PACIFIC JOURNAL OF MATHEMATICS
Vol. 144, No. 2, 1990

AN ALMOST CLASSIFICATION OF COMPACT LIE GROUPS WITH BORSUK-ULAM PROPERTIES

WACŁAW MARZANTOWICZ

We say that a compact Lie group G has the Borsuk-Ulam property in the weak sense if for every orthogonal representation V of G and every G-equivariant map $f\colon S(V) \to S(V)$, $V^G = \{0\}$, of the unit sphere we have $\deg f \neq 0$.

We say that G has the Borsuk-Ulam property in the strong sense if for any two orthogonal representations V, W of G with $\dim W = \dim V$ and $W^G = V^G = \{0\}$ and every G-equivariant map $f\colon S(V) \to S(W)$ of the unit spheres we have $\deg f \neq 0$. In this paper a complete classification, up to isomorphism, of group with the weak Borsuk-Ulam property is given. A classification of groups with the strong Borsuk-Ulam property does not cover nonabelian p-groups with all elements of the order p. In fact we deal with a more general definition admitting a nonempty fixed point set of G on the sphere $S(V)$.

1. The main theorems. In order to formulate our main results we introduce the following notation. Let G be a compact Lie group. We denote by G_0 the component of identity of and by Γ the quotient group G/G_0. We use standard notation of the theory of compact transformation groups (see for instance [**4**] or [**5**]). In particular, for every subgroup $H \subset G$, the fixed point set of H on a G-space X is denoted by X^H. Also, for a G-equivariant map $f\colon X \to Y$ between two G-spaces, we denote by f^H its restriction to the space X^H. The symbol (n, m) stands for the greatest common divisor of the integers n, m with the notation $(0, 1) = 0$ and $|G|$ for the rank of the (finite) group G. We will work with the following definition of the Borsuk-Ulam property (cf. [**11**]).

DEFINITION I. (A) We say that G has the Borsuk-Ulam property in the weak sense A if for every orthogonal representation V of G and every G-equivariant map $f\colon S(V) \to S(V)$ if $(\deg f^G, |\Gamma|) = 1$ then

$$\deg f \neq 0.$$

REMARK. Note that in the case $V^G = \{0\}$ there is no condition on $\deg f^G$ and the property then requires that $\deg f \neq 0$ for every G-equivariant map. Also, if $G = G_0$ then the condition $(\deg f^G, |\Gamma|) = 1$ means that $\deg f^G \neq 0$.

(B) We say that G has the Borsuk-Ulam property in the weak sense B if for every pair or orthogonal representations $W \subsetneq V$ of G, with $V^G = \{0\}$, there is no G-equivariant map $f: S(V) \to S(W)$.

DEFINITION II. (A) We say that G has the Borsuk-Ulam property in the strong sense A if for every pair of orthogonal representations V, W of G, with $\dim W = \dim V$, $\dim V^G = \dim V^G$, and every G-equivariant map

$$f: S(V) \to S(W)$$

if $(\deg f^G, |\Gamma|) = 1$ then $\deg f \neq 0$. (If $W^G = \{0\}$ or $G = G_0$, the same remark as above applies.)

(B) We say that G has the Borsuk-Ulam property in the strong sense B if there is no G-equivariant map $f: S(V) \to S(W)$ where W is an orthogonal representation of G such that $W \subsetneq U$ for an orthogonal representation U, $\dim U = \dim V$ and $U^G = \{0\}$.

To shorten notation, we say that G has property I.A (respectively I.B, II.A, and II.B) if G has the Borsuk-Ulam property in the sense of Definition I.A (respectively Def. I.B, Def. II.A and Def. II.B).

We now state a simple observation we shall frequently use.

1.1. PROPOSITION. *With the above notation we have the following implications*:

$$\begin{array}{ccc} G \text{ has II.A} & \Rightarrow & G \text{ has II.B} \\ \Downarrow & & \Downarrow \\ G \text{ has I.A} & \Rightarrow & G \text{ has I.B} \end{array}$$

Proof. Indeed, as in the classical Z_2-case, composing f with the inclusion of $S(W)$ into $S(U)$ (or into $S(V)$) we get a G-equivariant map from $S(V)$ into $S(V)$ (or into $S(U)$) of degree 0, which gives the horizontal implications. The vertical implications are evident.

We shall use the symbol T^k to denote the k-dimensional torus $S^1 \times \cdots \times S^1$ and Z_p^r for the p-torus $Z_p \times \cdots \times Z_p$, p-prime.

Our main result formulates as follows.

THEOREM 1. *A compact Lie group G has the Borsuk-Ulam property* I.A. *if and only if it is an extension*

$$T^k \to G \to \Gamma$$

of the k-dimensional torus, $k \geq 0$, by a p-group Γ, where p is a prime or q.

THEOREM 2. (a) *If a compact Lie group G has the Borsuk-Ulam property* II.A, *then it is an extension*

$$T^k \to G \to \Gamma$$

of T^k, $k \geq 0$, by a p-group Γ such that every element of Γ has the order p, p-prime or 1.

(b) *The product $G = T^k \times Z_p^r$ of the torus T^k, $k \geq 0$, by the p-torus Z_p^r, $r \geq 0$, has the Borsuk-Ulam property* II.A.

In a first step we will show that it is sufficient to study a connected compact Lie group and a nontrivial finite group only.

1.2. LEMMA. *Let $G_0 \to G \to \Gamma$ be the canonical extension of a compact Lie group G. Then if G_0 has* I.A. (*resp.* II.A) *and Γ has* I.A. (*resp.* II.A) *then G has this property too.*

Proof. Since G_0 is the normal subgroup, $\Gamma = G/G_0$ acts on V^{G_0} and $V^G = (V^{G_0})^\Gamma$. Moreover; for every G-map $f: S(V) \to S(V)$ its restriction $f|_{V^{G_0}}: S(V^{G_0}) \to S(V^{G_0})$ is a Γ-equivariant map. Suppose that $(\deg f^G, |\Gamma|) = 1$ or $V^G = \{0\}$, then from property I.A. of Γ it follows that $\deg f^{G_0} \neq 0$, and consequently $\deg f \neq 0$ by property I.A. of G_0. The proof for property II.A. is similar.

1.3. LEMMA. *Suppose that there exists an orthogonal representation V of G_0 with $V^{G_0} = \{0\}$ and a G_0-map $f: S(V) \to S(V)$ of degree 0, or that Γ fails to have property* I.A. *Then G fails to have property* I.A, *and consequently* II.A.

Proof. Suppose first, contrary to our claim, that for a group G with property I.A., there exist an orthogonal representation of Γ with $V^\Gamma = \{0\}$ and a Γ-equivariant map $f: S(V) \to S(V)$, $\deg f^\Gamma \neq 0$, $(\deg f^\Gamma, |\Gamma|) = 1$ and $\deg f = 0$. Note that the natural projection $G \to \Gamma$ makes V an orthogonal representation of G with $V^G = V^\Gamma$. Moreover the map $f: S(V) \to S(V)$ is G-equivariant with respect to this action of G, which contradicts property I.A. of G.

In order to prove the lemma under the first part of assumption we have to use the induced representation from a subgroup. This fact was explored in the work of M. Atiyah and D. Tall ([**1**], II §2. Prop. 2.2).

1.4. Let V, W be orthogonal representation of a compact Lie group H and $f: S(V) \to S(W)$ an H-map of degree k. If H is a subgroup of a group G with the finite quotient space $|G/H| = m$, then f induces a G-equivariant map

$$\mathrm{ind}_* f\colon S(\mathrm{ind}\, V) \to S(\mathrm{ind}\, W)$$

of the spheres of the induced representations of G, which is of degree k^m.

As a matter of fact, in [**1**] the above statement was proved only for the case of unitary representations of a finite group H. Nevertheless, the same proof remains valid in the orthogonal case.

To construct a G-map contradicting property I.A. of G, it is sufficient to use (1.4) for $H = G_0$ together with the observation that $(\mathrm{ind}\, V)^G = V^0 = \{0\}$. Indeed, the degree of the G-equivariant map

$$\mathrm{ind}_* f\colon S(\mathrm{ind}\, V) \to S(\mathrm{ind}\, W)$$

is $(\deg f)^{|\Gamma|} = 0$. This ends the proof of the lemma.

We can state now the following fact.

1.5. PROPOSITION. *An infinite compact Lie group* G *has property* I.A. *and* II.A. *if and only if its component of the identity* G_0 *is isomorphic to the torus* T^k, $k \geq 1$.

Proof. In 1966 W. C. Hsiang and W. Y. Hsiang ([**7**]) proved that for every connected nonabelian compact Lie group G there exist an orthogonal representation V of G with $V^G = \{0\}$ and a G-equivariant map $f\colon S(V) \to S(V)$ such that $\deg f = 0$. From this and Lemma 1.3 it follows that G_0 must be abelian.

On the other hand, in [**11**] the author proved that the torus T^k, $k \geq 1$, has property II.A, which completes the proof in respect of Proposition 1.1.

1.6. PROPOSITION. *Every finite p-group* Γ, *p-prime, has the Borsuk-Ulam property* I.A.

Proof. The statement follows from the results of T. tom Dieck (see [**5**] for the complete bibliography of Dieck's result on this subject) or from author's results of [**9**] partially published in [**10**]. Indeed, the mentioned result says that for a G-equivariant map f of a G-CW complex X ($S(V)$ is a G-CW complex being a smooth G-manifold [**5**]) we have the congruence of the Lefschetz numbers

$$L(f) \cong L(f^G) \qquad (\mathrm{mod}\, p)$$

if G is a p-group. For a G-map of the sphere this gives the statement.
In the author's work **[11]** it is shown that

1.7. PROPOSITION. *The p-torus Z_p^r, $r \geq 1$, p-prime has property* II.A.

Later we shall show that if a p-group contains an element of the order p^2 then it fails to have property II.A.

Summing up the above, we are left with the task to show that if a finite group is not p-group then it fails to have property I.A. and if a p-group contains an element of the order p^2 then it fails to have property II.A.

1.8. *Historical remarks.* In our terminology, the classical Borsuk-Ulam theorem states that $G = Z_2$ has property I.A., hence also I.B, if $V^G = \{0\}$ ([**2**]). Observe that since Z_2 has only one nontrivial irreducible representation, property I.B. coincides with property II.B, and I.A. with II.A. It is known and not difficult to check that Z_2 has property I.A.

The case $G = Z_p$, p-prime, was studied by many authors and it is known as the Smith theorem. For example, in **[8]** it is shown that $\operatorname{def} f \cong \deg f^G \pmod p$ for every G-equivariant map $f\colon S(V) \to S(W)$ as in Definition II.A. It follows that Z_p has property II.A.

The case $G = S^1$ was extensively studied by several authors with a view to its applications to nonlinear analysis. In **[10]** it is shown that the torus $T^k = S^1 \times \cdots \times S^1$, $k \geq 1$, has property I.A. Using analytic methods, L. Nirenberg proved that $G = S^1$ has property II.A. ([**12**]). This was also proved independently by E. Fadell, S. Husseini and P. Rabinowitz. They use the relative cohomological S^1-index introduced by them in **[6]**. It is worth pointing out that for $G = S^1$ the above result is covered by an earlier, unfortunately unpublished result of P. Traczyk. In 1977 he showed in his MSc. work that S^1 has property II.A. Moreover his simple geometrical method of proof gives also, as in **[12]**, a relation between the characters of representations V and W for which a G-map $f\colon S(V) \to S(W)$ exists. This method is exposed in the work **[11]**.

On the other hand, the mentioned Hsiangs result (compare with 1.5) states that every connected nonabelian compact Lie group does not have property I.A. Earlier, E. Floyd showed that for Z_{pq}, p, q prime, $p \neq 1$, there exist a unitary representation V of G of complex dimension 2, with $V^G = \{0\}$, and a G-equivariant map

$f\colon S(V) \to S(V)$ of degree 0. The proof is presented in Bredon's book **[4]** (**[4]** I.8). This means that such a group fails to have property I.A.

2. Finite groups without the Borsuk-Ulam properties I.A. and II.A. We wish to show first that a group G fails to have property I.A if its order has two distinct prime divisors, and secondly that a p-group G containing the cyclic group Z_{p^2} does not have property II.A.

The main idea of the proof is to find an example of an H-equivariant map of degree 0, for some subgroups $H \subset G$, and next lift it to a G-equivariant map of the induced representation (1.4).

2.1. THEOREM. *Let G be a finite group such that two distinct prime numbers p, q divide the order of G.*

Then there exists an orthogonal representation V of G with $V^G = \{0\}$ and a G-equivariant map $f\colon S(V) \to S(W)$ such that $\deg f = 0$.

In the proof of Theorem 5, the basic geometric ingredient is the Conner-Floyd operation

$$\square\colon \operatorname{Map}(X, X) \times \operatorname{Map}(Y, Y) \to \operatorname{Map}(X * Y, X * Y)$$

(see **[4]** I.). It allows us to produce, for $|G| = p^l q$, p, q primes, $p \neq q$, a G-equivariant map f of the sphere a representation with $\deg f = 0$. The proof falls naturally into two parts. We first examine the case of $|G|$ odd. We begin by proving the following lemma.

2.2. LEMMA. *Suppose that the statement of Theorem* 2.1 *holds for every group of order $p^l q$, where p, q are distinct odd primes.*

Then it also holds for every group G of odd order such that at least two distinct primes divide the order of G.

Proof. We shall prove the lemma by induction on the order of G. Assume that, for a given $n \in N$, Theorem 5 holds for every group G of odd order smaller than n and let G be a group of odd order $|G| = n$. By the Feit-Thompson theorem G is solvable.

Let $G_0 = \{e\} \subset G_1 \subset \cdots \subset G_{m-1} \subset G_m = G$ be a resolution of G with the factors being cyclic of prime order.

If $|G_{m-1}| = p^l$ for some prime p, then we are reduced to the case $|G| = p^l q$. Otherwise the statement of Theorem 2.1 holds for G_{m-1} by the induction assumption. This means that there exist an orthogonal representation V of G_{m-1} with $V^{G_{m-1}} = \{0\}$ and a G_{m-1}-equivariant map $f\colon S(V) \to S(V)$ of degree zero. Using the induced representation $\operatorname{ind} V$ of $G_m = G$ we construct a G-equivariant map $f\colon S(\operatorname{ind} V) \to S(\operatorname{ind} V)$ of degree 0 (see 1.4), which proves the lemma.

We shall have established Theorem 2.1 for odd order groups if we prove the following proposition.

2.3. PROPOSITION. *Suppose that the order of G is of the form $p^l q$, p, q distinct odd primes, $l \geq 1$.*

Then there exist an orthogonal representation V of G with $V^G = \{0\}$ and a G-equivariant map $f\colon S(V) \to S(V)$ of degree 0.

Proof. Observe first that we need only consider the case of G being an extension of a p-group H by Z_q. Indeed, let $G_0 = \{e\} \subset G_1 \subset G_2 \subset \cdots \subset G_{m-1} \subset G_m = G$ be a resolution of G with cyclic factors. If we take i_0 to be the smallest number for which the numbers $|G_i/G_{i-1}|$, $i < i_0$, and $|G_{i_0}/G_{i_0-1}|$ are relatively prime, then we get a subgroup of G of the desired form. Using once more the induced representation we get the conclusion for G. Recall, that for every $k \in Z$ there exist an orthogonal representation V of Z_q with $V^{Z_q} = \{0\}$ and a Z_q-equivariant map $f\colon S(V) \to S(V)$ such that $\deg f = kq + 1$. Indeed, we can take V to be the one-dimensional unitary representation V^1, and $f(z) = z^{kq-1}$. Since G has an epimorphism φ onto Z_q, f can be considered as a G-equivariant map of $S(V)$ with respect to the G-structure given by φ. On the other hand, H being p-group, it has an epimorphism ψ onto Z_p.

Proceeding as above we can construct an orthogonal representation $\widetilde{W}$ of H with $\widetilde{W}^H = \{0\}$ and an H-equivariant map $\tilde{h}\colon S(\widetilde{W}) \to S(\widetilde{W})$ of degree $mp + 1$, for every $m \in Z$. Taking the induced representation $W = \operatorname{ind} \widetilde{W}$, we get a G-equivariant map $h\colon S(W) \to S(W)$ of degree $(mp+1)^q$ (see 1.4).

We shall use the following well-known fact

2.4. For any orthogonal representation V, W, V_1, W_1 of G, with $\dim V_1 = \dim V$, $\dim W_1 = \dim W$, and G-equivariant maps $f\colon S(V) \to S(V_1)$ and $h\colon S(W) \to S(W_1)$, the join $S(V) * S(W)$ is G-homeomorphic to $S(V \in W)$ and $\deg(f * h) = \deg f \cdot \deg h$.

With the maps constructed at the beginning of the proof, we can use the Conner-Floyd operation $\square$ ([4] II.8.3). We recall this construction for the convenience of the reader. For given CW-complexes X and Y, there is a map

$$\lambda\colon X * Y \to X * Y \cup_Y Y * X \cup_X X * Y$$

defined by

$$(ta, (1-t)b) = \begin{cases} (3ta, (1-3t)b) & \text{for } 0 \le t \le 1/3, \\ ((3t-1)b, (2-3t)a) & \text{for } 1/3 \le t \le 2/3, \\ ((3t-2)a(3-3t)b) & \text{for } 2/3 \le t \le 1. \end{cases}$$

If $f: X \to X$ and $h: Y \to Y$ are any maps, then one may form the union

$$f * 1 \cup \tau \cup 1 * h: X * Y \cup_Y Y * Y \cup_X X * Y \to X * Y,$$

where $\tau: Y * X \to X * Y$ is the canonical map $\tau(tb, (1-t)a) = ((1-t)a, tb)$. The operation $\square$ is defined by

$$f \square h = (f * 1 \cup \tau \cup 1 * h) \cdot \lambda.$$

If X and Y are spheres, then $\deg(f * 1) = \deg f$ and $\deg(1 * h) = \deg h$ (2.4) so that

$$\deg(f \square h) = \deg f + \deg h - 1. \tag{2.5}$$

Note that if X and Y are G-spaces and f, h are G-equivariant then $f \square h$ is also equivariant, because all maps used in the definition of $\square$ are G-equivariant.

Adding t times f and s times h in the sense of the operation $\square$ we get

$$\deg\left(\overset{t}{\underset{1}{\square}} f \square \overset{s}{\underset{1}{\square}} h\right) = t(\deg f - 1) + s(\deg h - 1) + 1. \tag{2.6}$$

Let us apply this to the G-equivariant maps f and h we have defined earlier. We get

$$\deg\left(\overset{t}{\underset{1}{\square}} f \square \overset{s}{\underset{1}{\square}} h\right) = s(\deg h - 1) + tkq + 1.$$

What is left is to show that we can find $m, k \in Z$, and $s, t \in N$ such that

$$s(\deg h - 1) + tkq = -1.$$

Put $t = 1$. It is sufficient to find $s \in N$, $k \in Z$, such that

$$s(\deg h - 1) + kq = -1.$$

This will be proved by showing that $((\deg h - 1), q) = 1$ and $\deg h - 1 > 1$ for some m. But

$$\deg h - 1 = (mp+1)^q - 1 = m^q p^q + \binom{q}{1} m^{q-1} p^{q-1} + \cdots + q \cdot mp.$$

Let m be any prime number greater than p and q. Then $q|\binom{q}{1}$ and $q \nmid m^q p^q$, which shows that $((\deg h - 1), q) = 1$ and $\deg h - 1 > q$.

We have to add that in our construction we get a representation with fixed point set of G equal to $\{0\}$, as follows from the face with $(V \oplus W)^G = V^G \oplus W^G$ and 2.4. This completes the proof of Proposition 2.3.

The task is now to prove Theorem 2.1 for groups of even order.

2.7. Proposition. *Suppose that G is a finite group of even order and an odd prime divides the order of G.*

Then there exist an orthogonal representation V of G with $V^G = \{0\}$ and a G-equivariant map $f: S(V) \to S(V)$ of degree zero.

Proof. The proof is similar to that of Proposition 2.3. Let $G_{(p)}$ be the Sylow p-subgroup of G, $p||G|$, and $G_{(2)}$ the Sylow 2-subgroup of G. Just as in the proof of Lemma 3, we can find an orthogonal representation $\widetilde{V}$ of $G_{(p)}$ with $\widetilde{V}^{G_{(p)}} = \{0\}$ and a $G_{(p)}$-equivariant map $f: S(\widetilde{V}) \to S(\widetilde{V})$ of degree $mp + 1$, for every $m \in Z$. Also, for every $k \in Z$ we can find an orthogonal representation $\widetilde{W}$ of $G_{(2)}$ with $\widetilde{W}^{G_{(2)}} = \{0\}$ and a $G_{(2)}$-equivariant map $h: S(\widetilde{W}) \to S(\widetilde{W})$ of degree $sk + 1$.

Taking the induced representations $V - \operatorname{ind} \widetilde{V}$, $W = \operatorname{ind} \widetilde{W}$, we get G-equivariant maps

$$f: S(V) \to S(V), \quad h: S(W) \to S(W)$$

of degrees $(mp+1)^a$ and $(2k+1)^b$ respectively, where $a = |G/G_{(p)}|$, $b = |G/G_{(2)}|$. Note that b is an odd number.

As in Proposition 2.3, we form the G-equivariant map $\overset{t}{\underset{1}{\square}} f \square \overset{s}{\underset{1}{\square}} h$ of the sphere

$$\underbrace{S(V) * \cdots * S(V)}_{t} * \underbrace{S(W) * \cdots * S(W)}_{s} = S(tV \oplus sW).$$

By (2.5),

$$\deg\left(\overset{t}{\underset{1}{\square}} f \square \overset{s}{\underset{1}{\square}} h\right) = t(\deg f - 1) + s(\deg h - 1) + 1.$$

Write $\deg f - 1 = \gamma$, $\deg h - 1 = \delta$.

Then $\deg\left(\overset{t}{\underset{1}{\square}} f \square \overset{s}{\underset{1}{\square}} h\right) = t\gamma + s\delta + 1$. We thus have to determine $m, k \in Z$ for which $(\gamma, \delta) = 1$ and $\delta < 0$.

Put $m = 1$; then $\gamma = (p+1)^a - 1$ is odd, and consequently $\gamma = p_1^{\alpha_1} \cdot p_M^{\alpha_M}$, where $p_i \neq 2$. Take k such that $2k + 1 = p_1 \cdot p_2 \cdots p_M$.

Then $\delta = (2k+1)^b - 1 = -(p_1 \cdots p_M)^b - 1 < 0$, and $(\gamma, \delta) = 1$, since $p_i | \delta$ for every $p_i \in \{p_1, p_2, \dots, p_M\}$. This shows that there exist $t, s \in N$ such that $t\gamma + s\delta = -1$, which proves Proposition 2.7. The proof of Theorem 2.1 is complete.

We emphasize that the proof gives more, namely the representation V in the statement of Theorem 5 can be taken complex unitary.

The remainder of this section will be devoted to the proof the following statement.

2.8. Theorem. *Let G be a finite p-group with at least one element of the order p^2, p-prime. Then there exist a pair of orthogonal representations V, W of G with* $\dim W = \dim V$, $W^G = V^G = \{0\}$, *and a G-equivariant map $f: S(V) \to S(W)$ of degree* 0.

We start with the observation that it is sufficient to consider the case $G = Z_{p^2}$, as it follows from 1.4. In order to prove Theorem 2.8 for $G = Z_{p^2}$, we need some further notation. We shall denote by V^1 the unitary representation of Z_{p^2} of complex dimension one given by the inclusion $Z_{p^2} \subset S^1 = U(1)$. Accordingly, V^p denotes the tensor product $\underbrace{V \otimes \cdots \otimes V}_{p}$ of V. Note that Z_{p^2} acts freely on $S(V^1)$, and for every $x \in S(V^p)$ the isotropy group is $Z_p \subset Z_{p^2}$.

We shall have established Theorem 6 if we prove the following

2.9. Proposition. *Let $V = mV^1$, $W = mV^p$ be unitary representations of Z_{p^2} with V^1, V^p described above.*

Then for every $k \in Z$, there exists a Z_{p^2}-equivariant map $f: S(V) \to S(W)$ of degree $p^m + kp^2$. In particular, there exists a Z_{p^2}-map $f: S(V) \to S(W)$ of degree 0, provided $m \geq 2$.

Proof. This is a particular case of a more general result proved by C. Bowszyc ([**3**], Th. 3.1). In our terminology his theorem says that the function:

$$\deg: [S(V), S(V)]_{Z_{p^2}} \to Z$$

assigning to an equivariant homotopy class $[f]$ the degree of f is a bijection onto the set $\{n \in Z : n = r + kp^2, k \in Z\}$ where r is the degree of one particular Z_{p^2}-map $f: S(V) \to S(W)$. Taking the Z_{p^2}-map $f: S(V) \to W \backslash \{0\}$ we get, after normalization, a

Z_{p^2}-equivariant map $f: S(V) \to S(W)$ of degree p^m. Substituting $r = p^m$ in Bowszyc's result we get the conclusion of Proposition 2.9. For a fuller treatment we refer the reader to **[3]**.

Proposition 2.9 has also a very illustrative and geometric proof, the idea of which was suggested by K. Geba. We present its main points without details. First observe that instead of a map of spheres we consider a Z_{p^2}-equivariant smooth map of pairs $f: (D(V), S(V)) \to (W, W\setminus\{0\})$ a study of degree of f at 0 relative $S(V)$.

Secondly, by homotopy arguments, we can assume that our f is equal to a particular Z_{p^2}-equivariant map f_0 on some smaller disc $D_\zeta(V) \subset D(V)$. It is most convenient to take $f(z_1, \ldots, z_m) = (z_1^p, \ldots, z_m^p)$ as this map f_0. Since Z_{p^2} acts freely on the ring $D(V)\setminus \mathring{D}_\zeta(V)$, by transversality arguments we can assume that zero is a regular value of $f: D(V)\setminus \mathring{D}_\zeta(V) \to W$ and $f^{-1}(0) \cap (S(V) \cup S_\zeta(V)) = \varnothing$. Moreover, composing f if necessary with some Z_{p^2}-diffeomorphism of $D(V)\setminus \mathring{D}_\zeta(V)$ close to the identity, we can assume that there is at least one nontrivial zero of f on every radius $\{tv\} \subset V$, $t \in R^+$. Having such a map, we can "kill" a particular zero orbit of f composing f with the Z_{p^2}-retraction of $D(V)$ onto $D(V)$ with some narrow conical region containing the orbit removed.

After this procedure we obtain a Z_{p^2}-equivariant map f' of degree $\deg f \pm p^2$ with the sign depending on the sign of orbit. On the other hand we can construct a Z_{p^2}-equivariant map $f: (D(V), S(V)) \to (W, W\setminus\{0\})$ of degree greater than any $k > 0$ (or smaller than any $k < 0$). In fact, the map

$$f(z_1, \ldots, z_m) = (z_1^{p+kp^2}, z_2^p, \ldots, z_m^p)$$

is Z_{p^2}-equivariant of degree $p^{m-1}(p+kp^2)$. Starting from such a map we can get a Z_{p^2}-equivariant map of degree required in Proposition 3.7, by the procedure described above.

With respect to Theorems 2.1 and 2.8 the proof of Theorems 1 and 2 is complete.

2.10. REMARK. As a matter of fact, every 2-group not being Z_2^r contains the cyclic group Z_4 which can be checked by an induction on the order of G. This means that a 2-group has II.A property if and only if it is the 2-torus.

2.11. *Problem.* Has a finite group which all elements are of the order p, p an odd prime, the Borsuk-Ulam property II.A?

It would be desirable to give a similar classification for groups with property I.B, or II.B, but we have not been able to do this. The only natural example known to the author is the Hopf fibration. The Hopf map $\mathscr{H}: S^3 \to S^2$ is an S^3-equivariant map for S^3 acting on S^2 by the projection $\pi: S^3 \to SO(3)$. This shows that S^3 fails to have property II.B. Knowing that S^3 is a subgroup of every connected non-abelian simply-connected Lie group G, one would like to use induced representation once more. But, unfortunately, the induced representation is infinite-dimensional, since S^3 is not cofinite. Nevertheless, this example shows that every finite subgroup of S^3 acting without fixed point on S^2 fails to have property II.B. Consequently every finite supergroup of such a group does not satisfy II.B too.

Recently **[15]**, S. Waner gave a necessary and sufficient condition for the existence of a G-equivariant map, G-finite, from $S(V)$ into $S(W)$, where $W \subseteq V$ are orthogonal representations of G and $V^G = \{0\}$. As a consequence of his result, he got that $G = Z_{pq}$, $p \neq q$, p, q primes, fails to have property I.B. This allow us to conjecture that G has I.A iff it has I.B.

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Received July 6, 1987. The work has been completed at the Research Center on Mathematics in Barcelona. The author wishes to express his thanks to professor Manuel Castellet for the invitation and hospitality..

UNIVERSITY OF GDAŃSK
UL. WITA STOWOSZA 57
80-952 GDAŃSK, POLAND