} L(\mu)$ using **[14]**, II, 7.8. Let $s_i = s_{\alpha_i}$ be the reflection with respect to α_i (for $i = 1, 2, 3$) and let $s \in W_p$ be the affine reflection with $s(r, s, t) = (r+s+t-p, p-t, p-s)$ for all $(r, s, t) \in X(T)$. Then the stabilizer of μ' (under the dot action) in W_p consists of $\{1, s_1, s, s_1 s, s s_1, s_1 s s_1\}$, that of λ' of $\{1, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$. The intersection of these groups is $\{1, s_1\}$, a system of representatives modulo this intersection is $\{1, s_2, s_1 s_2\}$ in $\mathrm{Stab}(\lambda')$ and $\{1, s, s_1 s\}$ in $\mathrm{Stab}(\mu')$.

The element $w \in W_p$ with $w(r, s, t) = (p-r, r+s+t, p-t)$ has the property that $\mu = w \cdot \mu'$ and $\lambda = w \cdot \lambda'$. One has

$$\begin{aligned} w s_2 \cdot \mu' &= (p - (i+2), p-1, -1), \\ w s_1 s_2 \cdot \mu' &= (p+i, p-(i+2), -1). \end{aligned}$$

These weights are not dominant, but become dominant after adding $\rho = (1, 1, 1)$. Therefore they contribute 0 to the sum in **[14]**, II, 7.12, and we get $T_{\lambda'}^{\mu'} L(\lambda) = T_{\lambda'}^{\mu'} V(\lambda) \cong V(\mu) \cong L(\mu)$. Furthermore

$$\begin{aligned} w s \cdot \lambda' &= (2p-2, p-1, -1), \\ w s_1 s \cdot \lambda' &= (0, 2p-2, -1). \end{aligned}$$

The same argument as above implies

$$T_{\mu'}^{\lambda'} L(\mu) = T_{\mu'}^{\lambda'} V(\mu) \cong V(\lambda) = L(\lambda). \qquad \square$$

For $\alpha \in R$, let $U_\alpha = \{x_\alpha(t) | t \in K\}$ be the root subgroup of G with root homomorphism $x_\alpha \colon K \mapsto G$. We set:

$$H := U_{\alpha_1} \circ U_{\alpha_1+\alpha_2} \circ U_{\alpha_2+\alpha_3} \circ U_{\alpha_1+\alpha_2+\alpha_3} \cong (K^+)^4.$$

If $\{u_1, u_2, \dots, u_n\}$, $\{v_1, v_2, \dots, v_{2n}\}$ denote bases of $\mathbb{F}_q$ and $\mathbb{F}_{q^2}$ over $\mathbb{F}_p$ respectively, then let $x_1(u_i) := x_{\alpha_2}(u_i)$,

$$x_2(v_j) := x_{\alpha_1+\alpha_2}(v_j) x_{\alpha_2+\alpha_3}(v_j^q), \quad x_3(u_k) := x_{\alpha_1+\alpha_2+\alpha_3}(u_k).$$

The group

$$E := H^F = \langle x_1(u_i) x_2(v_j) x_3(u_k) | 1 \le i, k \le n; 1 \le j \le 2n \rangle$$

is then an elementary abelian p-subgroup of rank $4n$ of $G^F = \mathrm{SU}_4(q^2)$.

(2.4) LEMMA. *Let* $\lambda \in X_1(T)$ *be one of the weights in Lemma* (2.1). *Then*

$$\dim V_E L(\lambda) \geq 4n - 2 \quad \textit{for } \lambda \neq (p-1)\rho,$$
$$\dim V_E L((p-1)\rho) \geq 4n - 4.$$

Proof. Let $d := \dim V_E L(\lambda)$; then by [3], Proposition 5.1, there exists a "shifted subgroup" $E^\frown \leq KE^*$, which is elementary abelian of order p^{4n-d} such that $L(\lambda)|_{E^\frown}$ is free. In particular p^{4n-d} divides $\dim L(\lambda)$, which proves the claim for the weights of type (d), (e) and (f) by (2.2).

By (2.3) we are left with the weights $(p-1, 0, p-2)$ and $(p-1)\rho$; in both cases $L(\lambda) = V(\lambda)$. Now let $J := \{\alpha_1, \alpha_2\} \subseteq \Pi$; then H can also be described as the unipotent radical U_J of the standard parabolic subgroup $P_J = U_J \rtimes L_J \leq G$. If $\lambda = (r, s, t)$, then by Smith's theorem $L(\lambda)|_{L_J} = L(\lambda)^{U_J} \oplus X$, where $L(\lambda)^{U_J}$ is the space of U_J-fixed points, which is also an irreducible module for $L'_J \cong A_1(K)_{\text{s.c.}} \times A_1(K)_{\text{s.c.}}$, with highest weight (r, t), hence has dimension $(r+1)(t+1)$.

Since $E \subseteq U_J$ and $E^\frown$ is generated by elements of the form $1+\sum a_i(x_i-1)$ with $a_i \in K$, $x_i \in E$, it is clear that $L(\lambda)^{U_J} \subseteq L(\lambda)^{E^\frown}$, which has dimension $\dim L(\lambda)/p^{4n-d}$ because $L(\lambda)$ is $E^\frown$-free. So we get $p^{4n-d} \leq \dim L(\lambda)/(r+1)(t+1)$. Now we apply Weyl's dimension formula and obtain $p^{4n-d} \leq p^2(p+1)/6$ for $\lambda = (p-1, 0, p-2)$ and $p^{4n-d} \leq p^4$ for $\lambda = (p-1)\rho$. □

(2.5) PROPOSITION. *If* M *is a periodic simple* $KSU_4(q^2)$*-module, then*

$$M \cong \mathrm{St}_n.$$

Proof. By [3] 5.6, $V_E(X \otimes Y) = V_E(X) \cap V_E(Y)$ for KE-modules X and Y. It is easy to see that $\dim V_E(L(\mu)^{(r)}) = \dim V_E(L(\mu))$, so if M ($\not\cong \mathrm{St}_n$) is written in the form of (1.2), then at least for one μ_i we must have $\dim V_E(L(\mu_i)^{(i)}) \geq 4n-2$ by (2.4). Since all rank varieties V_E are homogeneous affine subvarieties of K^{4n} the intersection formula for homogeneous varieties implies

$$\begin{aligned}\dim V_E M &= \dim \bigcap_{i=0}^{n-1} V_E(L(\mu_i)^{(i)}) \\ &\geq (n-1)(4n-4) + 4n - 2 - (n-1)4n \geq 2,\end{aligned}$$

which contradicts periodicity by [3], 7.6 and 8.1. □

REMARK. This proposition together with (1.4) implies Theorem (1.3), and it is clear, since $G^F = (G^F)'$ if type (a) of the exceptions is excluded, how to get (0.1).

3. The Ree and Suzuki groups. In this section we deal with the cases, where R is of type G_2, F_4 and B_2, which were excluded earlier. Here twisted groups of Lie type can exist only if the characteristic of K is 3, 2 and 2 respectively, and τ always has order 2, interchanging the long and short roots.

PROPOSITION (3.1). *Let R be G_2 or F_4, $\operatorname{char} K = 3$ or 2 respectively and $\Gamma = G^F$, $G^F/Z(G^F)$, or $(G^F)'$; then every periodic simple $K\Gamma$-module is projective, and either isomorphic to St_{G^F} or to a Clifford component of $\mathrm{St}_{G^F}|_\Gamma$ if $\Gamma = (G^F)' \triangleleft G^F$ and $\Gamma \neq G^F$.*

Proof. We first prove this for $\Gamma = G^F$. By a result of Steinberg **[18]**, §12, an analogue of (1.2) applies here, with $X_1(T)$ replaced by $X_1(T)' =: X_1' = \{\mu = \sum r_\alpha \lambda_\alpha \in X_1(T) | \alpha \in \Pi$ and $r_\alpha = 0$ if α is a long root$\}$.

Case 1. $G^F = {}^2G_2(q^2)_{\text{s.c.}}$ with $q^2 = 3^n$, $n = 2m+1$ and $\operatorname{char} K = 3$, $R = \{\pm\alpha, \pm(2\alpha + 3\beta), \pm(\alpha + 3\beta)$ (long roots), $\pm\beta, \pm(\alpha+\beta)$, $\pm(\alpha + 2\beta)$ (short roots)$\}$, and the Dynkin diagram is:

$$\alpha \circ\!\!\Rrightarrow\!\!\circ \beta$$

Here $X_1(T)' = \{(0,0), (0,1), (0,2)\}$ and $\dim L(\lambda) = 1, 7, 27$ for $\lambda \in X_1(T)'$ respectively. See **[18]**, §12.

Let $E = \{x(u, v) | u, v \in \mathbb{F}_{3^n}\}$ with

$$x(u, v) := x_{\alpha+3\beta}(u) x_{\alpha+\beta}(u^{3^m}) x_{2\alpha+3\beta}(v) x_{\alpha+2\beta}(v^{3^m}).$$

Then E is elementary abelian of rank $2n$. With regard to the dimensions it is enough to look at $\lambda = (0, 2)$. Then $L(\lambda)$ has the following "monomial basis" with highest weight vector v_λ^+:

$$\{X^{n_1}_{-(\alpha+\beta)} X^{n_2}_{-(\alpha+2\beta)} X^{n_3}_{-\beta} \circ v_\lambda^+ | 0 \leq n_i \leq 2\},$$

where $X_{-\gamma}$ are root vectors in $\mathrm{Lie}(G)$.

If γ is any positive root, $x_\gamma\colon K \to G$ the corresponding root homomorphism and V_μ a weight subspace to the weight μ of some G module V, then one knows:

$$x_\gamma(t) \circ v_\mu \in \bigoplus_{i \in \mathbb{N}} V_{\mu + i\gamma}.$$

Since λ is the highest weight of $L(\lambda)$ and E consists of unipotent elements, E must fix the three basis vectors v_λ^+, $X_{-\beta}v_\lambda^+$, $X_{-\beta}^2 v_\lambda^+$. If $d := \dim V_E(L(0, 2))$, then as in the proof of (2.4), a suitable shifted elementary abelian $E^\frown \leq (KE)^*$ with $|E^\frown| = 3^{2n-d}$ acts freely on $L(0, 2)$ and we get $\dim L(0, 2)/|E^\frown| = 3^{d-2n+3} \geq 3$ and hence $d \geq 2n-2$. By elementary properties of rank varieties, $V_E(L(\lambda)) \cong K^{2n}$ for $\lambda = (0, 0)$ or $(0, 1)$. The analogue of (1.2) together with the intersection theorem for affine varieties yields $\dim V_E(M) \geq 2$ for all $M \neq \mathrm{St}_{G^F}$.

Case 2. $G^F = {}^2F_4(q^2)$, $q^2 = 2^n$, $n = 2m+1$ and $\operatorname{char} K = 2$. Here $X_1(T)' = \{(0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 0, 1, 1)\}$ with dimensions for $L(\lambda)$: $1, 26, 246, 2^{12}$ respectively (see **[20]**). Let $\theta\colon K \to K, x \mapsto x^{2^m}$; we consider the following elements of R^+ ($\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$):

$$\begin{aligned}
\gamma_1 &:= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4; & \gamma_2 &:= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4;\\
\delta_1 &:= \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4; & \delta_2 &:= \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4;\\
\varepsilon_1 &:= \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4; & \varepsilon_2 &:= \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4;\\
\varphi_1 &:= \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4; & \varphi_2 &:= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4;\\
\psi_1 &:= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4; & \psi_2 &:= \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4.
\end{aligned}$$

Notice that $\gamma_1, \delta_1, \ldots$ are short and $\gamma_2, \delta_2, \ldots$ are long.
Let

$$\begin{aligned}
x_1(u_1) &:= x_{\gamma_1}(u_1) \circ x_{\gamma_2}(u_1^{2\theta}); & x_2(u_2) &:= x_{\delta_1}(u_2) \circ x_{\delta_2}(u_2^{2\theta});\\
x_3(u_3) &:= x_{\varepsilon_1}(u_3) \circ x_{\varepsilon_2}(u_3^{\theta}); & x_4(u_4) &:= x_{\varphi_1}(u_4) \circ x_{\varphi_2}(u_4^{\theta});\\
x_5(u_5) &:= x_{\psi_1}(u_5) \circ x_{\psi_2}(u_5^{2\theta}); & &\text{with } u_i \in \mathbb{F}_{2^n}.
\end{aligned}$$

Notice that these elements commute because of $\operatorname{char} K = 2$, and hence the group $E := \langle x_i(u_i) | i = 1, 2, 3, 4, 5;\, u_i \in \mathbb{F}_{2^n}\rangle$ is an elementary abelian subgroup of G^F of rank $5n$.

For $L(0, 0, 1, 1)$ there is a monomial basis, consisting of elements of the form: $X_{-\omega_1}^{n_1} \circ \cdots \circ X_{-\omega_{12}}^{n_{12}} v_\lambda^+$ with $n_i \in \{0, 1\}$, where $\{\omega_1, \omega_2, \ldots, \omega_{12}\}$ is the set of short positive roots.

The 7 short roots α_3, $\alpha_2+\alpha_3$, $\alpha_1+\alpha_2+\alpha_3$, α_4, $\alpha_3+\alpha_4$, $\alpha_2+\alpha_3+\alpha_4$, $\alpha_2+2\alpha_3+\alpha_4$ have not full support in Π. Since the generators of E are products of x_γ's where γ is a positive root with full support, an argument similar to that of Case 1 shows that basis vectors involving only monomials in those 7 roots have to be fixed by E and hence by any shifted subgroup $E^\frown$ of KE^*.

Since there are 2^7 such monomials, $\dim L(0,0,1,1)^{E^\wedge} \geq 2^7$. If $d = \dim V_E(L(0,0,1,1))$ then again we can choose $E^\wedge$ to act freely on $L(0,0,1,1)$ and to be of order 2^{5n-d}. So we get $\dim L(0,0,1,1)/|E^\wedge| = 2^{12-5+d} \geq 2^7$; hence $d \geq 5n-5$. Since all other $V_E(L(\lambda))$ for $\lambda \in X_1(T)'$ have dimension $\geq 5n-1$ (4 does not divide $\dim L(\lambda)$ for $\lambda \neq (0,0,1,1)$), the analogue of (1.2) yields:

$$\dim V_E(M) \geq 4 \quad \text{for all } M \neq \mathrm{St}_{G^F}.$$

Proof for $G^F/Z(G^F)$ and $G^{F'}$. As $Z(G^F)$ is a p'-subgroup of G^F the same results hold for $\Gamma = G^F/Z(G^F)$. Now let $G^{F'} \neq G^F$; then $m = 0$, $n = 1$ and $q^2 = |G^F/G^{F'}| = p = 3$ in Case 1 and $= 2$ in Case 2.

By Clifford's theorem $\mathrm{St}|_{G^{F'}}$ is completely reducible. Let V be a simple direct summand of $\mathrm{St}|_{G^{F'}}$, with inertia group $I := I_{G^F}(V) = \{g \in G^F | V^g \cong V \text{ as } G^{F'} \text{ modules}\}$. Then $I = G^{F'}$. Otherwise by **[8]** §9, (9.9), $V = W|_{G^{F'}}$ for a simple G^F module W, which is isomorphic to St_{G^F}, since V is projective; but then W had to be a direct summand of the induced module V^{G^F} since W is projective, which contradicts the indecomposability of V^{G^F} given by Green's theorem **[8]** §16. So $\mathrm{St}|_{G^F} \cong \bigoplus_{g\in G^F/G^{F'}} V^g$ and we claim that each periodic simple $G^{F'}$ module is isomorphic to one of these V^g's.

Let S be an irreducible periodic module for $G^{F'}$. If $I = I_{G^F}(S) = G^F$ then again $S = W_{G^{F'}}$ for a simple G^F module W. In Case 1, $E \subseteq G^{F'}$ and in Case 2 $E \cap G^{F'}$ has rank ≥ 4; hence 3 or 2^3 divides $\dim W$ respectively which forces W to be $\cong \mathrm{St}_{G^F}$, leading to a contradiction. Hence $I = G^{F'}$ which implies that S^G is simple periodic for G^F. So $S^G \cong \mathrm{St}_{G^F}$ and $S \cong V^g$ for a $g \in G^F$ by Frobenius reciprocity. In particular S is projective. □

We conclude the treatment of twisted groups of Lie type, by computing the rank varieties of simple modules for $G^F = {}^2B_2(q^2)_{\text{s.c.}}$, which includes a classification of periodic simple modules, i.e. those whose variety has dimension 1.

Let $q^2 = 2^n$, $n = 2m+1$, $\operatorname{char} K = 2$, $\theta := (?)^2$, $\Pi = \{\alpha, \beta\}$. The Dynkin diagram is $\circ\!\!=\!\!\!=\!\!\circ$. Let $P := \{x(t,u) | t, u \in \mathbb{F}_{2^{2m+1}}\} \in \mathrm{Syl}_2(G^F)$, with $x(t,u) = x_\alpha(t)x_\beta(t^\theta)x_{\alpha+2\beta}(u)x_{\alpha+\beta}(t^{1+\theta}+u^\theta)$; then $|P| = 2^{4m+2}$ and $Z(P) = \{x(0,u)\}$ is maximal elementary abelian;

let $D := Z(P)$. Notice that there is only one conjugacy class of maximal elementary abelian 2-subgroups in G^F. Here $X_1(T)' = \{(0, 0), (0, 1)\}$ with dimensions 1 and 4 respectively.

(3.2) PROPOSITION. *Let* $G^F = {}^2B_2(q^2)_{\text{s.c.}}$ *and* M *be an irreducible* KG^F*-module; then* $V_E(M)$ *is linear and* $\dim V_E(M) = n - s$, *where* s *is the number of factors* $\lambda_i = (0, 1)$ *in the tensor product*

$$M = \bigotimes_{i=0}^{n-1} L(\lambda_i)^{(i)}, \qquad \lambda_i \in X_1(T).$$

Proof. We need only to consider $L(0, 1)$ which has a basis $B = \{v^+, X_{-\beta}v^+, X_{-(\alpha+\beta)}v^+, X_{-\beta}X_{-(\alpha+\beta)}v^+\}$. A straightforward computation shows, that

$$\begin{aligned} &x(0, u)v^+ = v^+; \quad x(0, u)X_{-\beta}v^+ = X_{-\beta}v^+; \\ &x(0, u)X_{-(\alpha+\beta)}v^+ = u^\theta v^+ + X_{-(\alpha+\beta)}v^+ \\ &x(0, u)X_{-\beta}X_{-(\alpha+\beta)}V^+ = uv^+ + u^\theta X_{-\beta}X_{-(\alpha+\beta)}V^+. \end{aligned}$$

Hence $x(0, u)$ is represented with respect to B by

$$\begin{bmatrix} 1 & & & \\ 0 & 1 & \text{0} & \\ u^\theta & 0 & 1 & \\ u & u^\theta & 0 & 1 \end{bmatrix}$$

Let now $\{u_i\}$ be an $\mathbb{F}_2$-basis of $\mathbb{F}_{2^n}$, then $\{x(0, u_i) =: x_i\}$ is a basis of E. For $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in K^n$ we define $u_{\mathbf{a}} := 1 + \sum a_i(x_i - 1)$. Then we get, using Lemma 4.2 of [**3**]:

$$V_E(L(0, 1)) = \{\mathbf{a} \in K^n | \operatorname{rk} u_{\mathbf{a}} - 1 < \tfrac{1}{2}4\} = \left\{\mathbf{a} \in K^n | \sum a_i u_i^\theta = 0\right\}.$$

The analogue of (1.2) now gives the result. □

REMARK. It may be interesting to notice that non-projective simple periodic modules only occur in finite groups of Lie type, if the group has a split BN-pair of rank one. But ${}^2G_2$ shows, they need not necessarily occur in this case.

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