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AN ESTIMATE OF THE VOLUME OF A COMPACT SET IN TERMS OF ITS INTEGRAL OF MEAN CURVATURE

Y. D. CHAI

A geometric inequality for a compact set in euclidean 3 space is obtained. The inequality involves volume and integral of mean curvature. Also some property of the compact set is studied. The method of outer parallel bodies is used in the proof.

I. Introduction. For a planar compact set K with area A and perimeter L the classical isoperimetric inequality states:

$$L^2 - 4\pi A \geq 0.$$

Equality in above inequality holds for regular disk. For proof of the inequality above see Guggenheimer [**10**].

The volume and surface area of the compact set W in euclidean 3 space, R^3, will be represented by the functionals $V(W)$ and $S(W)$. The functional $M(W)$ is the integral of the mean curvature and $|M|(W)$ is the integral of absolute mean curvature [**3**].

Geometric inequality involving integral of absolute mean curvature $|M|(W)$ and surface area $S(W)$ in euclidean 3 space has been founded by Russia mathematician I. A. Danelich.

He found the following facts:

(1) $|M|(W_2) \leq |M|(W_1)$ if W_1 and W_2 are compact sets and W_2 is a convex set contained in W_1 [**4**].

(2) If W is a compact set with bounded integral of absolute mean curvature, then

$$S(W) \leq 2/\pi^2 |M|^2 \qquad [\mathbf{5}].$$

The gist both of Danelich's result (2) and the classical geometric inequality $36\pi[V(W)^2] \leq S(W)^3$, [**12**], involving volume $V(W)$ and surface area $S(W)$ is contained in the following inequality:

(3) $V(W) \leq \sqrt{2}/[3\pi^3\sqrt{\pi}]|M|^3(W)$.

In this paper we will derive a new upper bound for $V(W)$ with compact set W having somewhat restricted conditions. It will be a sharper inequality than (3) in the previous paragraph and equality will hold for regular balls in R^3. It should be noted that Danelich's inequality never decides when equality holds. We also improve Danelich's first result (1). In proving our results we will actually establish a bound for the volume of the outer parallel bodies of a given body.

II. Geometric inequality for a compact set and its properties. If the motion is determined by the position of the moving frame (Q, e_1, e_2, e_3), the kinematic density has the form

$$dK = dP \wedge d\sigma \wedge d\pi$$

where dP is the volume element of R^3 at the origin Q of the moving frame, $d\pi$ is the area element of the unit sphere corresponding to the end point of e_3, and $d\pi$ is the element of rotation about e_3. For a compact set W parallel body of W at the distance r is defined to be set

$$W_r = \{x \in R^3 \mid |x - y| \leq r,\ y \in W\}.$$

Now we state *the kinematic fundamental formula* in R^3 without proof which is the work of Blaschke.

PROPOSITION 2.1. *Let D_0 and D_1 be two domains of R^3 bounded respectively by the surface Σ_0 and Σ_1 which we assume to be of class C^2. Let V_i, χ_i be the volume and the euler characteristic of D_i and let F_i, M_i be the area and the integral of the mean curvature of Σ_i $(i = 0, 1)$ respectively. Suppose D_0 is fixed and D_1 is moving and let dK be the kinematic density for D_1. If $\chi(D_0 \cap D_1)$ denotes the euler characteristic of the intersection $D_0 \cap D_1$, then*

$$\int_{D_0 \cap D_1 \neq \varnothing} \chi(D_0 \cap D_1)\, dk = 8\pi^2(V_0\chi_1 + V_1\chi_0) + 2\pi(F_0, M_1 + F_1 M_0).$$

Proof. See [**1**].

Federer considered *sets with positive reach* in his article Curvature Measure [**6**]. *The reach of subset A of R^3 is the largest number ε such that if point x is in R^3 and the distance, $\delta(x)$, from x to A is smaller than ε, then A contains the unique point $\xi(x)$, nearest to x.* It can be easily checked that compact set in R^3 with C^2 surface as its boundary has positive reach. Assuming that reach of A, reach(A), is positive, *the Steiner's formula* is established in the following form; for each topological ball A with C^2-boundary in R^3 and for r, $0 \leq r <$ reach(A), volume of the parallel body A_r of A, $A_r = \{x \in R^3 \mid |x - a| \leq r,\ a \in A\}$, is given by a polynomial of degree 3 in r:

$$V(A_r) = V(A) + S(A) \cdot r + M(A) \cdot r^2 + (4/3)\pi \cdot r^3$$

where V is volume, M is integral of mean curvature.

In fact the above equality follows from Proposition 2.1 and the fact that $A \cap D_r$ is contractible for all the positions of regular ball D_r of radius r which is shown in [6]. This suggests how we get geometric inequality involving volume and integral of mean curvature for special kind of compact sets by the usage of the estimate of volume of parallel bodies and the well known Minkowski inequality [11]. So we define a concept which generalizes the definition of compact convex sets in R^3.

DEFINITION 2.2. A topological 3-ball W in R^3 with C^2-surface as its boundary is called an $M(t)$-compact set if each point in its boundary has at most one negative principal normal curvature with respect to the inward normal vector which is less than $-1/t$ where $t >$ diameter of W.

REMARK 2.3. (1) Every $M(s)$-compact set is an $M(t)$-compact set if $s \geq t$.

(2) Every compact convex set with C^2-surface is an $M(\infty)$-compact set.

(3) Every topological 3-ball with 2-convex surface, [14], is an $M(\infty)$-compact set.

(4) Every topological ball with next-to-convex surface, [9], is an $M(\infty)$-compact set.

LEMMA 2.4. *Let W be an $M(t)$-compact set and G be the set of all motions in R^3. Then for any $g \in G$ and $r < t - \mathrm{diam}(W)$,*

$$\chi(W \cap gD_r) \geq 1 \quad \textit{if } W \cap gD_r \textit{ is nonempty}$$

where D_r is a solid regular ball with radius r and gD_r is the image of D_r by the motion g.

Proof. We will prove that $\beta_i(W \cap gD_r) = 0$ for $i \geq 1$ where β_i is the ith Betti number. For then $\chi(W \cap gD_r) = \beta_0(W \cap gD_r) \geq 1$ if $W \cap gD_r$ is nonempty. But we will show even stronger facts that $H_i(W \cap gD_r) = 0$ for $i \geq 2$.

Consider the Mayer-Vietoris homology sequence with W and gD_r for any $g \in G$ and $r < t - \mathrm{diam}(W)$ where $W \cap gD_r$ is nonempty. Then we have the following exact homology sequence:

$$\begin{aligned} &\to H_{i+1}(W) \oplus H_{i+1}(gD_r) \to H_{i+1}(W \cup gD_r) \to H_i(W \cap gD_r) \\ &\to H_i(W) \oplus H_i(gD_r) \to \end{aligned}$$

Since $H_{i+1}(W) = H_{i+1}(gD_r) = H_i(W) = H_i(gD_r) = 0$ for $i \geq 2$, $H_i(W \cap gD_r) \cong H_{i+1}(W \cup gD_r) = 0$ for $i \geq 2$.

Now suppose $\chi(W \cap gD_r) < 1$ for some $g \in G$ and $r < t - \operatorname{diam}(W)$. Then $r > 0$ and $\beta_1(W \cap gD_r) > 0$ since $\beta_i(W \cap gD_r) = 0$ for $i \geq 2$. So $H_1(W \cap gD_r) \neq 0$. Now we consider the following Mayer-Vietoris exact homology sequence with W and gD_r:

$$\begin{aligned} &\to H_2(W) \oplus H_2(gD_r) \to H_2(W \cup gD_r) \\ &\to H_1(W \cap gD_r) \to H_1(W) \oplus H_1(gD_r). \end{aligned}$$

Since

$$H_2(W) = H_2(gD_r) = H_1(W) = H_1(gD_r) = 0,$$
$$H_2(W \cup gD_r) \cong H_1(W \cap gD_r).$$

Therefore

$$H_2(W \cup gD_r) \neq 0 \quad \text{and so} \quad H^2(W \cup gD_r) \neq 0.$$

By the Alexander duality theorem,

$$H^2(W \cup gD_r) \cong H_0^*(R^3 - W \cup gD_r).$$

So $R^3 - (W \cup gD_r)$ has at least two open components. In fact $R^3 - (W \cup gD_r)$ contains at least one bounded component whose boundary is the union of the subset of the boundary of gD_r and subset of the boundary of W. Let A be such a bounded component and g be the center of gD_r.

Consider the smallest sphere S_g with its center at g which circumscribes $\overline{A} \cup gD_r$ where $\overline{A}$ is the closure of A. Then the radius of S_g is greater than r and less than $r + \operatorname{diam}(W)$. Let q be a point in $S_g \cap (\overline{A} \cup gD_r)$. Then q must be a point in the boundary of W since S_g never intersects the boundary of gD_r.

This means that all the principal normal curvatures at q with respect to inward normal vector are less than $-1/[r - \operatorname{diam}(W)]$. But it is a contradiction because t is greater than $r + \operatorname{diam}(W)$ and W is an $M(t)$-compact set.

Now we use Proposition 2.1 and Lemma 2.4 to estimate the volume of $W_r = \{x \in R^3 \mid |x-y| \leq r,\ y \in W\}$ in terms of the volume $V(W)$ of W, area $S(W)$ of the boundary of W and the integral of mean curvature $M(W)$ defined by $\int_{\text{boundary of } W} [(k_1+k_2)/2]\, dw$, where k_1 and k_2 are functions of principal normal curvatures on the boundary of W.

LEMMA 2.5. *Let W be an $M(t)$-compact set in R^3.*
Then $V(W_r) \leq V(W) + S(W) \cdot r + M(W) \cdot r^2 + (4/3)\pi \cdot r^3$ for r, $0 \leq r < t - \mathrm{diam}(W)$.

Proof. Let W be fixed and D_r be moving by the motion g. So $W \cap gD_r$ is the intersection of W and D_r at position g.

After putting $\chi(W \cap gD_r) = 0$ if $W \cap gD_r$ is empty if we apply Lemma 2.4, then we have for r, $0 \leq r < t - \mathrm{diam}(W)$,

$$
\begin{aligned}
(1) \qquad \int_{g \in G} \chi(W \cap gD_r)\, dk &= \int_{W \cap gD_r \neq \varnothing} \chi(W \cap gD_r)\, dk \\
&\geq \int_{W \cap gD_r \neq \varnothing} dp \wedge d\sigma \wedge d\pi.
\end{aligned}
$$

On the other hand, $\chi(D_r) = \chi(W) = 1$, $V(D_r) = (4/3)\pi \cdot r^3$ and $M(D_r) = 4\pi r$. So by Proposition 2.1 we have

$$
\begin{aligned}
(2) \quad &\int_{W \cap gD_r \neq \varnothing} \chi(W \cap gD_r)\, dk \\
&\qquad = 8\pi^2[V(W) + (4\pi/3)r^3 + S(W) \cdot r + M(W) \cdot r^2].
\end{aligned}
$$

By (1) and (2), we have for r, $0 \leq r < t - \mathrm{diam}(W)$

$$V(W_r) \leq V(W) + S(W) \cdot r + M(W) \cdot r^2 + (4\pi/3)r^3.$$

COROLLARY 2.6. *If W is an $M(\infty)$-compact set, then*

$$V(W_r) \leq V(W) + S(W) \cdot r + M(W) \cdot r^2 + (4\pi/3) \cdot r^3 \quad \textit{for } r,$$

$0 \leq r < \infty$ and "=" holds if W is a compact convex subset in R^3.

Proof. Since W is a compact set diam(W) is finite. Now our corollary follows from Lemma 2.5. If W is a compact convex subset in R^3, (1) holds with equality since $\chi(W \cap gD_r) = 1$ if $W \cap gD_r$ is nonempty. So this fact and (2) prove the second statement.

One of our goals in this paper is to estimate the volume of $M(\infty)$-compact sets in terms of the integral of the mean curvature of the boundary of the set. For this purpose we may need the following inequality which is an estimate of the volume of the sum of sets from below.

PROPOSITION 2.7. *Let A and B be nonempty measurable sets in R^3. Then*

$$[V(A+B)]^{1/3} \geq [V(A)]^{1/3} + [V(B)]^{1/3}$$

where $A+B = \{a+b \mid a \in A, \ b \in B\}$.

Proof. See **[11]**.

Note that in the proposition above if B is a regular ball centered at the origin of radius r, then $A+B = A_r$.

THEOREM 2.8. *Let W be an $M(\infty)$-compact set in R^3. Then*

$$V(W) \leq (1/48\pi^2)[M(W)]^3.$$

Proof. By Proposition 2.7 we have for r, $0 \leq r < \infty$

$$V(W_r) \geq V(W) + [3 \cdot (4\pi/3)^{1/3} \cdot (V(W))^{2/3}]r + [3(4\pi/3)^{2/3}(V(W))^{1/3}]r^2 + (4\pi/3)r^3. \tag{3}$$

On the other hand, if W is an $M(\infty)$-compact set, by Corollary 2.6, we have

$$V(W_r) \leq V(W) + S(W)r + M(W)r^2 + (4\pi/3)r^3 \quad \text{for } r, \ 0 \leq r < \infty. \tag{4}$$

From (3) and (4) we have

$$[3(4\pi/3)^{1/3}(V(W))^{2/3}]r + [3(4\pi/3)^{2/3}(V(W))^{1/3}]r^2 \leq S(W)r + M(W)r^2 \quad \text{for } r, \ 0 \leq r < \infty. \tag{5}$$

In (5) if we send r to ∞, then we have

$$[3(4\pi/3)^{2/3}(V(W))^{1/3}] \leq M(W). \tag{6}$$

Equivalently we have

$$V(W) \leq (1/48\pi^2)[M(W)]^3.$$

Inequality (6) is sharper than Danelich's result **[4]** for the $M(\infty)$-compact sets and equality holds when W is a regular ball.

COROLLARY 2.9. *If W is an $M(\infty)$-compact set and K is a convex subset of W, then*

$$M(K) \leq M(W).$$

Proof. It follows from the fact that $V(K_r) \leq V(W_r)$ and Corollary 2.6 for $r \to \infty$.

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SUNG KYUN KWAN UNIVERSITY
300 CHUN CHUN DONG
SUWON, 440-746 KOREA