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SEIFERT SURFACES OF KNOTS IN S^4

DANIEL RUBERMAN

This paper uses some ideas from 3-dimensional topology to study knots in S^4. We show that the Poincaré conjecture implies the existence of a non-fibered knot whose complement fibers homotopically. In a different direction, we show that Gromov's norm is an obstruction to a knot having a Seifert surface made out of Seifert fibered spaces, and hence to being ribbon. We also prove that any 3-manifold is invertibly homology cobordant to a hyperbolic 3-manifold, so that every knot in S^4 has a hyperbolic Seifert surface.

One of the reasons that the study of knots in the 4-sphere has a special character is that the Seifert surfaces that such knots bound are 3-dimensional. Hence the peculiar nature of the topology of 3-manifolds can lead to interesting behavior of 2-knots. In this paper we give several examples of this principle. The first example is to show that the 3-dimensional Poincaré conjecture implies the existence of non-fibered (topological) knots in S^4 whose exteriors are homotopy equivalent to the exterior of a fibered knot. (Similar phenomena have been noticed by J. Hillman and C. B. Thomas [**12**, **13**] and S. Weinberger [**32**].) The second instance is to see how the existence of a "geometric structure" on a Seifert surface influences topological properties of the knot.

Restrictions on the possible geometric structures are obtained via "Gromov's norm" of a 2-knot, defined below. We show that a knot with non-zero norm cannot have a Seifert surface which is a connected sum of Seifert-fibered 3-manifolds. In particular, the norm is seen to be an obstruction to a knot in S^4 being ribbon. A similar obstruction has been found by Bruce Trace [**30**]. In contrast, we will show that any knot has a Seifert surface which is a hyperbolic manifold. This follows from Theorem 2.6, which states that any 3-manifold has an invertible homology cobordism to a hyperbolic manifold. A cobordism W from M to N is called invertible if there is another cobordism W', so that $W \cup_N W' \cong M \times \mathbf{I}$. Without the requirement that the homology cobordism be invertible, this theorem is due to R. Myers [**23**].

1. Non-fibered 4-manifolds. If W is a manifold of dimension 6 or greater, then the theorem of Farrell [7] gives complete criteria for deciding whether W is a fiber bundle over the circle. Roughly, W must have the homotopy type of such a fibration, and some additional K-theoretic conditions must be satisfied. The argument proceeds by first using ambient surgery to find a codimension-one submanifold M which will be a candidate for the fiber, and then to apply the s-cobordism theorem to the complement of M. *Smooth* manifolds of dimension five which do not fiber smoothly are constructed in [**17**]; these depend on Donaldson's theorem about the non-smoothability of certain definite 4-manifolds.

In a similar vein, we show that the 3-dimensional Poincaré conjecture implies the existence of topological 4-manifolds which don't fiber over S^1, although they do satisfy the hypotheses of Farrell's theorem. The reason will be that there is no 3-manifold to serve as the fiber. Taking infinite cyclic covers gives rise to smooth 4-manifolds of the proper homotopy type of (3-manifold $\times\mathbf{R}$) which are not $\mathbf{R}$ cross any 3-manifold. However there is an argument for this which avoids the Poincaré conjecture: In order for Freedman's fake $S^3\times\mathbf{R}$ to be a product, it would have to be a μ-invariant 1 homotopy sphere cross $\mathbf{R}$. But Casson's recent work [**5, 1**] implies that a homotopy sphere has trivial μ-invariant.

We use the same constructions to answer a question (in the topological case) posed by J. Hass [**17**], as to the existence of a 4-dimensional analogue of the sphere/projective-plane theorem of 3-dimensional topology. Specifically, if W is a 4-manifold with $\pi_2(W)=0$ and $\pi_3(W)\neq 0$, then Hass asks if there is an embedded spherical space-form carrying a non-trivial class in $\pi_3(W)$. We show that the manifolds constructed as non-fibering manifolds above also do not have any embedded space-form carrying π_3.

Our results depend on a recent theorem of H. Rubinstein [**25**] on $\mathbf{Z}_3$-actions on the 3-sphere.

THEOREM 1.1 (*Rubinstein*). *A free action of* $\mathbf{Z}_3$ *on a lens space is conjugate to a linear action. In particular, a* 3*-manifold with fundamental group* $\mathbf{Z}_{3^k}$ *whose universal cover is* S^3 *is a lens space.*

If M is a closed 3-manifold, and $\widetilde{M}\to M$ is a regular $\mathbf{Z}_n$-covering, then Atiyah-Singer [**2**] define the α-invariant $\alpha(\widetilde{M},\mathbf{Z}_n)$ in terms of the intersection form of some four manifold over which (some multiple of) the covering extends. The α-invariant may be viewed as an element of the ring $R(\mathbf{Z}_n) = \mathbf{Q}[\chi]/(1+\chi+\cdots+\chi^{n-1})$, where the coefficients are eigenspace signatures as defined, say, in **[6]**. If A is an element of the surgery group $L_4^h(\mathbf{Z}_n)$, then Wall **[31]** defines the multisignature $\rho(A)$, lying in the same ring $R(\mathbf{Z}_n)$. If W is a cobordism between 3-manifolds M and N, and the $\mathbf{Z}[\mathbf{Z}_n]$-valued intersection form of W is represented by A, then the α-invariants of M and N are related by $\alpha(N) - \alpha(M) = \rho(A)$. Thus if the realization theorem for elements of the surgery group worked in dimension 4, there would be many homotopy lens spaces with α-invariants different from those of any genuine lens space.

Unfortunately, it is known (using **[5]**) that the realization theorem fails in this dimension. However, it does work in the topological category in the next dimension up, for "small" fundamental groups. Combined with Theorem 1.1, this leads to our non-fibering result.

THEOREM 1.2. *Assume that the* 3*-dimensional Poincaré conjecture holds. Then there are topological* 4*-manifolds satisfying the hypotheses of Farrell's fibering theorem* [7] *which are not bundles over* S^1. *These can be chosen to be complements of knots in* S^4.

Proof. Suppose that a group G is given by a twisted extension of the integers:

$$G \cong K \times_t \mathbf{Z}.$$

Here t is an automorphism of the group K. Then Farrell and Hsiang give a computation of the L-groups of G in terms of those of K. If $K \cong \mathbf{Z}_n$, then we have an exact sequence:

$$\to L_5^h(K) \to L_5^s(G) \xrightarrow{\partial} L_4^h(K) \xrightarrow{1-t_*} L_4^h(K) \to$$

If t_*, the map induced on $L_4^h(K)$, is the identity, then $L_5^s(G)$ splits as the direct sum $L_5^h(K) \oplus L_4^h(K)$.

The "boundary map" ∂ from $L_5^s(G)$ to $L_4^h(K)$ is given by codimension-one splitting. In practice, this means the following. Suppose M is a 4-manifold with fundamental group G, and W is a 5-dimensional cobordism from M to M' realizing an element A of the L-group $L_5^s(G)$. The surjection from G to $\mathbf{Z}$ is induced by a map f from W to the circle S^1. Making f transverse to a point of S^1 yields a 4-dimensional manifold V embedded in W; the multisignature $\rho(\partial A)$ is then exactly the multisignature of the intersection form on V. By the remarks preceding the theorem, then, the action of A on

M is to change M by varying the α-invariant of an embedded 3-manifold dual to $H^1(M) = \mathbf{Z}$. (We will call the α-invariant of such a 3-manifold a codimension-one α-invariant.)

For example, suppose that M is $S^1 \times L$, where L is a lens space with fundamental group $\mathbf{Z}_{3^k}$. Since t in this case is the identity, $\partial: L_5^s(\mathbf{Z}\times\mathbf{Z}_{3^k}) \xrightarrow{\cong} L_4^h(\mathbf{Z}_{3^k})$. Choose $A \in L_5^s(\mathbf{Z}\times\mathbf{Z}_{3^k})$ such that $\alpha(L)+\rho(\partial A)$ is not the α-invariant of any lens space with $\pi_1 = \mathbf{Z}_{3^k}$. This can be done since there are only finitely many such lens spaces, but infinitely many possible multisignatures. Using topological surgery **[8]**, realize A by a 5-dimensional cobordism. The 4-manifold M' at the end of the cobordism cannot fiber over S^1, although it is simple homotopy equivalent to M.

To see this, note that if M' were to fiber over S^1, the fiber would have to be a homotopy lens space, with α-invariant $\alpha(L) + \rho(\partial A)$. Hence the fiber couldn't be a lens space. But (assuming the Poincaré conjecture), this would violate Rubinstein's theorem.

To obtain examples of the same phenomenon where the 4-manifolds are knot complements, we use the same argument, but with different groups. Start with a 2-bridge knot in S^3 whose double branched cover is a lens space $L = L(3^k, q)$. The exterior, X, of the 2-twist spin of this knot is fibered with fiber $L_0 = L$ punctured, and the monodromy of the fibration acts by multiplying by -1 on $\mathbf{Z}_{3^k}$. It is easy to verify that $(-1)_*$ is the identity on $L_4^h(\mathbf{Z}_{3^k})$, so the construction of the previous paragraph works as well to get a manifold Y simple homotopy equivalent to X, but which cannot fiber over S^1. None of the operations affect the boundary of X, so that $S^2 \times D^2$ may be attached to Y to get a knot in S^4 which doesn't fiber. □

Doing the same sort of construction with a little more care, we obtain a negative answer to Hass's question about a 4-dimensional version of the sphere/projective-plane theorem of 3-dimensional topology by finding a 4-manifold with $\pi_2 = 0$, $\pi_3 \neq 0$ and with no spherical space form carrying a non-trivial class in π_3. The construction we give works in the topological case; it would be interesting to have a similar example in the smooth case. If we relax the requirement that $\pi_2 = 0$, the problem becomes easier, even in the smooth case. For instance, we show (Theorem 1.5 below) that there is no embedded 3-manifold with finite fundamental group carrying a nontrivial element of $\pi_3(S^2 \times T^2) \cong \mathbf{Z}$. These results are independent of the Poincaré conjecture.

By definition, a spherical space form is S^3/G, where G is a finite group acting linearly and freely on S^3. The spherical space-forms have been classified [33], and we rely on this classification in the proof of the next theorem. The fundamental groups are all well-known subgroups of SO(4), and are in the list: cyclic groups $\mathbf{Z}_n$, generalized quaternion groups D^*_{4m}, binary dihedral groups $D'_{(2m+1)2^r}$, tetrahedral groups T^*_v, binary octahedral 0^*, binary icosahedral I^*, and the products of these groups with cyclic groups of coprime order. Except for the cyclic groups, there is only one homeomorphism type with a given fundamental group.

THEOREM 1.3. *There is a topological manifold W with $\pi_2 = 0$ and π_3 non-trivial, in which there is no embedded spherical space-form carrying a non-zero class in π_3. W may be chosen simple homotopy equivalent to $S^1 \times L(3, 1)$.*

Proof. We will construct W as in the previous theorem, by acting on $S^1 \times L(3, 1)$ by an element A of $L^s_5(\mathbf{Z} \times \mathbf{Z}_3) = L^s_4(\mathbf{Z}_3)$. There are two α-invariants we wish to avoid; one is the α-invariant α_0 of the lens space $L(3, 1)$. The other comes from the group $G = D^*_8 \times \mathbf{Z}_3$. There is a 3-fold covering $S^3/D^*_8 \to S^3/G$ which has an α-invariant $\alpha_1 \in R(\mathbf{Z}_3)$. Choose A in $L^s_5(\mathbf{Z} \times \mathbf{Z}_3)$ so that the codimension-one α-invariant of W is neither α_1 nor α_0.

We claim that the resulting manifold contains no space-form carrying a non-zero class in π_3. The proof consists of checking the possibility of embedding for each of the space-forms enumerated in [33]. The majority of them can be eliminated by an essentially homological argument.

Note first that $\pi_3(W) = H_3(W) = \mathbf{Z}$, and that the Hurewicz map is given by multiplication by 3. A codimension-one submanifold of any oriented manifold represents a primitive homology class, so any 3-manifold embedded in W and carrying a non-trivial element of $\pi_3(W)$ represents a generator of $H_3(W)$. Let $\widetilde{W}$ be the infinite cyclic cover of W; it is proper homotopy equivalent to $L(3, 1) \times \mathbf{R}$. Any 3-manifold M representing the generator of $H_3(W)$ lifts to $\widetilde{W}$, and thus has a map of degree ± 1 to $L(3, 1)$. In particular, its fundamental group surjects to $\mathbf{Z}_3$.

This eliminates some possibilities, such as the binary icosahedral group I^* (or $I^* \times \mathbf{Z}_n$ with $(n, 120) = 1$). To make more progress, we use the linking form of the 3-manifold M. For any group which surjects onto $\mathbf{Z}_3$, consider the restriction of the linking form of M to the kernel of the map $H_1(M)$ to $\mathbf{Z}_3$. An argument similar to that in [10] shows that if M were to embed in $\widetilde{W}$, this linking form would be hyperbolic.

It is not hard to compute the homology of all the groups and decide which have surjections to $\mathbf{Z}_3$. The only ones which have such surjections such that the kernel supports a hyperbolic linking form are the groups $D^*_{4m} \times \mathbf{Z}_3$ where m is even. To eliminate these manifolds, we must work harder, since in fact the linking form in question is hyperbolic, at least when m is of the form $4k+2$.

The group D^*_{4m} is presented as

$$\{x, y : y^{2m} = 1, \; y^m = x^2, \; x^{-1}yx = y^{-1}\}.$$

Thus its homology is given by $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, generated by the images of x and y under abelianization. Suppose $S^3/D^*_{4m} \times \mathbf{Z}_3$ embeds in W. Then $N = S^3/D^*_{4m}$ embeds in a 4-manifold proper homotopy equivalent to $S^3 \times \mathbf{R}$, and hence (topologically and locally flat) in S^4. We will use the obstructions to such embeddings given in [10] to see that this cannot happen.

According to that paper, for appropriate homomorphisms φ of $H_1(N)$ to $\mathbf{Z}_2$, the associated α-invariant will satisfy an inequality. More precisely, there are three non-trivial homomorphisms of $H_1(N)$ to $\mathbf{Z}_2$, which we will denote φ_x, φ_y, and φ_{xy}. The map φ_x sends x to 1 and y to 0; the map φ_y does the opposite, and φ_{xy} is the sum of the other two. Applying Theorem 2.1 of [10], we find that for two of the three homomorphisms, we must have: $\alpha(N, \varphi) = \pm 1$.

We can calculate the α-invariant of the involution on the double cover $\widetilde{N}$ of N by using a formula due to Hirzebruch [12]. He observes that when a manifold such as $\widetilde{N}$ is covered by S^3, the α-invariant of an action on $\widetilde{N}$ may be computed as the average of the α-invariant's of a coset of $\ker[\pi_1(N) \to \mathbf{Z}_2]$ whose elements act on S^3. Carrying out this computation, we find that

$$\alpha(N, \varphi_y) = -1,$$
$$\alpha(N, \varphi_x) = \alpha(N, \varphi_{xy}) = -\frac{1}{2} - \frac{1}{2m}\sum_{j=1}^{m} \cot^2\left(\frac{\pi j}{2m}\right) = -\frac{m}{2}.$$

The last equality may be found in [34]. Hence for m greater than two, N cannot embed in S^4, and the manifold it 3-fold covers cannot embed in W. For $m = 2$, N does in fact embed in S^4, as the boundary of a tubular neighborhood of an embedded projected plane. But we prevented the corresponding 3-manifolds $S^3/(D^*_8 \times \mathbf{Z}_3)$ from embedding in W by choosing the codimension-one α-invariant of W to be different from the α-invariant of this 3-manifold. Hence there is no spherical space-form embedded in W, carrying a non-trivial class in π_3.

REMARK 1.4. The same technique gives manifolds simple homotopy equivalent to $S^1 \times L(3^k, q)$ with no embedded space-form carrying a non-zero class in π_3, for any k and q. Similarly, the knot complements in Theorem 1.2 can be chosen so that there is no (punctured) space-form as Seifert surface of the knot.

As mentioned above, if one does not require the vanishing of π_2, then it is easier to find 4-manifolds (even smooth ones) with no space-form carrying a non-trivial element of π_3. We give one such example, which has a somewhat stronger property.

THEOREM 1.5. *There is no* 3*-manifold with finite fundamental group embedded in the manifold* $S^2 \times T^2$ *carrying a non-trivial element of* $\pi_3(S^2 \times T^2)$.

Proof. Let $f \colon M \to S^2 \times T^2$ be a map whose induced map on π_3 is non-trivial. If $\pi_1(M)$ is finite, then the induced map on fundamental groups is trivial, and so f lifts to the universal cover $S^2 \times \mathbf{R}^2$. If f was an embedding and non-trivial on π_3, then the lifted map has these properties as well, so it suffices to show that there is no embedding of M in $S^2 \times \mathbf{R}^2$ which is non-trivial on π_3.

Now since $\pi_1(M)$ is finite, M has the rational homology of S^3, and its universal cover is homotopy equivalent to S^3. Therefore there is a well-defined, $\mathbf{Q}$-valued linking number between 1-cycles in M. Using this, one defines a $\mathbf{Q}$-valued Hopf invariant of a map f from M to S^2 as the linking number between the inverse images of two distinct regular values of f in S^2. This rational Hopf invariant has the property that the Hopf invariant of the induced map $\widetilde{M} \simeq S^3 \to M \xrightarrow{f} S^2$ is $|\pi_1(M)|$ times the Hopf invariant of f. Since $\pi_3(S^2)$ is detected by the usual Hopf invariant, it follows that a map $f \colon M \to S^2$ is trivial on π_3 if and only if it has trivial Hopf invariant.

Now if M happens to be the boundary of a 4-manifold V with the rational homology of a ball, then linking numbers of 1-cycles in M may be calculated in term of intersections of surfaces which they bound in V. In particular, if $f \colon M \to S^2$ is a map which extends over a rational ball then the inverse images (in V) of two regular values of the extended map give surfaces with boundary the cycles whose linking number is the Hopf invariant of f. The surfaces are disjoint, so it follows that the Hopf invariant of a map which extends in this way must be zero.

To apply these remarks in our situation, notice that $\pi_3(S^2 \times \mathbf{R}^2) = \pi_3(S^2)$ by projection. Moreover, an easy Mayer-Vietoris calculation (compare **[10]**) shows that if a rational homology sphere such as M embeds in $S^2 \times \mathbf{R}^2$, it separates, and bounds a rational ball V. But the projection to S^2 provides an extension of the map $M \to S^2$ over V; thus by the previous paragraph the induced map on π_3 must be trivial. □

2. Geometric structures on Seifert surfaces. The previous section used surgery-theoretic constructions in dimension 4 to restrict the possible Seifert surfaces of knots in S^4. In this section, we use more specifically 3-dimensional ideas to give restrictions of a different nature. In contrast to the previous section, the knots here will be smooth.

A compact 3-manifold is hyperbolic if its interior has a complete hyperbolic structure. In a mild abuse of language, we will say that a manifold with some 2-sphere boundary components is hyperbolic if the manifold obtained by filling in 3-balls is hyperbolic. Similarly, we will talk about punctured 3-manifolds being Seifert-fibered. The main results of this section are that every knot in S^4 has a hyperbolic Seifert surface, but not every knot has a Seifert-fibered Seifert surface. This second fact can be used to demonstrate that certain knots are not ribbon knots.

To demonstrate that certain knots have no Seifert Seifert surface, we will use Gromov's norm **[11, 28]**. If $z = \sum r_i \sigma_i$ is a real singular chain in a space X, then Gromov defines the norm of z as $\sum |r_i|$. The norm of a homology class in X is the infimum of the norms of chains representing that class. In particular, Gromov's norm of a closed orientable manifold is defined to be the norm of the fundamental class of the manifold.

The norm of a hyperbolic manifold is a constant times its volume in the hyperbolic metric. Gromov's norm adds under connected sum, and moreover if M is a 3-manifold which is a union along incompressible tori of hyperbolic manifolds and Seifert-fibered manifold, the norm of M is the sum of the norms of the hyperbolic pieces. If M is a punctured 3-manifold, then we will define its norm as the norm of the filled-in manifold.

DEFINITION 2.1. Let K be a knot in S^4. The Gromov norm of K, $|K|$, is the infimum of the Gromov norm of all Seifert surfaces of K.

There are other reasonable definitions of $|K|$; for instance one might define $|K|_0$ as the norm of the generator of $H_3(Y) \cong \mathbf{Z}$, where Y is the surgered manifold $S^4 - K \times D^2 \cup S^1 \times D^3$. It is easy to see that $|K|_0 \leq |K|$, and one might conjecture that they are equal. The proof of Proposition 2.4 shows that the two norms coincide for fibered knots. Gabai **[9]** has shown equality for the analogous quantities defined for knots in S^3.

LEMMA 2.2. *If K is a knot in S^4 which has a Seifert-fibered Seifert surface, then its norm is 0. The same holds if the Seifert surface is a graph manifold.*

Proof. This follows directly from the fact that the norm of a Seifert-fibered manifold, or a sum of such manifolds along tori or 2-spheres, is trivial. □

One observes directly that the Gromov norm thus provides an obstruction to a 2-knot being ribbon.

COROLLARY 2.3. *If K is a ribbon knot in S^4, then $|K| = 0$.*

Proof. It is well known that a ribbon knot in S^4 has a Seifert surface which is a connected sum of $S^2 \times S^1$'s. Such a manifold has norm 0 by the lemma. □

To use this corollary to find non-ribbon knots, we must find some knots with non-trivial norm. This is done in the following proposition.

PROPOSITION 2.4. *Suppose K is a fibered knot in S^4, whose fiber M is hyperbolic. Then the norm of K is the norm of M: $|K| = |M|$. In particular, $|K| \neq 0$, the knot is not ribbon, and has no Seifert-fibered Seifert surface.*

Proof. Let Y be the surgered manifold $S^4 - K \times D^2 \cup S^1 \times D^3$. Then Y is fibered over S^1 with fiber M. The infinite cyclic cover of Y, $\widetilde{Y}$, is diffeomorphic to $M \times \mathbf{R}$ and is therefore homotopy equivalent to M. If N is any other Seifert surface for K, then it lifts to $\widetilde{Y}$ and represents a generator of $H_3(\widetilde{Y}) \cong \mathbf{Z}$. The homotopy equivalence of $\widetilde{Y}$ with M restricts to a degree ± 1 map from N to M. Therefore **[28]** the norm of N is greater then or equal to that of M. It follows that $|K| = |M|$. □

Of course there are many knots in S^4 with hyperbolic fibers. For example, if K is a hyperbolic knot in S^3, then for large enough p, the p-fold cover of S^3 branched along K is a hyperbolic manifold. This manifold (punctured) is the fiber of the p-twist spin of K which is thus the desired knot. We remark that any invariant of 3-manifolds which is non-increasing under degree-one maps will provide obstructions to knots being ribbon in exactly the same way, provided it vanishes on connected sums of $S^2 \times S^1$'s. For example the "Seifert volume" of Brooks and Goldman **[4]** has this property. Using this observation, we can show that many fibered knots in S^4 whose fibers are Seifert-fibered 3-manifolds are not ribbon knots. For example, the Brieskorn homology spheres $\Sigma(p, q, r)$ have non-vanishing Seifert volume if p, q and r are sufficiently large. Thus the p-twist spun (q, r) torus knot is not a ribbon knot.

Since there are knots in S^4 with no Seifert-fibered Seifert surface, it seems reasonable to ask if there are further restrictions on the type of geometric structure a Seifert-surface might have. The discussion above shows that the reason Seifert-fibered spaces can be prohibited is that the knot is too complicated, at least as measured by Gromov's norm. This suggests that perhaps every knot has a hyperbolic Seifert surface. In the rest of this section, we show that this is indeed the case. This is accomplished by constructing a special sort of cobordism between any 3-manifold and a hyperbolic 3-manifold.

DEFINITION 2.5. Let M and N be 3-manifolds. A 4-manifold W with boundary $M \cup N$ is called a homology cobordism if $H_*(W, M) = H_*(W, N) = 0$. A cobordism W is invertible from M if there is a cobordism W' from N to M with $W \cup_M W' = N \times \mathbf{I}$. We say that M splits $N \times \mathbf{I}$.

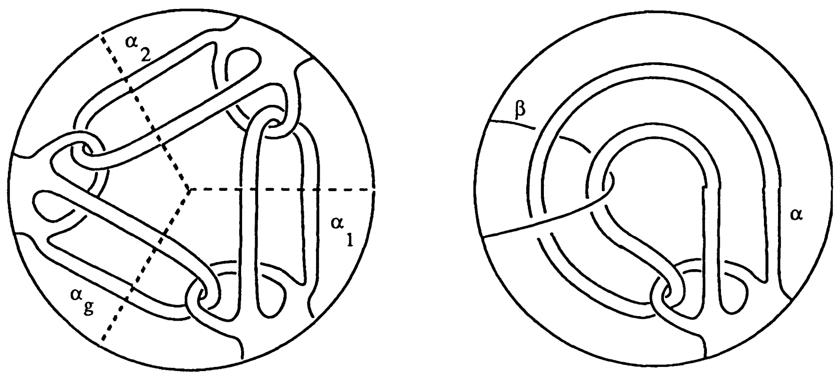
Not every invertible cobordism is a homology cobordism, nor is every homology cobordism invertible. Note that if M splits $N \times \mathbf{I}$, there is a degree-one map from M to N obtained by collapsing $N \times \mathbf{I}$ to N. In the remainder of this section, we show that for every 3-manifold N, there is a hyperbolic manifold M such that M splits $N \times \mathbf{I}$.

THEOREM 2.6. *Let N be a closed orientable 3-manifold. Then there is a hyperbolic 3-manifold M, and an invertible homology cobordism from M to N.*

We will show this shortly, but first draw some corollaries. We remark that Myers [23] showed that any 3-manifold is homology-cobordant to a hyperbolic 3-manifold. The cobordism he constructs will not in general be invertible, however. Several other authors [3, 4, 26] have shown that any 3-manifold is the target of a map of a non-zero degree from a hyperbolic manifold. From the collapsing map of the invertible cobordism we obtain:

COROLLARY 2.7. *For any 3-manifold N, there is a hyperbolic manifold M, and a degree-one map from M to N.*

From the construction of invertible cobordisms, we deduce the existence of hyperbolic Seifert surfaces for any knot in S^4:



(a) (b)

FIGURE 1

COROLLARY 2.8. *Let K be a knot in S^4. Then there is a Seifert surface M for K which is a hyperbolic 3-manifold.*

Proof. Let N_0 be an arbitrary Seifert surface for K; thus $N_0 \times \mathbf{I}$ is embedded in S^4. By Theorem 2.6, find a hyperbolic 3-manifold M splitting $N \times \mathbf{I}$. Thus M_0 embeds in $N_0 \times \mathbf{I}$, with boundary K. □

The proof of Theorem 2.6 follows the basic idea of [18, 23]: For each number g, find a hyperbolic 3-manifold M_g with boundary which splits $N_g \times \mathbf{I}$, where N_g is the orientable handlebody of genus g. The cobordism from M_g to N_g will be a product on the boundary, and will be a homology cobordism. By construction, M_g (for $g \geq 3$) will have the property that if it is glued to itself via any diffeomorphism of its boundary, the resulting closed manifold will be hyperbolic. Since any 3-manifold has a Heegaard splitting of genus

≥ 3, Theorem 2.6 will follow by gluing together two copies of M_g via the diffeomorphism which glued the two halves of the Heegaard splitting together.

The manifold we use for $M - g$ is the complement of the g arcs $\alpha_1, \ldots, \alpha_g$ embedded in B^3 as in Figure 1(a). Note that $\alpha_1, \ldots, \alpha_g$ are the lifts of the arc α in Figure 1(b) under the branched covering $B^3 \to B^3$, branched along the arc β drawn in Figure 1(b). This branched cover description will be the key to showing that M_g is hyperbolic. For $g = 1$, we have the solid torus $M_1 = B^3 - \nu(\alpha)$, which we will abbreviate to just M. Also, we will refer to (B^3, α, β) as a "tangle" and denote it by "T".

Recall the following definition **[21]** which captures the topological data inherent in a hyperbolic 3-manifold with non-torus boundary.

DEFINITION 2.9. Let X be an irreducible, compact 3-manifold, and $P \subset \partial x$ a union of essential tori and annuli. Then (X, P) is a pared manifold if:

1. Every abelian, non-cyclic subgroup of $\pi_1(X)$ is conjugate to a subgroup of $\pi_1(P)$.
2. Every map $\varphi\colon (S^1 \times \mathbf{I}, S^1 \times \partial\mathbf{I}) \to (X, P)$ which injects on π_1 deforms $(\mathrm{rel}\,\partial)$ into P.

If (X, P) is a pared manifold, write $\partial_0 X = \partial X - P$.

One should think of P as a maximal subsurface of ∂X carrying the parabolic elements of $\pi_1(X)$. We note further, that as a consequence of the torus-annulus theorems of Jaco-Shalen and Johannson **[15, 16]**, as long as (X, P) is not a Seifert pair, we may restrict to embedded tori and annuli in verifying the above conditions.

For β a properly embedded arc in a 3-manifold M, let X be the complement of a regular neighborhood $\nu(\beta)$. Let P be $\partial\nu(\beta)$; then we can write $\partial X = P \cup_{\partial P} \partial_0 X$. We will show that for $X = B^3 - \nu(\alpha \cup \beta) = M - \nu(\beta)$, and P as above, that (X, P) is a pared manifold.

THEOREM 2.10. *Suppose $\beta \subset M^3$ is a properly embedded arc such that $X = M - \nu(\beta)$, $P = \partial\nu(\beta)$ have the following properties:*

1. *(X, P) is a pared manifold.*
2. *$\partial_0 X$ is incompressible, and any essential annulus with boundary in $\partial_0 X$ V deforms,* $(\mathrm{rel}\,\partial)$ *into $\partial_0 X$.*

If $p\colon M_k \to M$ is a cyclic cover branched along β and $k \geq 3$, then M_k is irreducible, atoroidal, and anannular, and ∂M_k is incompressible.

Proof. We use the equivariant sphere theorem and Dehn's Lemma/loop theorem of **[20]**, and the equivariant versions of the annulus, and torus theorems **[19]**. The argument for each part of the conclusion is similar: a surface upstairs in M_k gives rise to a surface downstairs in M or X, which contradicts properties of X. Let g be a generator of the covering translations of M_k over M, and X_k be the unbranched cover $p^{-1}(X)$.

First we show that M_k is irreducible. By the equivariant sphere theorem, if there is a 2-sphere in M_k which does not bound a ball, then either there is an invariant such 2-sphere, or one for which $g^r(S^2) \cap S^2 = \emptyset$ for all $r \leq k$. In the latter case, the sphere misses the fixed point set $\tilde{\beta}$ and hence projects to an embedded sphere in X. Since X is irreducible, the sphere bounds a ball B, and so S^2 bounds $p^{-1}(B)$. If S^2 is invariant, then $S^2 \cap \tilde{\beta}$ is two points and $S^2 \cap X_k$ is an annulus $\widetilde{A}$. The annulus $\widetilde{A}$ projects to an embedded annulus A in X with boundary in P. Since (X, P) is pared, there is a solid torus in X with boundary $= A \cup A'$, where A' is an essential annulus in P. Lifting this solid torus to X_k, and gluing in $\widetilde{B}$ gives a ball with boundary S^2.

The proof that ∂M_k is incompressible follows a similar line, with the sphere replaced by a disk D. If $g^r(D)$ misses D for all $r \leq k$, then projecting down into X gives rise to a contradiction. If there is a disk which is invariant, then its intersection with X_k projects to an annulus A in X running from P to $\partial_0 X$. The end of A lying in P may be isotoped to lie in $\partial_0 X$. By hypothesis, there is a solid torus in X with boundary $A \cup A'$ where $A' \subset \partial_0 X$. As above, this solid torus lifts to X_k, where it can be used to find a ball in M_k pushing D into the boundary of M_k.

The hypothesis that $k \geq 3$ enters into the proof that M_k is atoroidal. The point is that if $\mathbf{Z}_k$ acts on T^2 with non-empty fixed-point set, then k must be 2. (This is easily shown by an Euler characteristic argument.) Hence if there is an essential torus $\widetilde{T}$ in M_k, then either there is one which is disjoint from all of its translates, or there is one which is invariant under $\mathbf{Z}_k$ and on which the group acts freely. In either case, the projection of $\widetilde{T}$ to its image T in M is a covering map. Since $\pi_1(\widetilde{T})$ injects into $\pi_1(M_k)$, it certainly injects into $\pi_1(X_k)$. Therefore the composition $\mathbf{Z} \oplus \mathbf{Z} \cong \pi_1(\widetilde{T}) \to \pi_1(X_k) \to \pi_1(X)$ is an injection. This contradicts the fact that (X, P) is a pared manifold.

The proof that M_k is anannular uses essentially the same argument, and will be omitted. □

In establishing the hypotheses of Theorem 2.10, we will use some easily derived properties of the tangle T of Figure 1(b).

LEMMA 2.11. *The tangle T has the following properties*:

1. *Both components are unknotted arcs which can be interchanged by an isotopy of B^3.*
2. *Any disk D^2 in $B^3 - T$ with $\partial D \subset S^2 - (\partial\alpha \cup \partial\beta)$ is isotopic (in $B^3 - T$) to one lying in S^2.*

Proof. The first statement is easily seen from the picture of T. If there were a disk as in 2, then it would have to separate α from β. Since both components are unknotted, then T would be a trivial tangle, and any knot gotten by closing the tangle would be a 2-bridge knot. One such knot is the true-lover's knot (9_{46} in Rolfsen's table **[24]**). But this is not a 2-bridge knot; for example its 2-fold branched cover doesn't have cyclic first homology. □

We are now in a position to verify the first hypothesis of Theorem 2.10 for the complement of the tangle T.

LEMMA 2.12. *Let X be the complement of T, i.e., $X = B^3 - \nu(\alpha\cup\beta)$ and P be the boundary of a regular neighborhood of P. Then (X, P) is a pared manifold.*

Proof. Since X is a compact submanifold of $\mathbf{R}^3$ with connected boundary, it is irreducible. Suppose that T is an incompressible torus in X. It is compressible in M, so it either bounds a solid torus in M, or it bounds a ball minus a knotted arc. If T bounds a solid torus in M, then it does so in X and is therefore not incompressible. If T bounds a ball minus an arc then β must go through the tunnel dug out by the arc, for T would compress in X if not. If T doesn't bound a solid torus, the arc must be knotted. But then the knot in S^3 obtained by capping off β would then be non-trivial. This contradicts the fact that β is itself trivial. □

Verifying the hypothesis of Theorem 2.10 concerning the incompressibility of $\partial_0 X$ and the existence of annuli with boundary in P is, unfortunately, more complicated. We divide X into two submanifolds X_1 and X_2 meeting in a common surface F. The submanifolds X_1 and X_2 are pictured below; each of their boundaries is divided into pieces: $\partial X_i = F \cup G_i \cup P_i$. The P_i are of course the pieces of P

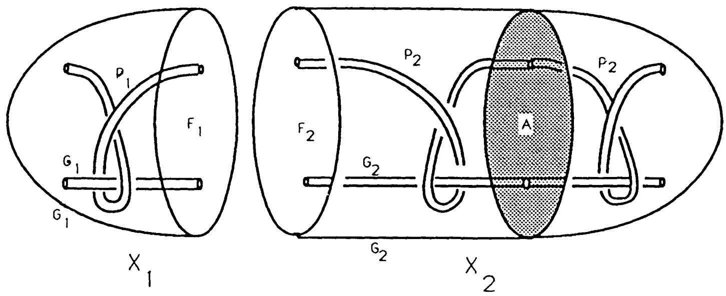


FIGURE 2. X split into pieces

in the X_i. The idea is to analyze a disc or annulus in X by considering its intersection with the X_i. A straightforward argument in the style of [**22**, **23**] shows that to verify that there are no discs or annuli, it suffices to demonstrate the following facts.

LEMMA 2.13. *For the manifolds* X_i *in Figure* 2, *the following hold*:

1. *Both* X_i *are irreducible, and the surfaces* F_i, G_i, *and* P_i *are all incompressible.*
2. *Any disc* D *in* X_i *with* $\partial D \cap P_i = \varnothing$ *and* $\partial D \cap F$ *a single arc is boundary-parallel.*
3. *There is a disk* D_1 *in* X_1 *with* $\partial D_1 \cap F =$ *two arcs, which is not boundary-parallel. Any disk* D *in* X_1 *with* $\partial D \cap P_1 = \varnothing$ *and* $\partial D \cap F =$ *two arcs is boundary-parallel or parallel to* D_1.
4. *There is no* (*non-trivial*) *disk* D *in* X_2 *with* $\partial D \cap P_2 = \varnothing$ *and* $\partial D \cap F = \partial D_1 \cap F$.
5. *Any essential annulus* $(A, \partial A)$ *in* $(X_i, \partial X_i - \partial F - \partial G_i - \partial P_i)$ *is homotopic* (*rel boundary*) *to an annulus in* $\partial X_i - \partial F - \partial G_i - \partial P_i$.

Proof. The statements in 1 about X_1 are straightforward, using the easily verified fact that X_1 is a genus-2 handlebody. To prove part 1 for X_2, note that X_2 is the union of two genus-2 handlebodies along the 3-punctured sphere A in Figure 2(b). The 3-punctured sphere is incompressible, as are G_2, P_2 and F in the subhandlebodies. Therefore they are incompressible in X_2 as well, and X_2 is irreducible by a standard theorem.

The proofs of the other statements share a similar pattern of argument. The pieces of the boundary of the X_i where the boundary of the disk (for 2–4) or annulus (for 5) lie are 3-punctured spheres. The main point is that there are only a few (rel boundary) homotopy types of properly embedded arcs in a 3-punctured sphere. For the case of an annulus, we also need the fact that a simple closed curve in a

3-punctured sphere is peripheral. Because of this, it is elementary to find all the possible arcs which could be part of the boundary of a disk (say for 2–4), or circles which could be boundary components of an annulus. Each case is then eliminated by an elementary knot-theoretic argument.

Rather than go through all of the (numerous) cases, we will just illustrate the idea in proving part 2 for X_1. So suppose that D is a disk in X_1 whose boundary misses P_1, and whose boundary meets F_1 in a single arc γ. Since it does not intersect P_1, γ must be in one of the 3 relative homotopy classes drawn in the following picture. The

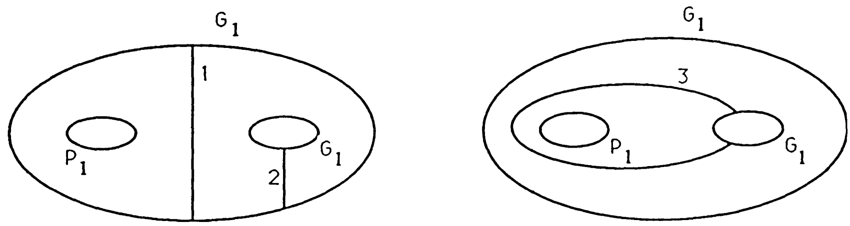


FIGURE 3. Arcs in 3-punctured sphere

letters (P_1, G_1) in the figure label a boundary component according to which surface shares that boundary component with F_1. Suppose that the arc γ is in the first relative homotopy class. The relative homotopy class of the other arc in the boundary of D is determined by its endpoints, but the homotopy class of the whole boundary of D is only determined up to twisting about the curves $\partial F_1 \cap \partial G_1$. So the possible boundaries curves in this case are the curves γ_n drawn below. But if one of these curves were to bound a disk in X_1, then the link with one component γ_n and the other component gotten by joining the ends of β in S^3 would be a trivial link. (See the figure below.)

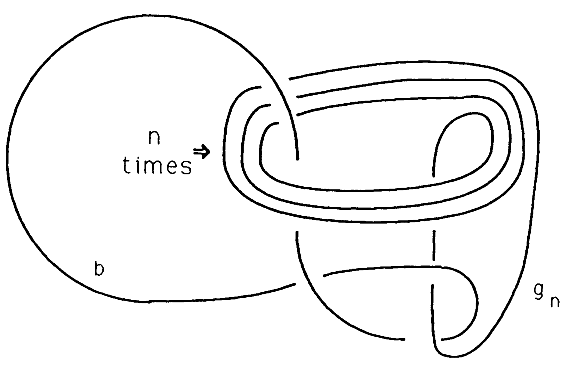


FIGURE 4. Link arising from supposed disk

Now one computes some link invariants to show that the link is in fact non-trivial. In this case, the two components have linking number $n-1$ (with respect to some orientation). So for there to be a disk, n would have to be 1. But for $n = 1$, the link is the Whitehead link, which is certainly non-trivial. The other relative homotopy classes of arcs are treated similarly.

The pattern is the same in all of the parts of the lemma concerning the existence of disks. In each case, as it turns out, the link whose components are the boundary of D and β with its ends joined can be shown to be non-trivial by using the one-variable Alexander polynomial. There is exactly one case where this doesn't work, and one gets the disk D_1 whose boundary is drawn below in Figure 5.

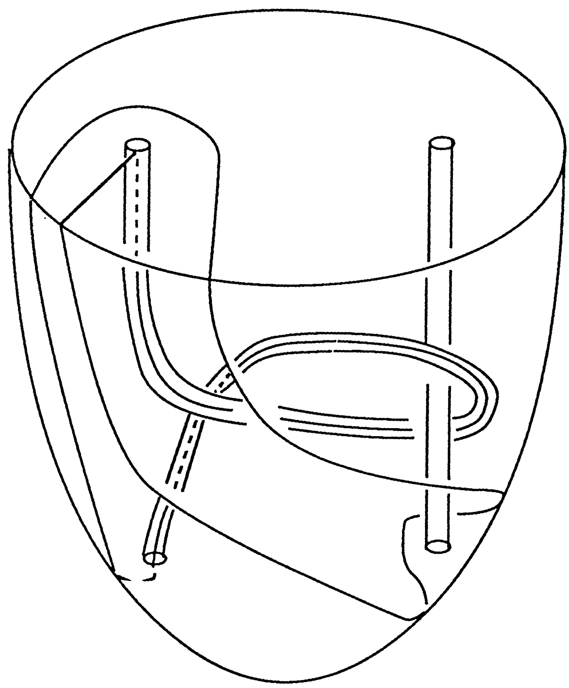


FIGURE 5. Boundary of disk in X_1

However one shows using the same argument that there are no disks in X_2 meeting in $\partial D_1 \cap F$. Part 5, concerning annuli in the X_i, follows a similar line, but is easier. Up to isotopy in ∂X_i, there are three possibilities for boundary curves in $\partial X_i - \partial^2 X_i$. None of the three are even homologous in X_i, so any annulus would have to go from a curve to itself. But such an annulus would have to be boundary-parallel, by the following argument. Note that filling in either α or

β makes X_1 (or X_2) a solid torus). Each of the possible boundary curves becomes isotopic to a longitude of a solid torus after filling in one of α or β. Therefore the annulus spanned by two copies of the boundary curve separates the solid torus and is parallel to the boundary torus on both sides. The arc (α or β) that was taken out lies on one side or the other, so the annulus is still boundary-parallel when the arc is taken out again to get X_1 or X_2. □

By analyzing the intersection of a disc or annulus with the boundary of the X_i, we see that (X, P) satisfies hypothesis 2 of Theorem 2.10.

COROLLARY 2.14. *If X is the complement of the tangle T, and P is the boundary of a regular neighborhood of β, then any essential annulus with boundary in $\partial_0 X = \partial X - P$ deforms into $\partial_0 X$. Moreover, $\partial_0 X$ is incompressible.*

So from Theorem 2.10, we obtain:

COROLLARY 2.15. *For $g \geq 3$, the branched cover M_g is irreducible, astroidal, annanular, and has incompressible boundary.*

From this corollary we see that any manifold gotten by gluing copies of M_g together via *any* homeomorphism of the boundary is an atoroidal Haken manifold. By Thurston's theorem [**29**], such a manifold is a hyperbolic manifold. The hyperbolic manifold M which we will use to prove Theorem 2.6 will have this form.

Proof of Theorem 2.6. Any closed 3-manifold N has a Heegaard splitting of genus $g \geq 3$, i.e. is obtained by gluing a genus-g solid handlebody H_g to itself via some homeomorphism φ. We will show that there is an invertible homology cobordism W_g from M_g to H_g which is a product on the boundary. Gluing two such cobordisms together via $\varphi \times \mathrm{id}|_{\mathbf{I}}$ gives an invertible homology cobordism from $M = M_g \cup_\varphi M_g$ to N. Therefore the hyperbolic manifold M splits $N \times \mathbf{I}$.

The cobordism W_g will be the exterior of an invertible tangle-concordance in $B^3 \times \mathbf{I}$ (using the obvious definition) from the tangle $T_g = \alpha_1, \ldots, \alpha_g \subset B^3$ (see Figure 1(b)) to the trivial g-string tangle. The complement of the trivial tangle is a handlebody of genus g, so the exterior (in $B^3 \times \mathbf{I}$) of the tangle concordance yields the desired cobordism between M_g and H_g.

To construct the concordance, note that there is an obvious surface in Figure 1(a) whose boundary is T_g union some arcs in ∂B^3. Note further that each component of the surface has genus one, and that the obvious generators (say a_j, b_j) of the homology are each unknotted and have zero self-twisting on the surface. The collection of a_j form an unlink in B^3, as do the collection of b_j. View B^3 as the $\frac{1}{2}$-level in $B^3 \times \mathbf{I}$, then the a_j may be surgered in, say $B^3 \times [\frac{1}{2}, 1]$. Surgery on an a_i yields a disk in $B^3 \times [\frac{1}{2}, 1]$ with boundary $\alpha_i \cup$ an arc in B^3. The collection of disks coming from surgering the a_j's may be regarded as a tangle-concordance in $B^3 \times [\frac{1}{2}, 1]$ to the trivial tangle in $B^3 \times 1$. Likewise, surgerying the b_j's in $B^3 \times [0, \frac{1}{2}]$ yields a concordance to the trivial tangle in $B^3 \times 0$.

Since the a_j are geometrically dual to the b_j, the two concordances fit together to be the product concordance from the trivial tangle to itself, just as in Sumners' original work on doubly null-cobordant knots [27]. Therefore, the tangle T_g admits an invertible tangle-concordance to the trivial tangle, and the theorem follows. □

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BRANDEIS UNIVERSITY
WALTHAM, MA 02254