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THE MODULI SPACE OF GENUS FOUR DOUBLE COVERS OF ELLIPTIC CURVES IS RATIONAL

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This paper is devoted to a proof of the rationality of the moduli space of those genus four smooth complex projective curves which are double covers of some elliptic curve. The study of the canonical model of a genus four curve as above allows to reduce the initial moduli problem to a simple one in plane projective geometry; this last formulation leads to compute an explicit representation of a certain group on a vector space and its corresponding field of invariants.

Let C be an irreducible, smooth, projective curve defined over the field of complex numbers.

We call C elliptic-hyperelliptic (e.h. for short) if it admits a degree two morphism $\pi: C \to E$ onto an elliptic curve. We denote by $\mathscr{M}_g^{\mathrm{eh}}$ the moduli space of e.h. curves of genus g. The aim of this note is to present a proof of the following:

THEOREM. *$\mathscr{M}_4^{\mathrm{eh}}$ is rational.*

We proceed as follows.

In §1 the canonical model of a generic e.h. curve C (of genus 4) is shown to be complete intersection of a unique cubic cone R and a unique quadric. By looking at the tangent space to the canonical space at the vertex of R, in §2, we associate to C a pair (Z, γ), where Z and γ are smooth coplanar curves of degree 3 and 2 respectively, and we are able to show that $\mathscr{M}_4^{\mathrm{eh}}$ is birational to

$$\{(Z, \gamma)\}/\mathbf{PGL}(3).$$

After fixing a quadratic form defining γ we can prove that $\{(Z, \gamma)\}/\mathbf{PGL}(3)$ is birational to

$$H^0(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(6))/G_{l_0}$$

where G_{l_0} is a $\mathbf{C}^*$-extension of $\mathbf{Z}_2$.

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In §3 we compute the representation of G_{l_0} on $H^0(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(6))$ and we show that its G_{l_0}-invariant field is purely transcendental over $\mathbf{C}$, completing the proof of the theorem. We wish to thank the referee for pointing out a mistake that we made in the first draft of the paper and for his helpful comments.

NOTATION. As usual we denote by $\mathscr{O}_X$ and Ω_X the structure sheaf and the canonical sheaf of the irreducible, smooth, projective variety X. For any invertible sheaf $\mathscr{F}$ on X we denote by $|\mathscr{F}|^*$ the projectivized dual of $H^0(X, \mathscr{F})$. We denote by $\mathscr{M}_g$ the moduli space of smooth projective curves of genus g, and by [C] the point of $\mathscr{M}_g$ representing the isomorphism class of the smooth curve C of genus g.

If a is an element of a certain group we denote by $\langle a \rangle$ the subgroup generated by a.

1. Some geometry of elliptic-hyperelliptic curves. Let us start with some recalls on elliptic-hyperelliptic curves of genus $g \geq 4$ (see [3] for details).

Let $\pi: C \to E$ be the ramified double cover associated to an elliptic involution i on C (E is the quotient of C by $\langle i \rangle$). To such a cover one can associate the branch locus B on E, which is an effective divisor of even degree on E, and a half $\mathscr{A}$ of the divisor class of B, defined uniquely so that $\pi^*\mathscr{A} = \mathscr{O}_C(\pi^{-1}(B))$. The canonical model $\widetilde{C} \subset |\Omega_C|^*$ of C lies on the elliptic normal cone

$$R = \bigcup_{p \in \widetilde{C}} \overline{PiP}$$

where $\overline{PiP}$ denotes the line joining conjugated points under i. From the natural decomposition

$$H^0(C, \Omega_C) \cong H^0(E, \Omega_E) \oplus H^0(E, \mathscr{A})$$

it follows that R is the cone with vertex $|\Omega_E|^*$ and projecting the elliptic normal curve $\widetilde{E} \subset |\mathscr{A}|^*$. $|\Omega_E|^*$ and $|\mathscr{A}|^* \subset |\Omega_C|^*$ are the fixed subspaces for the projective transformation of $|\Omega_C|^*$ inducing i on $\widetilde{C}$.

$\widetilde{C}$ is also the complete intersection of R and of a suitable quadric containing $\widetilde{C}$.

The branch points on $\widetilde{E}$ are exactly the intersection $|\mathscr{A}|^* \cap \widetilde{C}$ so that on $\widetilde{E}$ they are cut by any quadric through $\widetilde{C}$ not containing $\widetilde{E}$.

We set

$$\mathscr{M}_g^{\text{eh}} = \{[C] \in \mathscr{M}_g : C \text{ is e.h.}\}.$$

By considering the family of all elliptic normal cones in $\mathbf{P}^{g-1}$ and the family of all quadrics in the same $\mathbf{P}^{g-1}$ one can construct the flat family of all e.h. smooth genus g canonical curves $\mathfrak{F} \to \mathscr{B}$. Therefore there is a natural morphism $c: \mathscr{B} \to \mathscr{M}_g$ and we get $\mathscr{M}_g^{\text{eh}} = \operatorname{Im} c$. It follows that $\mathscr{M}_g^{\text{eh}}$ is a subvariety of $\mathscr{M}_g$ which by [2] is irreducible and of dimension $2g-2$. We observe, for the benefit of the reader, that $\mathscr{M}_g^{\text{eh}}$ coincides, in the notation of [2] with $S(2, 1; 1, \dots, 1)$. Obviously $\mathscr{M}_g^{\text{eh}}$ can be regarded as the coarse moduli space for (families of) genus g e.h. curves.

Furthermore [2, Thm. 1] implies immediately the following statement:

1.2. *The generic e.h. curve of genus* $g \geq 3$ *carries exactly one elliptic involution.*

Here by "generic" we mean "outside a Zariski closed set."

2. A birational model of $\mathscr{M}_4^{\text{eh}}$. Let U be the open subset of $\mathscr{M}_4^{\text{eh}}$ corresponding to curves admitting a unique elliptic involution.

For C a canonical curve such that $[C] \in U$ we set:

$$\begin{aligned} Q &= \{\text{the unique quadric containing } C\}, \\ R &= \{\text{the unique cubic cone containing } C\}, \\ V &= \{\text{the vertex of } R\}, \\ TC_V(R) &= \{\text{the tangent cone to } R \text{ at } V\}, \\ T_V(R) &= \{\text{the tangent space to } R \text{ at } V\}. \end{aligned}$$

So in $\mathbf{P}T_V(R)$ we find an elliptic curve $Z = \mathbf{P}TC_V(R)$ and a conic $\gamma = \mathbf{P}(v \in T_V(R): l_v \cdot Q \text{ is not reduced})$, where l_v is the line $\{\lambda v\}$.

It is clear that if C' is isomorphic to C the pair (Z', γ') gotten from C' is projectively equivalent to the pair (Z, γ). Hence we get a rational map

$$U \to \{(Z, \gamma)\}/\mathbf{PGL}(3).$$

In order to construct an inverse of the map above we start with a pair (Z, γ), we consider the set $\{P_1, \dots, P_6\} = Z \cap \gamma$ and the double cover C' of Z branched at $\{P_1, \dots, P_6\}$ with associated line bundle $\mathscr{A} = \mathscr{O}_Z(1)$.

It is straightforward to check that this construction gives the inverse we were looking for, so we have proved

LEMMA 2.1. *$\mathscr{M}_4^{\text{eh}}$ and $\{(Z, \gamma)\}/\mathbf{PGL}(3)$ are birationally isomorphic.*

It is clear that our theorem will follow by proving the rationality of $\{(Z, \gamma)\}/\mathbf{PGL}(3)$.

Let $(x_0:x_1:x_2)$ be homogeneous coordinates in $\mathbf{P}T_V(R) = \mathbf{P}^2$. We fix once and for all the quadratic form $\Gamma_0 = x_0^2 - 4x_1x_2$ in $H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$, let M be its matrix and γ_0 be the conic of equation $\Gamma_0 = 0$.

Since $\mathbf{PGL}(3)$ acts transitively on an open dense subset of $\mathbf{P}H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$, we get the following birational isomorphism:

$$\begin{aligned}\{(Z, \gamma_0)\}/\mathrm{Aut}\,\gamma_0 &= \mathbf{P}H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(3))/\mathrm{Aut}\,\gamma_0 \\ &\overset{\simeq}{\dashrightarrow} \{(Z, \gamma)\}/\mathbf{PGL}(3).\end{aligned}$$

Let G be the special orthogonal group of the quadratic form Γ_0 i.e.:

$$G = \{A, 3 \times 3 \text{ matrices}: AM{}^tA = M, \det A = 1\}.$$

It is a well known fact that:

$$G \cong \mathrm{Aut}\,\gamma_0$$

so we get

$$\{(Z, \gamma_0)\}/\mathrm{Aut}\,\gamma_0 \cong \{(Z, \gamma_0)\}/G.$$

Since G acts on the conic $\gamma_0 \cong \mathbf{P}^1$ and equivariantly on $H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(3))$, we get an induced action of G on $H^0(\gamma_0, \mathcal{O}_{\gamma_0}(3))$ and so an action of G on $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(6))$.

Proposition 2.2. *Let G_{l_0} be the subgroup of G which fixes the line of equation $x_0 = 0$. Then $\{(Z, \gamma_0)\}/G$ and $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(6))/G_{l_0}$ are birationally isomorphic.*

Proof. We will proceed in several steps.

(1) From the exact sequence

$$0 \to \mathcal{I}_{\gamma_0}(3) \to \mathcal{O}_{\mathbf{P}^2}(3) \to \mathcal{O}_{\gamma_0}(3) \to 0$$

where $\mathcal{I}_{\gamma_0}$ denotes the ideal sheaf of γ_0 in $\mathbf{P}^2$ (so $\mathcal{I}_{\gamma_0} = \mathcal{O}_{\mathbf{P}^2}(-2)$), we get the exact sequence:

$$0 \to H^0(\mathbf{P}^2, \mathcal{I}_{\gamma_0}(3)) \to H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(3)) \to H^0(\gamma_0, \mathcal{O}_{\gamma_0}(3)) \to 0.$$

Let $S = H^0(\mathbf{P}^2, \mathcal{I}_{\gamma_0}(3))$: we remark that $S \cong H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$.

One can see immediately that S is an invariant subspace for the action of G on $H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(3))$.

Therefore by [**4**, Chap. iv] there is a subspace $W \subset H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(3))$ complementary to S and invariant for G.

Clearly

$$W \cong H^0(\gamma_0, \mathscr{O}_{\gamma_0}(3))$$

and the action of G on both spaces is equivariant with respect to this isomorphism.

(2) We fix once and for all the linear form x_0 in $H^0(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(1))$ and we let l_0 be the line of equation $x_0 = 0$.

Set

$$G_{l_0} = \{A \in G\colon {}^tAx_0 = \lambda x_0, \lambda \in \mathbf{C}^*\},$$
$$G_0 = \{A \in G\colon {}^tAx_0 = x_0\}.$$

One can readily see that

$$G_0 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix}, \ \alpha \in \mathbf{C}^* \right\}.$$

After calling

$$G_0' = \left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & \alpha^{-1} & 0 \end{pmatrix}, \ \alpha \in \mathbf{C}^* \right\}$$

one finds that $G_{l_0} = G_0 \cup G_0'$ and also that $G_0' = G_0\langle m\rangle$ where

$$m = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We consider the subspace

$$\mathbf{1} \oplus W = \{\alpha\Gamma_0 x_0 + w\colon \alpha \in \mathbf{C}, w \in W\} \subset S \oplus W.$$

$\mathbf{1} \oplus W$ is invariant under the induced action of G_{l_0} and we have a (natural) rational map

$$\phi\colon \mathbf{P}(\mathbf{1} \oplus W)/G_{l_0} \to \mathbf{P}(S \oplus W)/G.$$

We claim that ϕ is birational.

ϕ is dominant: in fact for $\Gamma_0 l + w \in \mathbf{P}(S \oplus W)$ there is an element σ in G such that $\sigma(l) = l_0$ and so $\sigma(\Gamma_0 l + w) = \Gamma_0 l_0 + w' \in \mathbf{P}(\mathbf{1} \oplus W)$. ϕ is also injective on the set of elements for which $\alpha \neq 0$: we may assume $\alpha = 1$, if $\Gamma_0 x_0 + w'$ and $\Gamma_0 x_0 + w''$ are equivalent with respect to G, there are $\sigma \in G$ and $\alpha \in \mathbf{C}^*$ such that $\sigma(\Gamma_0 x_0 + w') = \alpha(\Gamma_0 x_0 + w'')$ in $S \oplus W$.

Since $\sigma(x_0) = \alpha x_0$, $\sigma \in G_{l_0}$ and we are done.

(3) There is an obvious birational map

$$W/G_{l_0} \xrightarrow{\sim} \mathbf{P}(\mathbf{1} \oplus W)/G_{l_0}$$

and by using the isomorphisms

$$W = H^0(\gamma_0, \mathscr{O}_{\gamma_0}(3)) \cong H^0(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(6))$$

(which are equivariant for the G_{l_0}-action) we conclude the proof.

3. The rationality of $\mathscr{M}_4^{\mathrm{eh}}$. To finish the proof of our theorem we have just to prove that $H^0(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(6))/G_{l_0}$ is rational.

Let $(t_1 : t_2)$ be homogeneous coordinates in $\mathbf{P}^1$ and consider the Veronese map

$$\mathbf{P}^1 \to \gamma_0 \subset \mathbf{P}^2$$

given by:

$$\begin{aligned} x_0 &= 2t_1t_2, \\ x_1 &= t_1^2, \\ x_2 &= t_2^2. \end{aligned}$$

Since the action of G_0 on $\mathbf{P}^2$ is given by:

$$\begin{aligned} x_0 &\mapsto x_0, \\ x_1 &\mapsto \alpha x_1, \\ x_2 &\mapsto \alpha^{-1} x_2 \end{aligned}$$

(for $\alpha \in \mathbf{C}^*$), we get on the t_i's:

$$t_1^{6-i} t_2^i \mapsto \alpha^{3-i} t_1^{6-i} t_2^i, \qquad 0 \le i \le 6,$$

so that the induced action of G_0 on $H^0(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(6))$ can be represented by the diagonal matrix:

$$\begin{pmatrix} \alpha^3 &&&&&& \\ & \alpha^2 &&&&& \\ && \alpha &&&& \\ &&& 1 &&& \\ &&&& \alpha^{-1} && \\ &&&&& \alpha^{-2} & \\ &&&&&& \alpha^{-3} \end{pmatrix}$$

By the same argument, the action of the element $m \in G_{l_0}$ can be represented by the matrix

$$\begin{pmatrix} & & & & & & 1 \\ & & & & & -1 & \\ & & & & 1 & & \\ & & & -1 & & & \\ & & 1 & & & & \\ & -1 & & & & & \\ 1 & & & & & & \end{pmatrix}$$

Now let $z_{3-i} = t_1^{6-i} t_2^i$ for $i = 0, 1, 2, 3$ and $z_{10-i} = t_1^{6-i} t_2^i$ for $i = 4, 5, 6$ so that the G_{l_0}-action is given by the standard product of the two matrices above by the vector ${}^t(z_3, z_2, z_1, z_0, z_6, z_5, z_4)$. Then the G_0-invariant field of $H^0(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(6))$ is

$$\mathbf{C}(w_1, w_2, w_3, w_4, w_5, w_6)$$

where

$$\begin{aligned} w_1 &= z_0, \\ w_2 &= z_4 z_3, \\ w_3 &= z_5 z_2, \\ w_4 &= z_6 z_1, \\ w_5 &= z_2 z_1^{-2}, \\ w_6 &= z_3 z_1^{-3}. \end{aligned}$$

In fact, it is enough to observe that the orbit of an element w' with $z_1 \neq 0$ contains a unique element w with $z_1 = 1$ (which is gotten by acting on w' via the element of G_0 having $\alpha = z_1^{-1}$) and that the coordinates of w are

$$(z_3 z_1^{-3}, z_2 z_1^{-2}, 1, z_0, z_6 z_1, z_5 z_1^2, z_4 z_1^3).$$

The G_0-invariant functions $w_1, \dots, w_6$ given above are gotten easily from these coordinates of w.

Now we consider the action of $\langle m \rangle$ on the w_i's. It is immediately seen that m acts as the identity on w_2, w_3, w_4, whereas

$$\begin{aligned} m(w_1) &= -w_1, \\ m(w_6) &= \frac{w_2}{w_4^3} \cdot \frac{1}{w_6}, \\ m(w_5) &= -\frac{w_3}{w_4^2} \cdot \frac{1}{w_5}. \end{aligned}$$

Let us consider the functions

$$f_1 = w_6 + \frac{w_2}{w_4^3} \cdot \frac{1}{w_6},$$
$$f_2 = w_5 - \frac{w_3}{w_4^2} \cdot \frac{1}{w_5},$$
$$f_3 = \left(w_6 - \frac{w_2}{w_4^3} \cdot \frac{1}{w_6} \right) w_1,$$
$$f_4 = \left(w_5 + \frac{w_3}{w_4^2} \cdot \frac{1}{w_5} \right) w_1,$$

and

$$w_1^2.$$

They are clearly invariant for the action of $\langle m \rangle$; furthermore the following relations hold:

$$f_1^2 - 4\frac{w_2}{w_4^3} = \left(\frac{f_3}{w_1}\right)^2; \qquad f_2^2 + 4\frac{w_3}{w_4^2} = \left(\frac{f_4}{w_1}\right)^2, \tag{3.1}$$

$$\begin{cases} w_6^2 - f_1 w_6 + \dfrac{w_2}{w_4^3} = 0, \\ w_1 = f_3 / \left(w_6 - \dfrac{w_2}{w_4^3} \cdot \dfrac{1}{w_6} \right), \\ w_5 = \dfrac{1}{2} \left(f_2 + \dfrac{f_4}{w_1} \right). \end{cases} \tag{3.2}$$

By (3.2) $\mathbf{C}(w_1, \dots, w_6)^{\langle m \rangle} \cong \mathbf{C}(w_1^2, w_2, w_3, w_4, f_1, f_2, f_3, f_4)$, and by (3.1) this last field is isomorphic to $\mathbf{C}(w_1^2, w_4, f_1, f_2, f_3, f_4)$ which is purely transcendental over $\mathbf{C}$: thus yielding the proof of our theorem.

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