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AN ALGEBRAICALLY DERIVED q-ANALOGUE OF A CHARACTER SUM ASSOCIATED WITH A CLASS OF SEMIREGULAR PERMUTATIONS

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The group algebra of the symmetric group can be used to determine the cycle structure of permutations which are obtained as products of designated conjugacy classes. Such matters arise, for example, in certain topological questions and in the embedding of graphs on orientable surfaces. We consider a set of permutations restricted by cycle structure, and use basic hypergeometric series to derive q-analogues associated with the generating functions for the numbers of such permutations. The expressions which are derived pose a number of combinatorial questions about their connexion with the Hecke algebra of the symmetric group.

1. Introduction and background to the problem. A permutation is said to be *p-semiregular* if all of its cycles have the same length p. In this paper we derive a q-analogue for the number $e(k, p)$ of p-semiregular permutations, with k cycles, which are the product of a designated full-cycle and a fixed point free involution. Such permutations occur in several areas of combinatorial theory ([7], [8]).

The q-analogue involves a new summable almost poised terminating ${}_3\phi_2$, discovered independently by Bressoud [4]. This (see (9)) is given in §2, together with other known basic hypergeometric summation theorems included for completeness. In §§3, 4 and 5 we consider the cases $p = 3$, 4, 6, respectively. In a sense to be explained in §6, these are the most interesting cases. Of the expressions we give for q-analogues, namely, M_q in Theorem 3.1, J_q in Theorem 4.1, G_q in Theorem 4.2 and L_q in Theorem 5.1, the one which specialises precisely when $q = 1$ to the correct expression (1) is given in Theorem 4.2.

For permutation problems, q-analogues of generating series often appear when a set of permutations is enumerated with respect to the inversion number or the major index (see [6]), marked by q. Since these are not class functions for the symmetric group, they cannot be used in conjunction with the group algebra of S_n to derive our result. However, the group algebra of the symmetric group is abstractly isomorphic to the Hecke algebra associated with S_n, and explicit (inequivalent) isomorphisms have been given by Wenzl **[12]** and Lusztig **[10]**. The latter involves the Kazhdan-Lusztig polynomials. Our result indicates that combinatorial information may be preserved by one or the other of these explicit isomorphisms.

If θ is a partition of N we write $\theta \vdash N$, and the corresponding conjugacy class if S_N is denoted by $\mathscr{C}_\theta$. This is the set of all permutations in S_N with i_j j-cycles, $j = 1, \ldots$, where $\theta = [1^{i_1} 2^{i_2} \ldots]$. The size of $\mathscr{C}_\theta$ is denoted by h^θ. The irreducible (ordinary) character associated with $\mathscr{C}_\theta$ is denoted by χ^θ, its value at any element of $\mathscr{C}_\theta$ where $\alpha \vdash N$ is denoted by χ^θ_α, and its degree is denoted by f^θ.

To derive an expression for $e(k, p)$, we use the following combinatorial facts which can be deduced from the group algebra $\mathbb{C}S_N$ over $\mathbb{C}$. The proofs are given in **[8]** where use is made of the fact that $K_\theta = \sum_{g \in \mathscr{C}_\theta} g \in \mathbb{C}S_n$ can be expressed as a linear combination of orthogonal idempotents in the centre of $\mathbb{C}S_n$ since the latter is semisimple **[11]**. Explicit use is also made of properties of χ^θ at $g \in \mathscr{C}_{[kp]}$, $g \in \mathscr{C}_{[k^p]}$, and $g \in \mathscr{C}_{[2^{\frac{1}{2}kp}]}$.

PROPOSITION 1.1. *Let*

$$T(k, p) = \sum_{\theta \vdash kp} \frac{1}{f^\theta} \chi^\theta_{[kp]} \chi^\theta_{[k^p]} \chi^\theta_{[2^{kp/2}]}.$$

Then

$$e(k, p) = h^{[kp/2]} h^{[k^p]} \frac{T(k, p)}{(kp)!}. \qquad \square$$

LEMMA 1.2.

(1) $T(k, p)$

$$= \sum_{i=0}^{kp-1} \frac{(-1)^i}{\binom{kp-1}{i}} \left([y^i] \frac{(1-y^2)^{\frac{1}{2}kp}}{1+y} \right) \left([y^i] \frac{(1-(-y)^p)^k}{1+y} \right) \qquad \square$$

Explicit expressions for $e(k, p)$ can be deduced from (1). For example

$$T(2n-1, 4) = 4^{3n-2} \binom{2n}{n} \binom{8n-4}{4n-2}^{-1}, \tag{2}$$

$$T(4n+2, 3) = 12(2^4 \cdot 3^3)^n \binom{4n+2}{n+1} \binom{12n+6}{6n+3}^{-1}. \tag{3}$$

Our purpose now is to derive a q-analogue of the expression for $T(k, p)$. We shall see that Theorem 5.2 is an exact q-analogue of Lemma 2.2. For this, the basic hypergeometric summation theorems given in §2 are needed.

2. Basic hypergeometric formulas. To evaluate the sums considered in the later sections, we require some observations on the basic hypergeometric series ${}_3\phi_2$ defined by

$$(4)\qquad {}_3\phi_2\left(\begin{matrix} a, b, c; q, t \\ d, e \end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n (c; q)_n t^n}{(q; q)_n (d; q)_n (e; q)_n},$$

where

$$(5)\qquad (A; q)_n = (A)_n = (1 - A)(1 - Aq) \cdots (1 - Aq^{n-1}).$$

We also require the q-binomial theorem (Thm. 2.1, [**3**]);

$$(6)\qquad (A; q)_N = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix} (-A)^j q^{j(j-1)/2}$$

where

$$(7)\qquad \begin{bmatrix} N \\ j \end{bmatrix}_r = \frac{(1 - q^{Nr})(1 - q^{(N-1)r}) \cdots (1 - q^{(N-j+1)r})}{(1 - q^{jr})(1 - q^{(j-1)r}) \cdots (1 - q^r)}$$

and we write just $\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]$ when $r = 1$.

Two ${}_3\phi_2$ summations are needed. The first is the q-analogue of Dixon's Theorem ([**5**], [**9**], [**2**, eq. (5.7), p. 216]). If N is even and α is 2 or 4 then

$$(8)\qquad {}_3\phi_2\left(\begin{matrix} q^{-N}, b, c; q, q^{(\alpha-N)/2}/bc \\ q^{1-N}/b, q^{1-N}/c \end{matrix}\right) = \frac{q^{N(\alpha-4)/4} \left(q^{1+N/2}\right)_{N/2} \left(q^{N/2}bc\right)_{N/2}}{\left(bq^{N/2}\right)_{N/2} \left(cq^{N/2}\right)_{N/2}}.$$

The second result is due to Bressoud ([**4**]; eq. (1.4) for $\alpha = 1$, eq. (1.10) for $\alpha = 3$). If N is odd and α is 1 or 3 then

$$(9)\qquad {}_3\phi_2\left(\begin{matrix} q^{1-N}, b, c; q, q^{(\alpha-N)/2}/bc \\ q^{1-N}/b, q^{1-N}/c \end{matrix}\right) = \frac{\left(q^{(N+1)/2}\right)_{(N-1)/2} \left(q^{(N+1)/2}bc\right)_{(N-1)/2}}{\left(bq^{(N+1)/2}\right)_{(N-1)/2} \left(cq^{(N+1)/2}\right)_{(N-1)/2}}.$$

The integer-valued functions $(\varepsilon(n, i), \lambda(n, i), \psi(n, i), \mu(n, i))$, which are introduced without further comment in §§3, 4 and 5, have been constructed to be consistent with the application of whichever of the above summation theorems is appropriate to the particular case.

3. The modulus 3 case. We begin by considering $\varepsilon(n, i)$ for integers $n \geq 1$ and $i \geq 0$ which satisfy the following conditions: for all $n \geq 1$:

$$\begin{aligned}(10)\quad \varepsilon(n, 3i+2) &= \varepsilon(n, 3(n-i-1)) + 6\binom{n-i}{2} - 6\binom{i+1}{2} \\ &= \varepsilon(n, 3(n-i-1)) + 6\binom{n}{2} - 6ni;\end{aligned}$$

for n even and positive

$$\begin{aligned}(11)\quad \varepsilon(n, 3i+1) &= \varepsilon(n, 3(n-i-1)) + 6\binom{n-i}{2} - 6\binom{i+1}{2} \\ &= \varepsilon(n, 3(n-i-1)) + 6\binom{n}{2} - 6ni;\end{aligned}$$

for n odd and positive with $\alpha = 1$ or 3

$$\begin{aligned}(12)\qquad \varepsilon(n, 3i) &= -6(n-1)i - 6i + 3i(3n-1) \\ &\quad - \binom{3i}{2} + \frac{3}{2}(\alpha - n)i - 3i \\ &= \frac{3}{2}(\alpha + n)i - \binom{3i}{2} - 6i;\end{aligned}$$

and for n odd and positive with $\beta = 2$ or 4

$$\begin{aligned}(13)\quad \varepsilon(n, 3i+1) &= -6(n-1)i + (3i+1)(3n-1) \\ &\quad - 6i - \binom{3i+1}{2} + \frac{3}{2}(\beta - n + 1)i - 6i \\ &= \frac{3}{2}(\beta + n - 1)i + 3n - 1 - \binom{3i+1}{2} - 6i.\end{aligned}$$

The first portion of each of (10)–(13) is designed to facilitate our proof of Theorem 3.1. We also note that (10) and (11) do not fully define $\varepsilon(n, i)$ when n is even; indeed, when n is even the $\varepsilon(n, i)$ can take any values which fulfill (10) and (11).

THEOREM 3.1. *Let* $\alpha = 1$ *or* 3, $\beta = 2$ *or* 4 *and*

$$(14)\qquad M_q(n) = \sum_{i=0}^{3n-1} \frac{(-1)^i}{\left[{3n-1 \atop i}\right]} \left\{ [y^i] \frac{(y^3; q^3)_n}{1-y} \right\}^2 q^{\varepsilon(n, i)}$$

where the $\varepsilon(n, i)$ *satisfy* (10)–(13). *Then, for* n *even*

$$(15)\qquad M_q(n) = 0,$$

and for n odd

$$(16)\quad M_q(n) = \left(2 - \frac{(1-q)\,q^{(3/4)(n-1)\beta+2}}{1-q^{(3n+1)/2}}\right) \times \frac{\left(q^{(3/2)(n+1)}; q^3\right)_{(n-1)/2}\left(q^{(3/2)(n+3)}; q^3\right)_{(n-1)/2}}{\left(q^{(3n+5)/2}; q^3\right)_{(n-1)/2}\left(q^{(3n+7)/2}; q^3\right)_{(n-1)/2}}.$$

Proof. Clearly

$$(17)\quad \begin{aligned} M_q(n) &= \sum_{i=0}^{3n-1} \frac{(-1)^i q^{\varepsilon(n,i)}}{\left[{3n-1 \atop i}\right]} \left\{[y^i](1+y+y^2)(y^3q^3;q^3)_{n-1}\right\}^2 \\ &= \sum_{i=0}^{n-1} \frac{(-1)^i q^{\varepsilon(n,3i)}}{\left[{3n-1 \atop 3i}\right]} \left(\begin{bmatrix} n-1 \\ i \end{bmatrix}_3 q^{3\binom{i+1}{2}}\right)^2 \\ &\quad - \sum_{i=0}^{n-1} \frac{(-1)^i q^{\varepsilon(n,3i+1)}}{\left[{3n-1 \atop 3i+1}\right]} \left(\begin{bmatrix} n-1 \\ i \end{bmatrix}_3 q^{3\binom{i+1}{2}}\right)^2 \\ &\quad + \sum_{i=0}^{n-1} \frac{(-1)^i q^{\varepsilon(n,3i+2)}}{\left[{3n-1 \atop 3i+2}\right]} \left(\begin{bmatrix} n-1 \\ i \end{bmatrix}_3 q^{3\binom{i+1}{2}}\right)^2. \end{aligned}$$

By (10), we see that when n is even the $(n-i-1)$ st term in the third sum cancels the i th term in the first sum, while the i th and $(n-i-1)$ st terms in the second sum cancel. Hence (15) is true.

On the other hand, when n is odd the first sum is identical with the third. Furthermore, we may rewrite each sum in terms of rising q-factorials since

$$(18)\qquad \begin{bmatrix} A \\ B \end{bmatrix}_r = \frac{(q^{-rA}; q^r)_B}{(q^r; q^r)_B} (-1)^B q^{rAB - r\binom{B}{2}}$$

and

$$(19)\quad \begin{aligned} (A;q)_{3m+t} &= (A)_t\,(Aq^t)_{3m} \\ &= (A)_t\,(Aq^t;q^3)_m (Aq^{t+1};q^3)_m (Aq^{t+2};q^3)_m. \end{aligned}$$

Therefore, when n is odd

$$(20)\quad M_q(n) = 2\,{}_3\phi_2\left(\begin{matrix} q^{3(1-n)}, q, q^2; q^3, q^{3((\alpha-n)/2-1)} \\ q^{2-3n}, q^{1-3n} \end{matrix}\right) + \frac{(1-q)}{(1-q^{1-3n})} \cdot {}_3\phi_2\left(\begin{matrix} q^{3(1-n)}, q^2, q^4; q^3, q^{3((\beta-n+1)/2-2)} \\ q^{4-3n}, q^{2-3n} \end{matrix}\right)$$

$$= 2\frac{\left(q^{(3/2)(n+1)}; q^3\right)_{(n-1)/2}\left(q^{(3/2)(n+3)}; q^3\right)_{(n-1)/2}}{\left(q^{(3n+5)/2}; q^3\right)_{(n-1)/2}\left(q^{(3n+7)/2}; q^3\right)_{(n-1)/2}} - q^{(3/4)(n-1)\beta+2}(1-q) \times \frac{\left(q^{(3/2)(n+1)}; q^3\right)_{(n-1)/2}\left(q^{(3/2)(n+3)}; q^3\right)_{(n-1)/2}}{\left(q^{(3n+1)/2}; q^3\right)_{(n+1)/2}\left(q^{(3n+5)/2}; q^3\right)_{(n-1)/2}}$$

by (9) and (8) respectively. □

4. The modulus 4 case. We begin by considering $\lambda(n, i)$, which may be arbitrary for i odd and is given by

$$(21)\quad \lambda(n, 4i) = -8i^2 + 2i + 2ni,$$

$$(22)\quad \lambda(n, 4i+2) = -8i^2 - 2i + 2ni$$

for i even.

THEOREM 4.1. *Let*

$$(23)\quad J_q(n) = \sum_{i=0}^{4n-2} \frac{(-1)^i q^{\lambda(n,i)}}{\left[\begin{smallmatrix} 4n-2 \\ i \end{smallmatrix}\right]} \left\{[y^i](y^2; q^2)_{2n-1}\right\} \times \left\{[y^i](y^4; q^4)_{n-1}(1+y^2)\right\}.$$

Then $J_q(n)$ *is zero if* n *is odd, and if* n *is even*

$$(24)\quad J_q(n) = \frac{2q\left(q^{2n}; q^4\right)_{n/2}\left(q^{2n+4}; q^4\right)_{(n-2)/2}}{\left(q^{2n+1}; q^2\right)_{n-1}}.$$

Proof. The terms in $J_q(n)$ with i odd must be zero since each term in braces is the ith coefficient of an even polynomial. Furthermore, the final term in brackets has a different form depending on whether

$i \equiv 0$ or $2 \pmod 4$. Therefore

$$(25) \qquad J_q(n) = \sum_{i=0}^{n-1} (-1)^i q^{\lambda(n,4i)+6i^2-4i} \frac{\left[{n-1 \atop i}\right]_4 \left[{2n-1 \atop 2i}\right]_2}{\left[{4n-2 \atop 4i}\right]} - \sum_{i=0}^{n-1} (-1)^i q^{\lambda(n,4i+2)+6i^2} \frac{\left[{n-1 \atop i}\right]_4 \left[{2n-1 \atop 2i+1}\right]_2}{\left[{4n-2 \atop 4i+2}\right]}$$
$$= \sum_{i=0}^{n-1} (-1)^i q^{2i(n-i-1)} \frac{\left[{n-1 \atop i}\right]_4 \left[{2n-1 \atop 2i}\right]_2}{\left[{4n-2 \atop 4i}\right]} - \sum_{i=0}^{n-1} (-1)^i q^{2i(n-i-1)} \frac{\left[{n-1 \atop i}\right]_4 \left[{2n-1 \atop 2i+1}\right]_2}{\left[{4n-2 \atop 4i+2}\right]}.$$

Now let us replace i by $n-i-1$ in the second sum. Thus

$$J_q(n) = (1-(-1)^{n-1}) \sum_{i=0}^{n-1} (-1)^i q^{2i(n-i-1)} \frac{\left[{n-1 \atop i}\right]_4 \left[{2n-1 \atop 2i}\right]_2}{\left[{4n-2 \atop 4i}\right]}.$$

Thus $J_q(n) = 0$ if n is odd. If n is even,

$$J_q(n) = 2 {}_3\phi_2 \left({q^{-4n+4}, q, q^3; q^4, q^{-2n} \atop q^{-4n+5}, q^{-4n+3}} \right) = \frac{2q \left(q^{2n}; q^4\right)_{n/2} \left(q^{2n+4}; q^4\right)_{(n-2)/2}}{\left(q^{2n+1}; q^2\right)_{n-1}}$$

by (8), and this establishes (24). □

While Theorem 4.1 relied only on the q-analogue of Dixon's summation, Theorem 4.2 requires the additional summation theorem given in (9).

Here we must define $\psi(n, i)$ for $n \geq 1$, $i \geq 0$ by

$$(26) \qquad \psi(n, 4i) = 2i(\alpha + n - 4i - 3),$$

$$(27) \qquad \psi(n, 4i+1) = 2i(-\beta + n - 4i) + (n-1)(\beta - 2),$$

$$(28) \qquad \psi(n, 4i+2) = 2i(\beta + n - 4i - 8) - (n-1)(\beta - 4) - 2,$$

$$(29) \qquad \psi(n, 4i+3) = 2i(-\alpha + n - 4i - 5) + 2\alpha(n-1) - 2.$$

THEOREM 4.2. *Let α be either 1 or 3, and β be either 2 or 4, and let*

$$(30) \qquad G_q(n) = \sum_{i=0}^{4n-1} \frac{(-1)^i q^{\psi(n,i)}}{\left[{4n-1 \atop i}\right]} \left\{ [y^i] \frac{(y^2; q^2)_{2n}}{1+y} \right\} \left\{ [y^i] \frac{(y^4; q^4)_n}{1+y} \right\}.$$

Then $G_q(n) = 0$ *if* n *is even, and if* n *is odd*

$$(31)\qquad G_q(n) = 2q\left(1-q^{2n}\right)\frac{\left(q^{2n+2};q^4\right)_{(n-1)/2}\left(q^{2n+6};q^4\right)_{(n-1)/2}}{\left(q^{2n+3};q^4\right)_{(n-1)/2}\left(q^{2n+1};q^4\right)_{(n+1)}}.$$

Proof. Noting that

$$[y^i]\frac{(y^2;q^2)_{2n}}{1+y} = [y^i](1-y)(y^2q^2;q^2)_{2n-1}$$

and

$$[y^i]\frac{(y^4;q^4)_n}{1+y} = [y^i](1+y^2)(1-y)(y^4q^4;q^4)_{n-1},$$

we see that we must split the sum for $G_q(n)$ into four parts depending on the residue of i (mod 4). Hence

$$(32)\qquad \begin{aligned} G_q(n) = {} & \sum_{i=0}^{n-1}(-1)^i q^{2i(\alpha+n-i-1)}\frac{\left[{n-1 \atop i}\right]_4\left[{2n-1 \atop 2i}\right]_2}{\left[{4n-1 \atop 4i}\right]} \\ & - \sum_{i=0}^{n-1}(-1)^i q^{2i(-\beta+n-i+2)+(\beta+2)(n-1)}\frac{\left[{n-1 \atop i}\right]_4\left[{2n-1 \atop 2i}\right]_2}{\left[{4n-2 \atop 4i+1}\right]} \\ & - \sum_{i=0}^{n-1}(-1)^i q^{2i(\beta+n-i-4)-(n-1)(\beta-8)}\frac{\left[{n-1 \atop i}\right]_4\left[{2n-1 \atop 2i+1}\right]_2}{\left[{4n-1 \atop 4i+2}\right]} \\ & + \sum_{i=0}^{n-1}(-1)^i q^{2i(-\alpha+n-i-1)+2\alpha(k-1)}\frac{\left[{n-1 \atop i}\right]_4\left[{2n-1 \atop 2i+1}\right]_2}{\left[{4n-1 \atop 4i+3}\right]}. \end{aligned}$$

If i is replaced by $n-i-1$ in the second and fourth sums, then

$$(33)\qquad \begin{aligned} G_q(n) = {} & \delta_n\sum_{i=0}^{n-1}(-1)^i q^{2i(\alpha+n-i-1)}\frac{\left[{n-1 \atop i}\right]_4\left[{2n-1 \atop 2i}\right]_2}{\left[{4n-1 \atop 4i}\right]} \\ & - \delta_n\sum_{i=0}^{n-1}(-1)^i q^{2i(\beta+n-i-4)-(n-1)(\beta-4)}\frac{\left[{n-1 \atop i}\right]_4\left[{2n-1 \atop 2i+1}\right]_2}{\left[{4n-1 \atop 4i+2}\right]}, \end{aligned}$$

where $\delta_n = 1+(-1)^{n-1}$. Therefore $G_q(n)$ is clearly 0 if n is even.

If n is odd,

(34)

$$\begin{aligned}G_q(n) &= 2\,{}_3\phi_2\left(\begin{matrix} q^{-4n+4}, q, q^3; q^4, q^{2(\alpha-n-2)} \\ q^{-4n+3}, q^{-4n+1} \end{matrix}\right) \\ &\quad - 2\frac{q^{(\beta-4)(1-n)}(1-q)}{(1-q^{4n-1})}\,{}_3\phi_2\left(\begin{matrix} q^{-4n+4}, q^5, q^3; q^4, q^{2(\beta-n-3)} \\ q^{-4n+3}, q^{-4n+5} \end{matrix}\right) \\ &= \frac{2(q^{2n+2};q^4)_{(n-1)/2}(q^{2n+6};q^4)_{(n-1)/2}}{(q^{2n+3};q^2)_{n-1}} \\ &\quad - \frac{2(1-q)(q^{2n+2};q^4)_{(n-1)/2}(q^{2n+6};q^4)_{(n-1)/2}}{(q^{2n+1};q^2)_n} \\ &= \frac{2(q^{2n+2};q^4)_{(n-1)/2}(q^{2n+6};q^4)_{(n-1)/2}((1-q^{2n+1})-(1-q))}{(q^{2n+1};q^2)_n} \\ &= \frac{2q(1-q^{2n})(q^{2n+2};q^4)_{(n-1)/2}(q^{2n+6};q^4)_{(n-1)/2}}{(q^{2n+1};q^2)_n}.\end{aligned}$$

as desired. □

5. The modulus 6 case. We now consider $\mu(n, i)$, which, for n, $i \geq 0$, satisfy the following requirements: if n is odd, then

$$\begin{aligned}(35)\quad \mu(n, 6i) &= \mu(n, 6(n-i)) + 15(n-i)^2 - 6(n-i) - 15i^2 + 6i \\ &= \mu(n, 6(n-i)) + 15n^2 - 6n - 30ni.\end{aligned}$$

If n is even then, for $\gamma = 2$ or 4,

$$(36)\qquad \mu(n, 6i) = 3(\gamma + n)i - \binom{6i+1}{2}.$$

THEOREM 5.1. *Let*

$$(37)\qquad L_q(n) = \sum_{i=0}^{6n} \frac{(-1)^i q^{\mu(n,i)}}{\binom{6n}{i}} \left\{[y^i](y^3; q^3)_{2n}\right\} \left\{[y^i](y^2; q^2)_{3n}\right\}.$$

Then $L_q(n)$ is zero if n is odd, and, if n is even

$$(38)\qquad L_q(n) = \frac{q^{\frac{3}{2}n(\gamma-4)}(q^{3n+6}; q^6)^2_{\frac{1}{2}n}}{(q^{3n+1}; q^6)_{\frac{1}{2}n}(q^{3n+5}; q^6)_{\frac{1}{2}n}}.$$

Proof. We begin by observing that the ith term in (37) must be zero unless i is a multiple of 6. Therefore

$$(39)\quad \begin{aligned} L_q(n) &= \sum_{i=0}^{n} \frac{q^{\mu(n,6i)}}{\left[{6n \atop 6i}\right]} \left\{[y^{6i}](y^3;q^3)_{2n}\right\}\left\{[y^{6i}](y^2;q^2)_{3n}\right\} \\ &= \sum_{i=0}^{n} \frac{(-1)^i q^{\mu(n,6i)}}{\left[{6n \atop 6i}\right]} \begin{bmatrix} 2n \\ 2i \end{bmatrix}_3 \begin{bmatrix} 3n \\ 3i \end{bmatrix}_2 q^{3\binom{2i}{2}+2\binom{3i}{2}} \\ &= \sum_{i=0}^{n} (-1)^i q^{\mu(n,6i)+15i^2-6i} \frac{\left[{2n \atop 2i}\right]_3 \left[{3n \atop 3i}\right]_2}{\left[{6n \atop 6i}\right]}. \end{aligned}$$

If n is odd, then by (35), the ith terms and the $(n-i)$th terms cancel. Therefore $L_q(n)=0$ if n is odd. If n is even, then by (36) and (39)

$$(40)\quad \begin{aligned} L_q(n) &= \sum_{i=0}^{n} q^{6i((\gamma-n)/2-1)} \frac{(q^{-6n};q^3)_{2i}(q^{-6n};q^2)_{3i}(q)_{6i}}{(q^2;q^2)_{3i}(q^3;q^3)_{2i}(q^{-6n})_{6i}} \\ &= \sum_{i=0}^{n} \frac{(q^{-6n};q^6)_i(q^{-6n+3};q^6)_i(q^{-6n};q^6)_i(q^{-6n+2};q^6)_i(q^{-6n+4};q^6)_i}{(q^2;q^6)_i(q^4;q^6)_i(q^6;q^6)_i(q^3;q^6)_i(q^6;q^6)_i} \\ &\quad\times q^{6i((\gamma-n)/2-1)} \frac{(q;q^6)_i(q^2;q^6)_i(q^3;q^6)_i}{(q^{-6n};q^6)_i(q^{-6n+1};q^6)_i(q^{-6n+2};q^6)_i} \\ &\quad\times \frac{(q^4;q^6)_i(q^5;q^6)_i(q^6;q^6)_i}{(q^{-6n+3};q^6)_i(q^{-6n+4};q^6)_i(q^{-6n+5};q^6)_i} \\ &= {}_3\phi_2\left({q^{-6n},\, q,\, q^5;\, q^6,\, q^{3(\gamma-n)-6} \atop q^{-6n+5},\, q^{-6n+1}} \right) \\ &= q^{(3/2)n(\gamma-4)} \frac{(q^{3n+6};q^6)_{n/2}(q^{3n+6};q^6)_{n/2}}{(q^{3n+1};q^6)_{n/2}(q^{3n+5};q^6)_{n/2}} \end{aligned}$$

by (8). Hence Theorem 5.1 has been established. □

6. Concluding remarks. The proofs of the theorems in §§3–5 rely on the reduction of certain q-binomial coefficient sums to ${}_3\phi_2$'s using the methods outlined in §5 of [**1**], and, in particular, the application of Theorem 5.1. The summands which occur with index i in the q-analogue of Lemma 2.2 have the form

$$\frac{\left[{rn-s \atop ri-t}\right]_{r'} \left[{jn-u \atop ji-v}\right]_{j'}}{\left[{kn-w \atop ki-y}\right]}.$$

It is therefore clear that the simplest q-hypergeometric series are obtained with $rr' = j\ \ j' = k$, to ensure the most extensive cancellation of factors after normalisation. However, factors of the form $(q^a;q^k)_i$ cannot be cancelled if $(a,k)=1$, so there will be at least $\phi(k)$ (where

ϕ is Euler's totient function) entries in the numerator of the resulting ${}_m\phi_{m-1}$ corresponding to the reduced residue class modulo k. Thus, the only ways the ${}_3\phi_2$ summations of §2 can be applied is with $\phi(k) < 4$, so $k = 1, 2, 3, 4, 6$. Examination of these shows that the truly nontrivial cases occur for $\phi(k) = 2$, in which case $k = 3, 4, 6$. These are indeed the instances of k we have considered. Other examples for $k = 3, 4, 6$ can be derived following our methods, and we have merely provided a sample.

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