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DIFFERENTIAL GEOMETRY OF SYSTEMS OF PROJECTIONS IN BANACH ALGEBRAS

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Let A be a Banach algebra, n a positive integer and $Q_n = \{(q_1, \dots, q_n) \in A^n : q_i q_k = \delta_{ik} q_i, q_1 + \cdots + q_n = 1\}$. The differential geometry of Q_n, as a discrete union of homogeneous spaces of the group G of units of A is studied, a connection on the principal bundle $G \to Q_n$ is defined and invariants of the associated connection on the tangent bundle TQ_n are determined.

Introduction. The structure of the set Q of all idempotent elements of a Banach algebra A plays a fundamental role in several aspects of spectral theory. This work deals with the differential structure of the space

$$Q_n = \left\{ (q_1, \dots, q_n) \in A^n : q_i q_k = \delta_{ik} q_i, \sum_{i=1}^{n} q_i = 1 \right\}$$

of systems of n "orthogonal" projections in A.

The manifold Q_n appears as a universal model when certain polynomial equations are considered. More precisely, if $\alpha_1, \dots, \alpha_n$ are *different* complex numbers and $\alpha(X)$ denotes the polynomial $(X - \alpha_1) \cdots (X - \alpha_n)$, then the set $A_\alpha = \{a \in A : \alpha(a) = 0\}$ is a closed submanifold which is diffeomorphic to Q_n. Thus Q_n is the model for all simple algebraic elements of A of degree n. Moreover, Q_n plays a role in the study of arbitrary algebraic (in particular, nilpotent) elements (see [**AS**]).

Section 1 contains the description of the differential structure of Q_n and A_α as closed analytic submanifolds of A^n and A, respectively; it contains also the proof that Q_n and A_α are diffeomorphic.

Using Kaplansky's notion of SBI-rings, we recover a result of Barnes [**Ba**] concerning the surjectivity of $A_\alpha \to B_\alpha$ when B is the quotient of A by its Jacobson radical. In §2 we show that Q_n is a discrete union of homogeneous spaces of G, the group of units of A; this fact, together with a classical result of Michael [**Mi**], shows that an epimorphism $f : A \to B$ of Banach algebras induces Serre fibrations $Q_n(A) \to Q_n(B)$ and $A_\alpha \to B_\alpha$. In §3 we obtain an explicit way of lifting differentiable curves in Q_n to G by solving a linear differential equation which we call the *transport equation*; this fact is due to Daleckii and S. G. Krein **[DK]** and T. Kato **[Ka1]** but its geometrical meaning is new. In fact, in §4 we define a connection in the principal bundle $G \to Q_n$ and show that the horizontal liftings of differentiable curves in Q_n are precisely the solutions of the transport equation.

Several invariants of the tangent bundle of Q_n are calculated in §5 (covariant derivative, curvature, geodesics, etc.). As observed by Kato **[Ka1]**, **[Ka2**, II.4] the lifting theorem has important applications in quantum mechanics (see **[Ga]**, **[GS]**). A remark about C^*-algebras is in order: our results extend to the case of some involution algebras, in particular to all C^*-algebras. For instance, the transport equation has a unitary solution if the curve has selfadjoint values; in a forthcoming paper the immersion of

$$P_n = \{p \in Q_n : p_i^* = p_i, \ i = 1, \dots, n\}$$

into Q_n will be studied, together with associated fibrations $Q_n \to P_n$.

Concerning the references, the reader may consult Rickart's book **[Ri]** for the literature up to 1960; the topology of the space of idempotents $Q = Q_2$ has been considered in **[PR1]**, **[Ra]**, **[Ko]**, **[Ze]**, **[Au]**, **[Gr]** and with special emphasis on the differential struture of Q in **[Ra]**, **[Gr]**, **[Ki]**, **[HK]**; for the transport equation the reader may consult **[Ka1]** and **[DK2]**; in **[PR2]** the differential geometry of $P = P_2$ is needed for the study of minimality of geodesics; see also **[CPR2]** for a related problem; finally, the case of algebraic operators on Hilbert space, the reader may consult the books **[He]** and **[AFVH]**. In particular, some problems concerning the set P_n in this context are discussed in **[CH]**. The set Q_n appears, implicitly or explicitly, in various works; we only mention **[Ja**, p. 54], **[Ka2**, II.5] and **[DK2**, Chapter IV].

1. Differential structure of systems of projections. Let A be a real or complex algebra with identity 1. Denote by $G = G(A)$ the group of units of A and by $Q = Q(A)$ the set of all idempotents of A.

Suppose that the polynomial $\alpha(X) = \prod_{i=1}^n (X - \alpha_i)$ has different roots $\alpha_1, \dots, \alpha_n$ in the field. Let $g_j(X) = \prod_{i \neq j}(X - \alpha_i)$ and $q_j(X) = g_j(X)/g_j(\alpha_j)$. Then $q_j(X)$ has degree $n-1$, $q_j(\alpha_i) = \delta_{ji}$, for $i \neq j$ $q_i(X)q_j(X) = h(X)\alpha(X)$ for some polynomial $h(X)$ and $\sum_{i=1}^n q_i(X) = 1$ (because $1 - \sum_{i=1}^n q_i(X)$ has degree $\leq n-1$ and it vanishes at n values, the α_j).

Let A_α denote the solution set of α, i.e., the set of all $a \in A$ with $\alpha(a) = 0$.

1.1. Proposition. *Let* $a \in A(\alpha)$. *Then*

(i) $\sum_{i=1}^{n} q_i(a) = 1$;
(ii) $q_i(a)q_j(a) = 0$ *if* $i \neq j$;
(iii) $q_i(a) \in Q$, $i = 1, \ldots, n$;
(iv) $q_i(a)a = aq_i(a) = \alpha_i q_i(a)$, $i = 1, \ldots, n$.

Proof. (i) follows from $\sum_{i=1}^{n} q_i(X) = 1$ and (ii) follows from the equality $q_i(X)q_j(X) = h(X)\alpha(X)$. From (i) and (ii),

$$q_i(a) = q_i(a)\sum_{k=1}^{n} q_k(a) = \sum_{k=1}^{n} q_i(a)q_k(a) = q_i(a)^2,$$

which gives (iii). Finally from $\alpha(X) = c(X - \alpha_i)q_i(X)$ (with $c = g_i(\alpha_i) \neq 0$) it follows that $0 = \alpha(a) = c(aq_i(a) - \alpha_i q_i(a))$ and this completes the proof because $q_i(a)$ commutes with a.

Let $Q_n = Q_n(A)$ denote the set of all n-tuples of idempotents q_i of A which satisfy $q_i q_j = 0$ if $i \neq j$ and $\sum_{i=1}^{n} q_i = 1$.

1.2. Proposition. *The mapping* $a \to (q_i(a), \ldots, q_n(a))$ *is a bijection from* A_α *onto* Q_n *whose inverse is* $(q_1, \ldots, q_n) \to \sum_{i=1}^{n} \alpha_i q_i$.

The proof is a straightforward application of Proposition 1.1. Thus, from a set-theoretical view point, Q_n is a universal model for the sets A_α. We shall extend this result to the differential geometry setting.

1.3. Remark. I. Kaplansky introduced the notion of SBI-rings (SBI = suitable for building idempotents) as those rings A such that the natural mapping $Q(A) \to Q(A/R)$ is onto, where R is the Jacobson radical of A.

It is known that for a SBI-ring A, the map $Q_n(A) \to Q_n(A/R)$ is also onto for each $n = 1, 2, \ldots$ (see [**Ja**, p. 54]).

It is also known that all Banach algebras are SBI [**Ri**, p. 58]. These facts and 1.2 imply that, for every $\alpha = (\alpha_1, \ldots, \alpha_n)$ (with $\alpha_l \neq \alpha_k$), $A_\alpha \to (A/R)_\alpha$ is onto, a result due to Barnes [**Ba**, Theorem 7].

From now on, we will assume that A is a real or complex Banach algebra with identity. For n-tuples $Z = (Z_1, \ldots, Z_n)$ in A^n we use the norm $\|Z\| = \max_{1 \leq i \leq n} \|Z_i\|$. The general facts on Banach algebras and Banach manifolds needed below can be found in [**Ri**] and [**La**], respectively.

1.4. THEOREM. *Let* $a \in A_\alpha$ *be a fixed element,* $q = q(a) = (q_1(a), \ldots, q_n(a)) \in Q_n$ *the corresponding system of idempotents. Set*

$$T = \{X \in A;\ q_i X q_i = 0 \text{ for all } i = 1, \ldots, n\},$$
$$S = \{Y \in A;\ q_k Y q_l = 0 \text{ for all } k \neq l\}.$$

1.4.(i) A *is the Banach space direct sum* $A = T \oplus S$.
1.4.(ii) *For each* $Z = X + Y$, $X \in T$, $Y \in S$, *set*

$$X' = \sum_{i \neq k} q_i X q_k / (\alpha_k - \alpha_i)$$

and define

$$\phi(Z) = \exp(X')(a + Y)\exp(-X').$$

Then ϕ *is a diffeomorphism from a neighborhood* U *of* $O \in A$ *onto a neighborhood* V *of* a. *Moreover,* $\phi|_{U \cap T}$ *is a homeomorphism onto* $V \cap A_\alpha$.

Proof. It is clear that every $Z \in A$ decomposes as $X + Y$, where

$$X = \sum_{j \neq k} q_j Z q_k \in T \quad \text{and}$$

$$Y = \sum_l q_l Z q_l \in S, \quad \text{for } \sum_{l=1}^{n} q_l = 1 \quad \text{and}$$
$$Z = \left(\sum q_l\right) Z \left(\sum q_l\right) = \sum_{j \neq k} q_j Z q_k + \sum_l q_l Z q_l.$$

It is also clear that the decomposition is topological, for T and S are respectively defined as the images of the projections

$$Z \to \sum_{j \neq k} q_j Z q_k \quad \text{and} \quad Z \to \sum_l q_l Z q_l.$$

An easy computation shows that the derivative of ϕ at O is the identity: in fact, for $Y \in S$ $D\phi(O)Y = Y$ obviously; for $X \in T$ $D\phi(O)X = [X', a] = X'a - aX' = X$; the assertion follows from the decomposition $A = T \oplus S$.

Then, by the inverse function theorem, there exist open neighborhoods U' of O and V' of a such that ϕ maps U' diffeomorphically onto V'. Consider next $Z = X + Y$ with $\phi(Z) \in A_\alpha$. Since

$\phi(Z) = M(a+Y)M^{-1}$, then $a+Y$ is also a root of α. Then $O = \prod_i(a+Y-\alpha_i)$ and using Prop. 1.1.(iv):

$$\begin{aligned} O &= q_j \prod_i (a+Y-\alpha_i) = q_j \prod_i (\alpha_j + Y - \alpha_i) \\ &= q_j Y L \end{aligned}$$

where $L = \prod_{j\neq i}(Y - (\alpha_i - \alpha_j))$. If Y has small norm ($\|Y\| < \min\{|\alpha_i - \alpha_j|,\ i \neq j\}$ suffices) then L is invertible and therefore $q_j Y = 0$ for each j. Hence $\phi(Z) \in A_\alpha$ with Y small implies $Z \in T$. This means that (perhaps for smaller neighborhoods) ϕ is a homeomorphism from $U' \cap T$ onto $V' \cap V_\alpha$.

Considering the maps ϕ as analytic local coordinates in A, we obtain:

1.5. COROLLARY. *A_α is a closed analytic submanifold of A whose tangent space at $a \in A_\alpha$ can be identified to the Banach space T.*

1.6. REMARKS. (i) The choice of the chart ϕ may seem rather artificial; for instance, the derivative at O of $\phi_1(X+Y) = \exp(X)(a+Y)\exp(-X)$ is $X + Y \to Xa - aX + Y = [X, a] + Y$ and the equalities $q_i[X, a]q_j = (\alpha_j - \alpha_i)q_i X q_j$ $(i \neq j)$ show that $D\phi_1(O)$ maps T onto T and S onto S. Thus, ϕ_1 also provides charts for the analytic structure of A_α. However, we have chosen the map ϕ because it is the exponential map of the natural connection to be studied later (see §4). This remarks applies also to the charts chosen below for Q_n.

(ii) An obvious consequence of 1.3 is that A_α is locally arcwise connected for all α as above. For the simpler case of $\alpha(X) = X(X-1)$ this is a result of Zemanek [**Ze**, 3.2] for complex Banach algebras, which was generalized for real algebras by Aupetit [**Au**, p. 413]. However both results have been also proved in [**PR1**, 4.3] (see also 2.2(iii) below).

1.7. THEOREM. *Q_n is a closed submanifold of A^n.*

Proof. Fix $q \in Q_n$ and define $T' = \{X = (X_1, \dots, X_n) \in A^n : q_r X_i q_s = 0$ for $r \neq i$ and $s \neq i$ or $r = s = i$, and $q_i X_i q_k + q_i X_k q_k = 0$ for $i \neq k\}$.

The map $\theta\colon A^n \to A^n$, $\theta(Z_i, \dots, Z_n) = (X_1, \dots, X_n)$ defined by

$$X_1 = \sum_{i>1} q_1 Z_1 q_i + q_i Z_1 q_1 ,$$

$$X_2 = \left(\sum_{i>2} q_2 Z_2 q_i + q_i Z_2 q_i\right) - (q_1 Z_1 q_2 + q_2 Z_1 q_1),$$

$$\vdots$$

$$X_k = \sum_{i>k} (q_k Z_k q_i + q_i Z_k q_k) - \sum_{i<k} (q_i Z_i q_k + q_k Z_i q_i) \qquad (k \le n-1),$$

$$X_n = -\sum_{k=1}^{n-1} X_k$$

is a projection onto T' whose kernel is the set S' of all $Y = (Y_1, \dots, Y_n) \in A^n$ with $q_r Y_i q_s = 0$ for $r = i$ and $s > i$ or $s = i$ and $r > i$.

Thus $T' \oplus S' = A^n$. For $X \in T'$ put

$$\widetilde{X} = \sum_{i \neq j} \widetilde{X}_{ij} \quad \text{where } \widetilde{X}_{ij} = \begin{cases} q_i X_j q_j & \text{if } j < i, \\ -q_i X_i q_j & \text{if } i < j. \end{cases}$$

Observe that $q_i \widetilde{X} q_i = 0$ for $i = 1, \dots, n$.

Consider now the map $\psi\colon A^n \to A^n$ defined by

$$\psi(Z)_i = \psi(X+Y)_i = \exp(\widetilde{X})(q_i Y_i) \exp(-\widetilde{X})$$

for $X \in T'$, $Y \in S'$. Then $D\psi(O)Y = Y$ for $Y \in S'$ and, calculating,

$$(D\psi(O)X)_i = [\widetilde{X}, q_i] = X_i \quad \text{for } X \in T', \ i = 1, \dots, n.$$

This means that $D\psi(O) =$ identity and ψ is a diffeomorphism from a neighborhood of O onto a neighborhood of q. For $Y \in S'$ such that $\|Y\| < 1$ it is easily shown that $q + Y \in Q_n$ if and only if $Y = O$. This completes the proof.

REMARK. According to Proposition 1.2, the bijections connecting A_α and Q_n are given by algebraic expressions.

The next result, whose proof follows easily from the theorems above, shows that Q_n is a universal model for the sets A_α of simple algebraic elements of degree n.

1.8. THEOREM. *The map $a \to (q_1(a), \dots, q_n(a))$ is a diffeomorphism from A_α onto Q_n whose inverse is given by $(q_1, \dots, q_n) \to$*

$\sum_{i=1}^{n} \alpha_i q_i$. *Consequently, for any other* $\beta = (\beta_1, \dots, \beta_n)$ *with* $\beta_i \neq \beta_j$ *the map* $a \to \sum_{i=1}^{n} \beta_i q_i(a)$ *is a diffeomorphism from* A_α *onto* A_β.

2. Fibrations. The group G of invertible elements of A acts on Q_n by inner automorphisms on each coordinate: if $g \in G$ and $q = (q_1, \dots, q_n) \in P_n$ then $gqg^{-1} = (gq_1g^{-1}, \dots, gq_ng^{-1}) \in Q_n$.

2.1. THEOREM. *Let* q *be a fixed element of* Q_n *and define* $\pi: G \to Q_n$ *by* $\pi(g) = gqg^{-1}$. *Then*

(i) *there exist an open neighborhood* U *of* q *in* Q_n *and a local section* $\sigma: U \to G$ *of* π;

(ii) *the orbit* $V_q = \{gqg^{-1}: g \in G\}$ *is open (and closed) in* Q_n;

(iii) $\pi: G \to V_q$ *is a principal fiber bundle with structure group* $G_0 = \{g \in G: gq_1 = q_1 g,\ i = 1, \dots, n\}$.

Therefore Q_n *is a discrete union of homogeneous spaces of* G.

Proof. Given $q' \in Q_n$ define

$$\sigma(q') = \langle q, q' \rangle = q'_1 q_1 + \cdots + q'_n q_n.$$

It is clear that $\sigma(q) = 1$ and $\sigma(q)q_i = q'_i \sigma(q')$. Thus, for every q' in a neighborhood U of q, we have $\sigma(q') \in G$ and $\sigma(q')q\sigma(q')^{-1} = q'$. This proves (i) and (ii) and the rest of the statement follows from standard arguments (see [**St**, §7]).

2.2. REMARKS. (i) An invertible element g belongs to G_0 if and only if $q_k g q_l = 0$ for all $k \neq l$. Thus, the Lie algebra of G_0 can be identified to $\{X \in A: q_k X q_l = 0 \text{ for all } k \neq l\}$.

(ii) With the notations of 2.1 and 1.6 it is easy to describe trivializations of the tangent bundle TQ_n and of a suplement NQ_n of TQ_n in the trivial bundle $\varepsilon: Q_n \times A^n \to Q_n$. We call NQ_n the "normal bundle" of Q_n. Given $q \in Q_n$, let $U_q = \{q' \in Q_n: \sigma(q') \in G\}$. Then $h: U_q \times A^n \to U_q \times A^n$, defined by

$$h(q', Z) = (q', \sigma(q')Z\sigma(q')^{-1}),$$

is a diffeomorphism which trivializes simultaneously $\tau: TQ_n \to Q_n$ and a bundle $\nu: NQ_n \to Q_n$ where $(NQ_n)_q = S'$ (as in 1.6).

(iii) Given $q \in Q_n$, its connected component (in Q_n) can be described as the set $\{gqg^{-1}: g \in G^0\}$, where G^0 is the connected component of 1 in G: in fact, it suffices to replace G by G^0 in the proof of 2.1. Of course, similar statements hold for A_α. This generalizes [**Ze**, Theorem 3.3] and [**Au**].

2.3. COROLLARY. *Consider a fixed* $q \in Q_n$ *and a continuous curve* $\gamma: [0, 1] \to Q_n$ *such that* $\gamma(0) = q$. *Then, there exists a continuous curve* $\Gamma: [0, 1] \to G$ *such that* $\Gamma(0) = 1$ *and* $\pi \circ \gamma = \gamma$, *where* $\pi(g) = gqg^{-1}$.

We consider now the behaviour of the functor Q_n under epimorphisms.

Let $f: A \to B$ be a continuous homomorphism of Banach algebras which preserves the identity

Clearly f induces maps $G(f): G(A) \to G(B)$, and $f_n: Q_n(A) \to Q_n(B)$. We shall prove that f_n is a Serre fibration when f is an epimorphism **[Sp]**.

2.4. THEOREM. *Let* $f: A \to B$ *be a (continuous) epimorphism of Banach algebras. Then* $f_n: Q_n(A) \to Q_n(B)$ *is a Serre fibration. In particular,* f_n *is onto if and only if its image intersects every connected component of* $Q_n(B)$.

Proof. Replacing A and B by $C(I^m, A)$ (= algebra of all maps $I^m \to A$) and $C(I^m, B)$ respectively (where $I = [0, 1]$), it suffices to show that if $\gamma: I \to Q_n(B)$ is such that $\gamma(0) = q' = f_n(q)$ for some $q \in Q_n(A)$ there exists a curve $\tilde{\gamma}: I \to Q_n(A)$ such that $f_n \circ \tilde{\gamma} = \gamma$.

For this, we consider the commutative diagram

$$\begin{array}{ccc} G(A) & \xrightarrow{f} & G(B) \\ {\scriptstyle \pi_q}\downarrow & & \downarrow{\scriptstyle \pi_{q'}} \\ Q_n(A) & \xrightarrow[f_n]{} & Q_n(B) \end{array}$$

where $\pi_q(g) = gqg^{-1}$, $\pi_{q'}(h) = hq'h^{-1}$ $(g \in G(A),\ h \in G(B))$. By the local triviality of $\pi_{q'}$ proved in 2.1, there is a curve $\delta: I \to G(B)$ with $\delta(0) = 1$ and $\pi_{q'}\delta = \gamma$. Michael **[Mi]** proved that $f: G(A) \to G(B)$ is a Serre fibration; therefore, there is a curve $\varepsilon: I \to G(A)$ such that $\varepsilon(0) = 1$ and $f \circ \varepsilon = \delta$. To finish the proof it suffices to define $\tilde{\gamma} = \pi_q \circ \varepsilon$, which satisfies $f_n \circ \tilde{\gamma} = \gamma$.

The next theorem extends results of Raeburn **[Ra]** concerning the set $\pi_0(P(A \hat{\otimes} B))$ of all connected components of the idempotents of $A \hat{\otimes} B$, where A is supposed to be commutative.

We omit its proof and that of the proposition below because they are simple combination of Raeburn's techniques without previous results.

2.5. PROPOSITION (*cf.* [**Ra**, p. 383]). *Let A be a Banach algebra and $B_1, \dots, B_n$ be open balls in $\mathbf{C}$ with pairwise disjoint closures, centered at $\alpha_1, \dots, \alpha_n$, respectively. Let $U = B_1 \cup \cdots \cup B_n$ and $A_U = \{a \in A:$ the spectrum of a is contained in $U\}$. Then A_U is open in A and $f = (f_1, \dots, f_n): A_U \to A^n$ is an analytic retraction onto Q_n, where $f_i: U \to \mathbf{C}$ is defined by $f_i(z) = \delta_{ik}$ for $z \in B_k$ and $f_n(a)$ is obtained by means of the holomorphic functional calculus.*

2.6. THEOREM (*cf.* [**Ra**, 4.5, 4.7]). *Let A and B be complex Banach algebras. Suppose that A is commutative with spectrum X. Then the Gelfand map $A \to C(X)$ induces bijections*

$$\pi_0(Q_n(A\hat{\otimes}B)) \to [X, Q_n(B)],$$
$$\{Q_n(A\hat{\otimes}B)\} \to \{Q_n(C(X, B))\}$$

where $[\ ,\ ]$ denotes homotopy classes of maps and $\{Q_n(C)\}$ is the set of orbits of the action of $G(C)$ on $Q_n(C)$.

2.7. REMARK. If A is the algebra of complex continuous functions on the 3-sphere, B is the algebra of all 2×2-matrices over $\mathbf{C}$ and $n = 2$, we reobtain the example of [**PR1**, 7.13].

3. Lifting C^1-curves. The transport equation. In this section we describe a method which leads to a lifting Γ of a curve $\gamma: [a, b] \to Q_n$, as in Corollary 2.3, valid when γ is rectifiable and continuous. For the sake of simplicity we only consider $n = 2$, the general case being similar and somewhat more involved. The reader can find the details (for $n = 2$) in [**PR1**]. Our present interest in this construction lies in that it leads to the transport equation.

Consider a continuous rectifiable curve $\gamma: [a, t] \to Q$ and a partition $\Pi: t_0 = a < t_1 < \cdots < t_n = t$ such that $\|\gamma_k - \gamma_{k+1}\| < 1$ $(k = 0, \dots, n-1)$, where $\gamma_k = \gamma(t_k)$; then

$$\sigma_k = \gamma_k\gamma_{k-1} + (1-\gamma_k)(1-\gamma_{k-1}) \in G \qquad (k = 0, \dots, n-1) \quad \text{and}$$

$$\sigma_k\gamma_0\sigma_1^{-1} = \gamma_1,$$
$$\sigma_2\sigma_1\gamma_0\sigma_1^{-1}\sigma_2^{-1} = \sigma_2\gamma_1\sigma_2^{-1} = \gamma_2, \dots, \sigma_n\cdots\sigma_1\gamma_0\sigma_1^{-1}\cdots\sigma_n^{-1} = \gamma_n.$$

Thus, σ can be thought of as a "discrete" curve of units which conjugates γ_0 with γ_n. Putting $u(\Pi) = \sigma_n \cdots \sigma_1$, it can be shown [**PR1**, §5] that the limit $\Gamma(t) = \lim u(\Pi)$, when the length of the partition Π tends to zero, exists and defines a unit of the algebra. Moreover

$\Gamma\colon [a, b] \to G$ is continuous and rectifiable. If the original curve γ has a continuous derivative, then the value

$$\begin{aligned}&(1/h)(\Gamma(t+h) - \Gamma(t) \quad \text{is, approximately,}\\ &(1/h)(\sigma_{t+h}\Gamma(t) - \Gamma(t)), \quad \text{where}\\ &\sigma_{t+h} = \gamma(t+h)\gamma(t) + (1-\gamma(t+h))(1-\gamma(t)).\end{aligned}$$

Then,

$$\begin{aligned}(1/h)(\Gamma(t+h) - \gamma(t)) &\cong (1/h)(\sigma_{t+h} - 1)\Gamma(t)\\ &= (1/h)(2\gamma(t+h)\gamma(t) - \gamma(t+h) - \gamma(t))\Gamma(t)\\ &= (1/h)\{\gamma(t+h)(\gamma(t) - \gamma(t+h)) + (\gamma(t+h) - \gamma(t))\gamma(t)\}\Gamma(t)\end{aligned}$$

and

$$\begin{aligned}\dot\Gamma(t) &= \lim_{h\to 0}(1/h)(\Gamma(t+h) - \Gamma(t))\\ &= \{-\gamma(t)\dot\gamma(t) + \dot\gamma(t)\gamma(t)\}\Gamma(t).\end{aligned}$$

Thus, the lifting Γ of γ constructed by the limiting process described above satisfies the initial values problem

$$\begin{aligned}&\dot\Gamma = (\dot\gamma\gamma - \gamma\dot\gamma),\\ &\Gamma(0) = 1.\end{aligned}$$

In the general case $n > 2$ the initial value problem is

$$\begin{aligned}&\dot\Gamma = \left(\sum_1^n \dot\gamma_k\gamma_k\right)\Gamma,\\ &\Gamma(0) = 1,\end{aligned}$$

where $\gamma = (\gamma_1, \dots, \gamma_n)\colon [a, b] \to Q_n$ is of class C^1. Observe that $\sum_1^2 \dot\gamma_k\gamma_k = \dot\gamma_1\gamma_1 - \dot\gamma_1(1-\gamma_1) = \dot\gamma_1\gamma_1 - \gamma_1\dot\gamma_1$ because $\gamma_2 = 1 - \gamma_2$ and $\dot\gamma_1 = \dot\gamma_1\gamma_1 + \gamma_1\dot\gamma_1$ (differentiate $\gamma_1^2 = \gamma_1$).

As we said before, we shall not justify all the assertions about Γ. Instead we include the proof of the following result due to Daleckii, Krein and Kato, for the sake of completeness (see **[DK2**, IV, Theorem 1.1]).

3.1. THEOREM. *Let* $\gamma\colon [a, b] \to Q_n$ *be a* C^1 *curve. Then, the unique solution in* A *of the initial conditions problem*

$$\begin{aligned}&\dot\Gamma = \hat\gamma\Gamma,\\ &\Gamma(a) = 1,\end{aligned}$$

where $\hat{\gamma} = \sum_{k=1}^{n} \dot{\gamma}_k \gamma_k$, *satisfies*

(i) $\Gamma(t) \in G \quad (t \in [a, b])$,

(ii) $\Gamma(t)\gamma(a)\Gamma(t)^{-1} = \gamma(t) \quad (t \in [a, b])$.

Proof. Existence and uniqueness of Γ follow from general facts [**La**, p. 71]. To prove (i) consider the companion problem

$$\begin{cases} \dot{\Delta} = -\Delta\hat{\gamma}, \\ \Delta(a) = 1, \end{cases}$$

and observe that $(\Delta\Gamma)^{\cdot} = \dot{\Delta}\Gamma + \Delta\dot{\Gamma} = 0$. Then $\Delta\Gamma$ is constant on $[a, b]$ and, since $\Delta(a) = \Gamma(a) = 1$, it is $\Delta\Gamma \equiv 1$. Thus $\Gamma(t)$ is left invertible in A; moreover, $\Gamma(t)$ belongs to the connected component of the identity in the set of left invertible elements. It is easy to see that this component is completely contained in G. This proves (i).

To see (ii) we compute $(\Gamma^{-1}\gamma_k\Gamma)^{\cdot} \quad (k = 1, \dots, n)$:

$$\begin{aligned} (\Gamma^{-1}\gamma_k\Gamma)^{\cdot} &= -\Gamma^{-1}\dot{\Gamma}\Gamma^{-1}\gamma_k\Gamma + \Gamma^{-1}\dot{\gamma}_k\Gamma + \Gamma^{-1}\gamma_k\dot{\Gamma} \\ &= -\Gamma^{-1}\{\hat{\gamma}\gamma_k - \dot{\gamma}_k - \gamma_k\hat{\gamma}\}\Gamma; \end{aligned}$$

observe that $\hat{\gamma}\gamma_k = (\sum \dot{\gamma}_i\gamma_i)\gamma_k = \dot{\gamma}_k\gamma_k$, because $\gamma_i\gamma_k = 0$ for $i \neq k$, and that $\gamma_k\hat{\gamma} = \gamma_k(\sum \dot{\gamma}_i\gamma_i) = -\gamma_k(\sum \gamma_i\dot{\gamma}_i) = -\gamma_k\dot{\gamma}_k$, because $\dot{\gamma}_k = \dot{\gamma}_k\gamma_k + \gamma_k\dot{\gamma}_k$ and $\sum \dot{\gamma}_k = (\sum \gamma_k)^{\cdot} = 1^{\cdot} = 0$. Thus

$$(\gamma^{-1}\gamma_k\Gamma)^{\cdot} = -\Gamma^{-1}\{\dot{\gamma}_k\gamma_k - \dot{\gamma}_k + \gamma_k\dot{\gamma}_k\}\Gamma = 0$$

and $\Gamma^{-1}\gamma_k\Gamma$ is constantly $\gamma_k(a)$. This completes the proof of (ii).

3.2. Remark. The proof of part (i) could have been omitted because it is a general fact that the solution of $\dot{\Gamma} = \varphi\Gamma$, $\Gamma(a) = 1$, where $\varphi: [a, b] \to A$ is a continuous curve, is a curve of invertible element of A.

If A is an involutive Banach algebra, i.e. there exists a continuous antilinear mapping $x \to x^*$ such that $(xy)^* = y^*x^*$, $1^* = 1$ and $x^{**} = x \quad (x, y \in A)$, we consider the unitary group of A

$$U = \{u \in G: u^{-1} = u^*\}$$

and the selfadjoint part of Q_n

$$P_n = \{p = (p_1, \dots, p_n) \in Q_n: p_k^* = p_k \quad (k = 1, \dots, n)\}.$$

For these algebras more specific results hold. We omit the details about the differential structure of P_n.

3.3. COROLLARY. *If $\gamma: [a, b] \to P_n$ is a C^1 curve then the solution of $\dot{\Gamma} = \hat{\gamma}\Gamma$, $\Gamma(a) = 1$, defines a curve $\Gamma: [a, b] \to U$ which conjugates the curve γ.*

Proof. It suffices to show that $\Gamma(t) \in U$ for every $t \in [a, b]$. Observe first that

$$\dot{\Gamma}^* = \left\{\left(\sum \dot{\gamma}_k \gamma_k\right) \Gamma^* = \Gamma^* \left(\sum \dot{\gamma}_k \gamma_k\right)^* \right. \\ = \Gamma^* \left(\sum \gamma_k \dot{\gamma}_k\right) = -\Gamma^* \left(\sum \dot{\gamma}_k \gamma_k\right),$$

because

$$\sum \dot{\gamma}_k \gamma_k + \gamma_k \dot{\gamma}_k = \sum \dot{\gamma}_k = \left(\sum \gamma_k\right)^{\cdot} = 1^{\cdot} = 0.$$

Thus $(\Gamma^*\Gamma)^{\cdot} = \dot{\Gamma}^*\Gamma + \Gamma^*\dot{\Gamma} = 0$ and $\Gamma^*\Gamma$ is constant. But $\Gamma(0) = \Gamma^*(0) = 1$, so $\Gamma^*\Gamma = 1$. Now, $\Gamma(t)$ is invertible for all t, by Theorem 3.1, so $\Gamma(t)^* = \Gamma(t)^{-1}$.

3.4. REMARK. Of course many liftings of γ may exist. But Γ is the unique horizontal lifting of γ with respect to the connection we shall define in the next section. This fact completes Kato's remark [**Ka**, II.4.2, Remark 4.4]. Moreover, if our σ's, used to obtain the transport equation, are multiplied (at left or at right) by $(1 - (\gamma_k - \gamma_{k-1})^2)^{-1/2}$, where $(1 - r)^{-1/2} = \sum_{m=0}^{\infty} \binom{-1/2}{m} (-r)^m$ for $\|r\| < 1$, we get a different "discrete" lifting of γ but in the limit it becomes the same continuous curve Γ. In this sense, the local solution [**Ka**, p. 102, (4.18)]

$$\Gamma_1(t) = (1 - (\gamma(t) - \gamma(0))^2)^{-1/2}(\gamma(t)\gamma(0) + (1 - \gamma(t)))(1 - \gamma(0))$$

is related to the global solution Γ.

4. The connection. Let $q \in Q_n$ be fixed and $\pi: G \to Q_n$ defined by $\pi(g) = gqg^{-1} = (gq_1g^{-1}, \dots, gq_ng^{-1})$. It is very easy to show that the derivative of π at $g \in G(T\pi)_g: (TG)_g: (TG)_g \to (TQ_n)_{\pi(g)}$ is given by

$$(T\pi)_g(X) = g[g^{-1}X, q]g^{-1} \qquad (X \in (TG)_g)$$

where $[Z, q] = ([Z, q_1], \dots, [Z, q_n])$ for all $Z \in A$.

We say that $X \in (TG)_g$ is *vertical* if $(T\pi)_g(X) = 0$ or, what is the same, if $[g^{-1}X, q] = 0$. Then, if $V_g = \{X \in (TG)_g: [g^{-1}X, q] = 0\}$, it is clear that $V_g = g \cdot V_1$ and that

$$\begin{aligned} V_1 &= \{X \in A = (TG)_g : [X, q] = 0\} \\ &= \{X \in A : q_k X q_i = 0 \text{ for all } i \neq k\} \\ &= \left\{ \sum_{i=1}^{n} q_i X q_i : X \in A \right\}. \end{aligned}$$

This shows that

$$\begin{aligned} H_1 &= \{X \in A : q_i X q_i = 0 \ (i = 1, \dots, n)\} \\ &= \left\{ \sum_{k \neq i} q_k X q_i : X \in A \right\} \end{aligned}$$

is a supplement of V_1 in A $(= (TG)_1)$ and, in general $H_g = gH_1$ is a supplement of V_g in A $(= (TG)_g)$. Moreover, $H_g \cdot h = H_{gh}$ $(g \in G$, $h \in H)$. Finally, the projections $h_g : (TG)_g \to H_g$, $v_g : (TG)_g \to V_g$ given by

$$\begin{aligned} h_g(X) &= g \sum_{i \neq k} q_k g^{-1} X q_i, \\ v_g(X) &= g \sum_{i=1}^{n} q_i g^{-1} X q_i, \end{aligned}$$

verify

$$\begin{aligned} h_g(X) &= g h_1(g^{-1} X), \\ v_g(X) &= g v_1(g^{-1} X). \end{aligned}$$

Clearly the mappings $g \to h_g$ and $g \to v_g$ from G into the bounded linear operators on A are differentiable. All these facts show that $g \to H_g$ defines a connection in the principal bundle $\pi : G \to Q_n'$.

For the theory of connections we refer the reader to [**KN**]. However, we are dealing with Banach manifolds and bundles, which requires a few notational changes.

From now on by "curve" we mean a C^∞ curve.

Given a curve $\gamma : [\alpha, \beta] \to Q_n$, a *horizontal lifting* of γ is a curve $\Gamma : [\alpha, \beta] \to G$ such that $\pi\Gamma = \gamma$ and $\dot{\Gamma}(t) \in H_{\Gamma(t)}$ $(t \in [\alpha, \beta])$.

It is a general fact that, for each $g_0 \in G$ such that $\gamma(\alpha) = g_0 p g_0^{-1}$, there is a unique horizontal lifting Γ such that $\Gamma(\alpha) = g_0$. In particular, if $\gamma(\alpha) = q$ there is a unique horizontal lifting Γ such that $\Gamma(\alpha) = 1$.

4.1. THEOREM. *Given a curve* $\gamma\colon [\alpha, \beta] \to Q_n$ *the horizontal lifting* Γ *such that* $\Gamma(\alpha) = 1$ *is the solution of the transport equation*

$$\dot{\Gamma} = \hat{\gamma}\Gamma, \quad \textit{where } \hat{\gamma} = \sum_{i=1}^{n} \dot{\gamma}_i \gamma_i, \tag{4.2}$$

with initial condition $\Gamma(\alpha) = 1$.

Proof. We have seen that the solution Γ of (4.2) is a lifting of π, i.e. $\pi \circ \Gamma = \gamma$ (see 3.1). By the uniqueness of both objects it suffices to show that the horizontal lifting Γ with $\Gamma(\alpha) = 1$ satisfies (4.2). We recall that Γ satisfies

$$\Gamma(t) q \Gamma(t)^{-1} = \gamma(t) \qquad (t \in [\alpha, \beta]), \tag{4.3}$$

$$\dot{\Gamma} \in H_\Gamma = \Gamma H_1, \text{ i.e. } \dot{\Gamma}(t) \in \Gamma(t) H_1 \qquad (t \in [\alpha, \beta]) \tag{4.4}$$

or, what is the same

$$\Gamma^{-1}\gamma\Gamma = q \tag{4.5}$$

and

$$\Gamma^{-1}\dot{\Gamma} \in H_1. \tag{4.6}$$

Differentiating (4.5) we get $0 = \Gamma^{-1}(-\dot{\Gamma}\Gamma^{-1}\gamma + \dot{\gamma} + \gamma\dot{\Gamma}\Gamma^{-1})\Gamma$ and cancelling Γ^{-1} and Γ, we get

$$\dot{\gamma} = [\dot{\Gamma}\Gamma^{-1}, \gamma]. \tag{4.7}$$

Now, (4.6) means that $q_i \Gamma^{-1}\dot{\Gamma} q_1 = 0$, $(i = 1, \dots, n)$, which can also be written as

$$q\Gamma^{-1}\dot{\Gamma} = \Gamma^{-1}\dot{\Gamma}(1 - q). \tag{4.8}$$

Replacing (4.5) in (4.8) we get $\Gamma^{-1}\gamma\dot{\Gamma} = \Gamma^{-1}\dot{\Gamma} - \Gamma^{-1}\dot{\Gamma}\Gamma^{-1}\gamma\Gamma$ which, after cancellation, gives

$$\gamma\dot{\Gamma}\Gamma^{-1} = \dot{\Gamma}\Gamma^{-1}(1 - \gamma) \tag{4.9}$$

and

$$\dot{\Gamma}\Gamma^{-1}\gamma = (1 - \gamma)\dot{\Gamma}\Gamma^{-1}. \tag{4.10}$$

Finally,

$$\begin{aligned} \hat{\gamma}\Gamma &= \left(\sum_i^n \dot{\gamma}_i \gamma_i\right)\Gamma \\ &= \sum_1^n [\dot{\Gamma}\Gamma^{-1}, \gamma_i]\gamma_i\Gamma \qquad \text{(by 4.7)} \\ &= \sum_1^n \{\dot{\Gamma}\Gamma^{-1}\gamma_i - \gamma_i\dot{\Gamma}\Gamma^{-1}\gamma_i\}\Gamma. \end{aligned}$$

This last expression coincides with $\dot{\Gamma}$ because $\gamma_i\dot{\Gamma}\Gamma^{-1} = \dot{\Gamma}\Gamma^{-1}(1-\gamma_i)$ by (4.9) and therefore $\gamma_i\dot{\Gamma}\Gamma^{-1}\gamma_i = \dot{\Gamma}\Gamma^{-1}(1-\gamma_i)\gamma_i = 0$. This proves the theorem.

4.11. REMARK. In general, if $\gamma: [\alpha, \beta] \to Q_n$ is a curve with origin $q' = g_0 q g_0^{-1}$ then Γ is the horizontal lifting with origin g_0 if and only if it is the solution of the problem $\dot{\Gamma} = \hat{\gamma}\Gamma$, $\Gamma(\alpha) = g_0$.

We compute next the 1-form, the 2-form and the curvature form of the connection.

We recall that the 1-form θ assigns to each $X \in (TG)_g$ the horizontal component of $g^{-1}X \in (TG)_1 = \mathscr{L}$, the Lie algebra of H. More explicitly,

$$\theta_g X = v_1(g^{-1}X) = g^{-1}v_g(X) = \sum_{i=1}^{n} q_i g^{-1} X q_i.$$

The 2-form $d\theta$ of the connection is defined by

$$d\theta(X, Y) = \tfrac{1}{2}\{X \cdot \theta Y - Y \cdot \theta X - \theta([X, Y])\},$$

where $X, Y \in (TG)_g$, $[\ ,\ ]$ denotes the Lie bracket and $Z \cdot W$ denotes the derivative of W in the direction of Z, i.e. W is extended to a vector field on a neighborhood of g and given a curve $\delta: (-\varepsilon, \varepsilon) \to G$ such that $\delta(0) = g$ and $\dot{\delta}(0) = Z$,

$$Z \cdot W = \frac{d}{dt_{t=0}} W(\delta(t)).$$

Although the notation is the same, the Lie bracket should not be confused with the commutator bracket of the algebra.

From the computations

$$\begin{aligned} X \cdot \theta Y &= X \cdot \left(\sum_{i=1}^{n} q_i g^{-1} Y q_i\right) \\ &= -\sum_{i=1}^{n} q_i g^{-1} X g^{-1} Y q_i + \sum_{i=1}^{n} q_i g^{-1} X \cdot Y q_i, \\ Y \cdot \theta X &= -\sum_{i=1}^{n} q_i g^{-1} Y g^{-1} X q_i + \sum_{i=1}^{n} q_i g^{-1} Y \cdot X q_i, \end{aligned}$$

and

$$\theta([X, Y]) = \sum_{i=1}^{n} q_i g^{-1}[X, Y] q_i,$$

we get

$$d\theta(X, Y) = \frac{1}{2}\sum_{i=1}^{n} q_i[g^{-1}Y, g^{-1}X]q_i$$
$$= \left(-\frac{1}{2}\right)\sum_{i=1}^{n} q_i[g^{-1}X, g^{-1}Y]q_i.$$

The *horizontal differential of* θ, also called the *curvature form of the connection* is $\Omega(X, Y) = d\theta(h_gX, h_gY)$ for $[X, Y] \in (TG)_g$. Explicitly

$$\Omega(X, Y) = \left(-\frac{1}{2}\right)\sum_{i=1}^{n} q_i[g^{-1}h_gX, g^{-1}h_gY]q_i$$
$$= \left(-\frac{1}{2}\right)\sum_{i=1}^{n} q_i\left[\sum_{k\neq l} q_k g^{-1}Xq_l, \sum_{r\neq s} q_r g^{-1}Yq_s\right] q_i$$
$$= \left(-\frac{1}{2}\right)\sum_{i=1}^{n} q_i g^{-1}\{X(1-q_i)g^{-1}Y - Y(1-q_i)g^{-1}X\}q_i$$
$$= \left(-\frac{1}{2}\right)\sum_{i=1}^{n} q_i g^{-1}\{X\bar{q}_i g^{-1}Y - Y\bar{q}_i g^{-1}X\}q_i,$$
$$\left(\text{where } \bar{q}_k = 1 - q_k = \sum_{i\neq k} q_i\right)$$
$$= \left(-\frac{1}{2}\right)\sum_{i=1}^{n} q_i g^{-1}(Xg^{-1}Y - Yg^{-1}X - Xq_i g^{-1}Y + Yq_i g^{-1}X)q_i.$$

The *structure equation* $\Omega(X, Y) = d\theta(X, Y) + (\frac{1}{2})[\theta X, \theta Y]$ is thus trivially satisfied.

5. Calculations on the tangent bundle, geodesics. Consider $q \in Q_n$ fixed and let $A_1 = \{X \in A: q_iXq_i = 0, i = 1, \dots, n\}$ (in §4 we called it H_1). It is clear that $H = \{g \in G: gq_i = q_ig, i = 1, \dots, n\}$ operates at left on A_1 by $h \cdot X := hXh^{-1}$.

Thus we define the associated bundle of $\pi: G \to Q_n$ with standard fibre A_1, denoted by $G \otimes A_1 \to Q_n$, where $G \otimes A_1 := G \times A_1/\sim$, $(g, X) \sim (gh, h^{-1}X)$ for $h \in H$ and the map $G \otimes A_1 \to Q_n$ is determined by $(g, X) \to \pi(g)$. It is a general fact that this vector bundle is isomorphic to the tangent bundle TQ_n, by means of $(g, X) \to (\pi(g), gXg^{-1}) \in (TQ_n)_{\pi(g)}$. Given a curve $\gamma: [\alpha, \beta] \to$

Q_n the parallel displacement of the fibre $(TQ_n)_{\gamma(\alpha)}$ along γ from α to $t \in [\alpha, \beta]$ is defined by $\tau_\alpha^t\colon (TQ_n)_{\gamma(\alpha)} \to (TQ_n)_{\gamma(t)}$, $\tau_\alpha^t(Z) = \Gamma(t)Z\Gamma(t)^{-1}$, where Γ is the horizontal lifting of γ with origin $\Gamma(\alpha) = 1$.

Given $X \in (TQ_n)_q$ and a vector field Z defined near q the *covariant derivative* $D_X Z$ is $D_X Z := X \cdot Z + [Z, \widetilde{X}]$, where

$$\widetilde{X} = \sum_{i=1}^{n} X_i q_i \quad \text{and} \quad X \cdot Z = \frac{d}{dt_{t=0}} Z(\delta(t))$$

for a curve $\delta\colon (-\varepsilon, \varepsilon) \to Q_n$ such that $\delta(0) = q$ and $\dot{\delta}(0) = X$.

5.1. Proposition. *For every curve* $a\colon [\alpha, \beta] \to A^n$ *the element* $Da/dt = \dot{a} + [a, \hat{\gamma}]$ *is well defined and has the following properties:*

(a) *if* $\gamma_i a \gamma_i = 0$ *for all* $i = 1, \dots, n$ *then* $\gamma_i (Da/dt) \gamma_i = 0$ *for all* $i = 1, \dots, n$ (*in other words,* Da/dt *is tangent if* a *is tangent*).

(b) *if* $\gamma_i a \gamma_k = 0$ *for all* $i \neq k$ *then* $\gamma_i (Da/dt) \gamma_k = 0$ *for all* $i \neq k$ (*i.e.* Da/dt *is normal if* a *is normal*).

Proof. (a) Differentiating $\gamma_i a \gamma_i = 0$ we get

$$0 = \dot{\gamma}_i a \gamma_i + \gamma_i \dot{a} \gamma_i + \gamma_i a \dot{\gamma}_i.$$

Multiplying by γ_i at right and left we have

$$\gamma_i \dot{\gamma}_i a \gamma_i + \gamma_i \dot{a} \gamma_i + \gamma_i a \dot{\gamma}_i \gamma_i = 0. \tag{5.2}$$

On the other hand

$$\begin{aligned} \gamma_i \frac{Da}{dt} \gamma_i &= \gamma_i \dot{a} \gamma_i + \gamma_i [a, \hat{\gamma}] \gamma_i \\ &= \gamma_i \dot{a} \gamma_i + \gamma_i \left(a \sum \dot{\gamma}_k \gamma_k - \sum \dot{\gamma}_k \gamma_k a \right) \gamma_i \\ &= \gamma_i \dot{a} \gamma_i + \gamma_i a \dot{\gamma}_i \gamma_i - \gamma_i \sum \dot{\gamma}_k \gamma_k a \gamma_i \end{aligned}$$

and $\gamma_i \sum_k \dot{\gamma}_k \gamma_k = \gamma_i \sum_k (1 - \gamma_k) \dot{\gamma}_k$ because $\dot{\gamma}_k = \dot{\gamma}_k \gamma_k + \gamma_k \dot{\gamma}_k$ (differentiate $\gamma_k^2 = \gamma_k$); thus

$$\gamma_i \sum_k \dot{\gamma}_k \gamma_k = \gamma_i \sum_k \dot{\gamma}_k - \gamma_i \sum \gamma_k \dot{\gamma}_k = -\gamma_i \dot{\gamma}_i,$$

because $\sum_k \dot{\gamma}_k = 0$ and $\gamma_i \gamma_k = 0$ if $i \neq k$.

This shows that

$$\gamma_i \frac{Da}{dt} \gamma_i = \gamma_i \dot{a} \gamma_i + \gamma_i a \dot{\gamma}_i \gamma_i + \gamma_i \dot{\gamma}_i a \gamma_i = 0, \qquad \text{by (4.2).}$$

The proof of (b) is similar.

This shows that for every vector field Y of Q_n along γ, the formula $Da/dt = \dot{Y} + [Y, \hat{\gamma}]$ defines another vector field of Q_n, the *covariant derivative* of Y.

The *torsion* of the connection, defined by $T(X, Y) = D_X Y - D_Y X - [X, Y]$ in general, turns out to be in our case

$$(5.3) \qquad T(X, Y) = [Y, \widetilde{X}] - [X, \widetilde{Y}],$$

where $X, Y \in (TQ_n)_g$ and $\widetilde{X} = \sum_{i=1}^n X_i q_i$, $\widetilde{Y} = \sum_{i=1}^n Y_i q_i$.

5.4. REMARK. For $n = 2$ the connection is symmetric, in the sense that its torsion is zero everywhere: in fact, for $n = 2$ we have $X_1 + X_2 = 0$, $Y_1 + Y_2 = 0$, $q_1 + q_2 = 1$, $q_i X_i = X_i(1 - q_i)$, $q_i X_j = -X_i q_j$.

These equalities, when replaced in (4.3), prove the assertion. However, for $n > 3$ this is no longer true.

The *curvature* of the connection, expressed by $R(X, Y)Z = D_X(D_Y Z) - D_Y(D_X Z) - D_{[X,Y]}Z$ for $X, Y, Z \in (TQ_n)_q$, is given, in our case, by

$$(5.5) \qquad R(X, Y)Z = \left[\sum_{i=1}^n [X_i, Y_i]q_i, Z\right]$$

or, abbreviating

$$(5.6) \qquad R(X, Y)Z = [[X, Y]^\sim, Z].$$

We study now the geodesic curves of the connection, that is, the curves $\gamma: [\alpha, \beta] \to Q_n$ such that $D\dot{\gamma}/dt = 0$. It is a well-known fact that this condition is equivalent to $\tau_\alpha^t(\dot{\gamma}(\alpha)) = \dot{\gamma}(t)$, $(t \in [\alpha, \beta])$. The equation defining the geodesic curves can be written as

$$(5.7) \qquad \ddot{\gamma}_k + [\dot{\gamma}_k, \hat{\gamma}] = 0, \qquad k = 1, \dots, n.$$

Using the commutation rules obtained from $\sum \gamma_i = 1$, $\gamma_i^2 = \gamma_i$ and $\gamma_i\gamma_k = 0$ for $i \neq k$, we get

(i) $\dot{\gamma}_i\gamma_i = (1 - \gamma_i)\dot{\gamma}_i$ $(i = 1, \dots, n)$;
(ii) $\dot{\gamma}_i\gamma_k + \gamma_i\dot{\gamma}_k = 0$ $(i \neq k)$;
(iii) $\sum_i^n \dot{\gamma}_k = 0$;
(iv) $\gamma_i\dot{\gamma}_i^2 = \dot{\gamma}_i^2\gamma_i$ $(i = 1, \dots, n)$;
(v) $\gamma_i\dot{\gamma}_i\gamma_i = 0$ $(i = 1, \dots, n)$.

These equalities imply that (5.7) is equivalent to

$$(5.8) \quad \ddot{\gamma}_k + \gamma_k\left(\sum_1^n \dot{\gamma}_i^2\right) + \left(\sum_1^n \dot{\gamma}_i^2\right)\gamma_k - 2\dot{\gamma}_k^2 = 0, \qquad (k = 1, \dots, n).$$

It is easy to exhibit all the solutions of (5.8) which satisfy $\gamma(t) \in Q_n$ for all t. In fact, for $q \in Q_n$, $X \in (TQ_n)_q$, $\gamma(t) = e^{t\widetilde{X}} q e^{-t\widetilde{X}}$ $(t \in R)$, satisfies (5.8) and all the solutions of (5.8) with the additional condition $\gamma(t) \in Q_n$, have this form. The connection is also complete, in the sense that its geodesics are defined for all $t \in R$, and the exponential map of the connection is given by

$$\mathrm{Exp}_q : (TQ_n)_q \to Q_n, \quad \mathrm{Exp}_q(X) = e^{\widetilde{X}} q e^{-\widetilde{X}}.$$

Properties of minimality of length of geodesics are studied in a forthcoming paper (**[CPR2]**).

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