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SOME COEFFICIENT PROBLEMS AND APPLICATIONS

Y. J. LEUNG AND G. SCHOBER

We combine Bombieri's method of second variation and Schiffer's variation by truncation to consider coefficient conjectures on $\log f'$, $\log f/z$, and $\log zf'/f$ where f is a normalized univalent analytic function in $|z|>1$. Related results apply to the geometric properties of extremal functions for more general linear problems. Another by-product is a simplified approach to the geometric structure of solutions to the N th diameter problem.

1. Introduction. Let Σ be the class of one-to-one analytic functions $f(z)=z+\sum_{k=0}^{\infty} b_k z^{-k}$ in $\Delta=\{z\colon |z|>1\}$. The function

$$K_n(z)=z(1+z^{-n})^{2/n}=z+\frac{2}{n}z^{1-n}+\cdots$$

belongs to Σ and maps Δ onto the complement on an n-star γ_n. For $k=2$ and $k=3$ it is true that $|b_{k-1}|\le 2/k$ holds for the full class Σ, and for a time it was hoped that this estimate might persist for all k. However, Bazilevich [1] disproved this conjecture for even integers $k\geq 4$, and it is now known [3, 14] to be false for all integers $k\geq 4$.

In this article we shall consider similar conjectures for the coefficients of

$$\log f'(z)=\sum_{k=2}^{\infty} c_k z^{-k}, \quad \log\frac{f(z)}{z}=\sum_{k=1}^{\infty} d_k z^{-k}, \quad \text{and}$$
$$\log\frac{zf'(z)}{f(z)}=\sum_{k=1}^{\infty} e_k z^{-k}.$$

In the full class Σ the coefficients of d_k and e_k are unbounded. For that reason we shall consider these coefficients only for the compact subclasses Σ' consisting of functions in Σ that never vanish and Σ_0 consisting of functions in Σ with $b_0=0$. Since $d_1=-e_1=b_0$, the bounds $|d_1|=|e_1|=|b_0|\leq 2$ are sharp for Σ', and these coefficients vanish for Σ_0. Thus we shall restrict attention from now on to coefficients with subscript $k\geq 2$.

For $n \geq 2$, the functions K_n belong to all three classes Σ, Σ', and Σ_0. Since

$$\log K_n'(z) = \frac{-2(n-1)}{n}\frac{1}{z^n} + \cdots, \quad \log \frac{K_n(z)}{z} = \frac{2}{n}\frac{1}{z^n} + \cdots, \quad \text{and}$$
$$\log \frac{zK_n'(z)}{K_n(z)} = \frac{-2}{z^n} + \cdots,$$

one might conjecture that

$$(1) \qquad |c_k| \leq \frac{2(k-1)}{k} \quad \text{for all } f \in \Sigma \text{ and all } k \geq 2,$$

$$(2) \qquad |d_k| \leq \frac{2}{k} \quad \text{for all } f \in \Sigma' \cup \Sigma_0 \text{ and all } k \geq 2,$$

$$(3) \qquad |e_k| \leq 2, \quad \text{for all } f \in \Sigma' \cup \Sigma_0 \text{ and all } k \geq 2.$$

Based on a result of Jenkins, we shall see in §4 that these bounds are correct if one requires the first half of the coefficients to vanish.

Since $c_2 = -b_1$, and $c_3 = -2b_2$, it follows that conjecture (1) is true without restriction for $k = 2$ and $k = 3$. However, in §4 we shall see that it fails for all $k \geq 4$. For $f \in \Sigma'$ it is a result of Fekete and Szegö that conjectures (2) and (3) fail already for $k = 2$. In §4 we shall show that they fail also for $k > 2$. For $f \in \Sigma_0$ the conjectures (2) and (3) are true for $k = 2$ and $k = 3$ since $d_2 = b_1$, $e_2 = -2b_1$, $d_3 = b_2$, and $e_3 = -3b_2$. In §4 we shall show that they fail for all $k \geq 4$.

General results of a similar nature are obtained in §§2 and 3 for functions defined in the complement of the n-star γ_n. Our method of proof uses Loewner's differential equation. Variations of the identity are constructed up to second order terms in much the same way as Bombieri varied from the Koebe function in his work **[2]** on the local Bieberbach conjecture. A similar variation was used also by the authors **[9, 10]** to establish some geometric properties of solutions to linear extremal problems for Σ.

Applications are given in §4 to the class Σ by making a change of variables. Although the results are phrased for n-fold symmetric functions, i.e., functions satisfying $f(z) = e^{-2\pi i/n} f(e^{2\pi i/n} z)$, they are new for 1-fold symmetric functions, which are unrestricted. More general linear problems are considered in §5. We show that the extremal functions for certain linear problems possess strong geometric properties. Finally, as a further application, in §6 we describe geometric properties of solutions to the Nth diameter problem.

2. Symmetric variations from the identity in $\mathbb{C}\backslash\gamma_n$. In this section we shall prove a number of lemmas, which may be of some independent interest, concerning certain coefficients of functions near to the identity mapping in $\mathbb{C}\backslash\gamma_n$. This will be accomplished by constructing variations up to second order using Loewner's differential equation. The calculations are somewhat technical. Ultimately, they involve judicious choices for the parameter function in Loewner's equation and asymptotic considerations. Most of the principal results in §§4 to 6 will follow directly from the lemmas of this section.

We shall denote also by $K_n(z) = \frac{1}{z}(1+z^n)^{2/n}$ the mapping from the punctured unit disk $\mathbb{D}\backslash\{0\}$ onto the complement of the n-star γ_n. Its inverse function

$$S_n(w) = 4^{-1/n}w[1-\sqrt{1-4w^{-n}}]^{2/n},$$

where $\sqrt{1}=1$, maps $\mathbb{C}\backslash\gamma_n$ onto $\mathbb{D}\backslash\{0\}$. We shall use the Loewner equation

$$\frac{\partial f}{\partial t} = z\left(\frac{1+\kappa z}{1-\kappa z}\right)\frac{\partial f}{\partial z}, \qquad t \geq 0, \quad z \in \mathbb{D},$$

where $\kappa = \kappa(t)$ is a continuous complex-valued function on $[0, \infty)$ with $|\kappa(t)| = 1$. This equation has a solution $f(z, t)$ which is one-to-one and analytic in z for each t and is normalized so that $e^{-t}f(z, t) = z + O(z^2)$ as $z \to 0$ [**11**, Sect. 6.1]. Define a nonvanishing n-fold symmetric function $F(\cdot, t)$ in the complement of γ_n by the relation $F(w, t) = f(S_n(w)^n, t)^{-1/n}$. We suppress the dependence of F on n. Then F satisfies the differential equation

$$(4) \qquad \frac{\partial F}{\partial t} = \frac{S_n(1+\kappa S_n^n)}{nS_n'(1-\kappa S_n^n)}\frac{\partial F}{\partial w}, \qquad w \in \mathbb{C}\backslash\gamma_n, \quad t \geq 0,$$

and $e^{t/n}F(w, t) = w + O(w^{1-n})$ as $w \to \infty$.

We choose $\kappa(t) = e^{i\varepsilon\theta(t)}$, where ε is a small real parameter and θ is a bounded continuous function of t. Then $\kappa = 1 + i\varepsilon\theta - \frac{1}{2}\varepsilon^2\theta^2 + O(\varepsilon^3)$ and the differential equation (4) is of the form

$$(5) \qquad \frac{\partial F}{\partial t} = \frac{S_n}{nS_n'}\left[\frac{1+S_n^n}{1-S_n^n} + \varepsilon\theta\frac{2S_n^n}{(1-S_n^n)^2} - \varepsilon^2\theta^2\frac{S_n^n(1+S_n^n)}{(1-S_n^n)^3} + O(\varepsilon^3)\right]\frac{\partial F}{\partial w}$$

as $\varepsilon \to 0$. Now F depends also on ε, and we may expand

$$(6) \qquad F(w, t, \varepsilon) = \phi(w, t) + \varepsilon V(w, t) + \varepsilon^2 Q(w, t) + O(\varepsilon^3).$$

For $\varepsilon = 0$ the differential equation (5) becomes

$$\frac{\partial \phi}{\partial t} = \frac{S_n(1+S_n^n)}{nS_n'(1-S_n^n)}\frac{\partial \phi}{\partial w}.$$

The function S_n satisfies both the differential equation $S_n' = S_n^2(S_n^n+1)^{(n-2)/n}/(S_n^n-1)$ and the identity

$$w = \frac{1}{S_n}(1+S_n^n)^{2/n}.$$

It follows that

$$\frac{\partial \phi}{\partial t} = \frac{-w}{n}\frac{\partial \phi}{\partial w} \quad \text{and} \quad e^{t/n}\phi(w, t) = w.$$

Thus the functions $e^{t/n}F(\cdot, t, \varepsilon)$ are normalized variations of the identity function in $\mathbb{C}\backslash\gamma_n$, as desired.

A comparison of the coefficients of ε and ε^2 in (5) leads to the differential equations

$$\begin{aligned}\frac{\partial V}{\partial t} &= \frac{S_n}{nS_n'}\left[\frac{1+S_n^n}{1-S_n^n}\frac{\partial V}{\partial w} + i\theta\frac{2S_n^n}{(1-S_n^n)^2}\frac{\partial \phi}{\partial w}\right] \\ &= \frac{-w}{n}\frac{\partial V}{\partial w} - \frac{2ie^{-t/n}\theta w S_n^n}{n(1-S_n^{2n})}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \frac{S_n}{nS_n'}\left[\frac{1+S_n^n}{1-S_n^n}\frac{\partial Q}{\partial w} + i\theta\frac{2S_n^n}{(1-S_n^n)^2}\frac{\partial V}{\partial w} - \theta^2\frac{S_n^n(1+S_n^n)}{(1-S_n^n)^3}\frac{\partial \phi}{\partial w}\right] \\ &= \frac{-w}{n}\frac{\partial Q}{\partial w} - \frac{2i\theta w S_n^n}{n(1-S_n^{2n})}\frac{\partial V}{\partial w} + \frac{e^{-t/n}\theta^2 w S_n^n}{n(1-S_n^n)^2}\end{aligned}$$

for the first and second variations.

It will be convenient to use the notation

$$p_k = \frac{(2k)!}{(k!)^2}$$

($p_0 = 1$) and remember that these binomial coefficients are positive integers. By writing

$$S_n(w)^n = \frac{1}{4}w^n\left[1 - \frac{1}{R(w)}\right]^2$$

$$\text{where} \quad R(w) = \frac{1}{\sqrt{1-4w^{-n}}} = \sum_{k=0}^{\infty} p_k w^{-kn},$$

the differential equations become

$$n\frac{\partial V}{\partial t} = -w\frac{\partial V}{\partial w} - 2ie^{-t/n}\theta R w^{1-n},$$
$$n\frac{\partial Q}{\partial t} = -w\frac{\partial Q}{\partial w} - 2i\theta R w^{1-n}\frac{\partial V}{\partial w} + e^{-t/n}\theta^2 R^2 w^{1-n}.$$

As $w \to \infty$, the difference $F(w, t, \varepsilon) - \phi(w, t)$ is bounded. Also recall that F is n-fold symmetric. Therefore the expansions of V and Q around infinity are of the form

$$V(w, t) = \sum_{j=1}^{\infty} V_j(t) w^{1-jn} \quad \text{and} \quad Q(w, t) = \sum_{j=1}^{\infty} Q_j(t) w^{1-jn}.$$

It follows from the system of differential equations that

$$nV_j' = (jn-1)V_j - 2ip_{j-1}\theta e^{-t/n} \quad \text{and}$$
$$nQ_j' = (jn-1)Q_j + 2i\theta\sum_{k=1}^{j-1}(kn-1)p_{j-k-1}V_k + 4^{j-1}\theta^2 e^{-t/n}.$$

If w is restricted to a large circle, then $e^t F(w, t, \varepsilon)$ is uniformly bounded. Therefore the coefficients $V_j(t)$ and $Q_j(t)$ vanish at $t = \infty$. Thus this system can be integrated to yield

$$(7) \qquad V_j(0) = \frac{2ip_{j-1}}{n}\int_0^\infty e^{-jt}\theta(t)\,dt \quad \text{and}$$
$$Q_j(0) = -\frac{4^{j-1}}{n}\int_0^\infty e^{-jt}\theta(t)^2\,dt + \sum_{k=1}^{j-1}\frac{4(kn-1)p_{k-1}p_{j-k-1}}{n^2} \times \int_0^\infty\int_t^\infty e^{-ks-(j-k)t}\theta(s)\theta(t)\,ds\,dt.$$

In the expansion of $F(w, 0, \varepsilon) = w + \sum_{j=1}^{\infty} B_j(0, \varepsilon)w^{1-jn}$, the coefficients are of the form $B_j(0, \varepsilon) = \varepsilon V_j(0) + \varepsilon^2 Q_j(0) + O(\varepsilon^3)$. Evidently, $V_j(0)$ is purely imaginary and $Q_j(0)$ is real, and so $\operatorname{Re} B_j(0, \varepsilon) = \varepsilon^2 Q_j(0) + O(\varepsilon^3)$. For each $j \geq 3$, we shall find a function θ such that $Q_j(0) > 0$. This will show the existence of a function $F(\cdot, 0, \varepsilon)$, nearby the identity function, with $\operatorname{Re} B_j(0, \varepsilon)$ positive. This is the purpose of Lemma 1.

If $n = 1$ and $j \geq 4$, substitute $\theta(t) = e^{\alpha t}$ for $\alpha < 2$ into (7), and discard from the sum all terms except the one for which $k = 2$. Then we have

$$Q_j(0) \geq \frac{-4^{j-1}}{j-2\alpha} + \frac{8p_{j-3}}{(2-\alpha)(j-2\alpha)}.$$

If α is sufficiently close to 2, it is clear that the second term dominates and leads to a positive $Q_j(0)$. Similarly, if $n \geq 2$ and $j \geq 2$, substitute $\theta(t) = e^{\alpha t}$ for $\alpha < 1$ into (7), and discard from the sum of all terms except the one for which $k = 1$. Then we have

$$Q_j(0) \geq \frac{-4^{j-1}}{n(j-2\alpha)} + \frac{4(n-1)p_{j-2}}{n^2(1-\alpha)(j-2\alpha)}.$$

If α is sufficiently close to 1, then the second term dominates and leads to a positive $Q_j(0)$.

We have reached our desired conclusions, but the functions θ that were chosen are not bounded. However, the truncations $\theta_N(t) = \min\{N, \theta(t)\}$ are bounded, and the dominated convergence theorem implies that each $Q_j(0)$ is positive for some θ_N. Thus we have proved the following lemma.

LEMMA 1. *Either if $n = 1$ and $j \geq 4$ or if $n \geq 2$ and $j \geq 2$, then there exists a nonvanishing univalent analytic function F in the complement of the n-star γ_n with expansion $F(w) = w + \sum_{k=1}^{\infty} B_k w^{1-kn}$ in a neighborhood of ∞ such that* $\operatorname{Re}\{B_j\} > 0$.

The special case $n = 1$ and $j \geq 4$ of this lemma was proved in **[10]**. It is our purpose in this article to study certain logarithmic coefficients.

LEMMA 2. *Either if $n = 1$ and $j \geq 4$ or if $n \geq 2$ and $j \geq 2$, then there exists a univalent analytic function F in the complement of the n-star γ_n with expansions*

$$F(w) = w + \sum_{k=1}^{\infty} B_k w^{1-kn} \quad \textit{and} \quad \log F'(w) = \sum_{k=1}^{\infty} C_k w^{-kn}$$

in a neighborhood of ∞ such that $\operatorname{Re}\{C_j\} < 0$.

Proof. We shall use the functions $F(w, t, \varepsilon)$ constructed above as solutions of the differential equation (5). Since $\phi(w, t) = e^{-t/n} w$, the expansion (6) implies that the function $\log F'(w, t, \varepsilon)$ satisfies

$$(8) \quad \log F'(w, t, \varepsilon) = \frac{-t}{n} + \varepsilon e^{t/n} \frac{\partial V}{\partial w} + \varepsilon^2 \left[e^{t/n} \frac{\partial Q}{\partial w} - \frac{1}{2} e^{2t/n} \left(\frac{\partial V}{\partial w} \right)^2 \right] + O(\varepsilon^3)$$

as $\varepsilon \to 0$. Thus the coefficients of the expansion

$$\log F'(w, t, \varepsilon) = \sum_{j=1}^{\infty} C_j(t, \varepsilon) w^{-jn}$$

near infinity satisfy

$$\begin{aligned} C_j(0, \varepsilon) &= \varepsilon(1 - jn)V_j(0) \\ &\quad + \varepsilon^2 \left[(1 - jn)Q_j(0) \right. \\ &\qquad \left. - \frac{1}{2} \sum_{k=1}^{j-1} (1 - kn)(1 - jn + kn) V_k(0) V_{j-k}(0) \right] \\ &\quad + O(\varepsilon^3). \end{aligned}$$

As before, the coefficient of ε is purely imaginary, and so it is sufficient to find a function θ so that the coefficient of ε^2 is negative. After substitution, this expression becomes

$$\begin{aligned} (9) \qquad & \frac{4^{j-1}(jn-1)}{n} \int_0^\infty e^{-jt} \theta(t)^2 \, dt \\ & + \sum_{k=1}^{j-1} \frac{2(kn-1) p_{k-1} p_{j-k-1}}{n^2} \\ & \times \left[(jn - kn - 1) \int_0^\infty \int_0^\infty e^{-ks - jt + kt} \theta(s) \theta(t) \, ds \, dt \right. \\ & \qquad \left. - 2(jn-1) \int_0^\infty \int_t^\infty e^{-ks - jt + kt} \theta(s) \theta(t) \, ds \, dt \right]. \end{aligned}$$

If $n = 1$ and $j \geq 4$, substitute $\theta(t) = e^{\alpha t}$ for $\alpha < 2$ into (9). Then (9) becomes

$$\begin{aligned} (10) \qquad & \frac{4^{j-1}(j-1)}{j - 2\alpha} \\ & + \sum_{k=2}^{j-1} \frac{2(k-1) p_{k-1} p_{j-k-1}}{(k - \alpha)} \left[\frac{(j-k-1)}{(j-k-\alpha)} - \frac{2(j-1)}{(j-2\alpha)} \right]. \end{aligned}$$

The sum of the terms in (10) corresponding to the indices $k = 2$ and $k = j - 2$ is

$$\frac{-8 p_{j-3} \{2(j-4) + (2-\alpha)[j(j-5) + 8]\}}{(2-\alpha)(j-2-\alpha)(j-2\alpha)}.$$

If $j \geq 5$, they are the only unbounded terms in (10) as α increases to 2, and so (10) tends to $-\infty$. If $j = 4$, then the term with index $k = 2$ reduces to $-16/(2-\alpha)^2$. It dominates the others and tends to $-\infty$ as α approaches 2. In any case the expression (10) becomes negative as α increases to 2.

Similarly, if $n \geq 2$ and $j \geq 2$, substitute $\theta(t) = e^{\alpha t}$ for $\alpha < 1$ into (9). Then (9) becomes

$$(11) \quad \frac{4^{j-1}(jn-1)}{n(j-2\alpha)} + \sum_{k=1}^{j-1} \frac{2(kn-1)p_{k-1}p_{j-k-1}}{n^2(k-\alpha)} \left[\frac{jn-kn-1}{j-k-\alpha} - \frac{2(jn-1)}{j-2\alpha}\right].$$

The sum of the terms corresponding to $k = 1$ and $k = j-1$ is

$$\frac{-4p_{j-2}\{(j-2)(n-1) + (1-\alpha)[(j-2)(jn-1) + 2(n-1)]\}}{n(1-\alpha)(j-1-\alpha)(j-2\alpha)}.$$

If $j \geq 3$, these terms are the only unbounded ones in (11) as α increases to 1, and so (11) tends to $-\infty$. If $j = 2$, the sum in (11) reduces to $-2(n-1)/n(1-\alpha)^2$. It dominates and tends to $-\infty$ as α approaches 1. In both cases the expression (11) becomes negative as α increases to 1.

All that is needed to complete the proof is to truncate these functions θ, as before, since they were originally assumed to be bounded.

LEMMA 3. *If $n \geq 1$ and $j \geq 2$, then there exists a nonvanishing univalent analytic function F in the complement of the n-star γ_n with expansions*

$$F(w) = w + \sum_{k=1}^{\infty} B_k w^{1-kn} \quad \textit{and} \quad \log\frac{F(w)}{w} = \sum_{k=1}^{\infty} D_k w^{-kn}$$

in a neighborhood of ∞ such that $\operatorname{Re}\{D_j\} > 0$.

Proof. Only a few steps change from the proof of Lemma 2. Corresponding to (8) we have

$$\log\frac{F(w,t,\varepsilon)}{w} = \frac{-t}{n} + \varepsilon e^{t/n}\frac{V}{w} + \varepsilon^2\left[e^{t/n}\frac{Q}{w} - \frac{1}{2}e^{2t/n}\left(\frac{V}{w}\right)^2\right] + O(\varepsilon^3)$$

as $\varepsilon \to 0$. Thus the coefficients of the expansion

$$\log \frac{F(w, t, \varepsilon)}{w} = \sum_{j=1}^{\infty} D_j(t, \varepsilon) w^{-jn}$$

near infinity satisfy

$$D_j(0, \varepsilon) = \varepsilon V_j(0) + \varepsilon^2 \left[Q_j(0) - \frac{1}{2} \sum_{k=1}^{j-1} V_k(0) V_{j-k}(0) \right] + O(\varepsilon^3).$$

The coefficient of ε is purely imaginary, and it is sufficient to find a function θ so that the coefficient of ε^2 is positive. The coefficient of ε^2 equals

$$(12) \qquad \begin{aligned} &\frac{-4^{j-1}}{n} \int_0^\infty e^{-jt} \theta(t)^2 \, dt \\ &+ \sum_{k=1}^{j-1} \frac{2p_{k-1} p_{j-k-1}}{n^2} \\ &\times \left[\int_0^\infty \int_0^\infty e^{-ks-jt+kt} \theta(s)\theta(t) \, ds \, dt \right. \\ &\qquad \left. + 2(kn-1) \int_0^\infty \int_t^\infty e^{-ks-jt+kt} \theta(s)\theta(t) \, ds \, dt \right]. \end{aligned}$$

Substitute $\theta(t) = e^{\alpha t}$ for $\alpha < 1$ into (12). Then (12) becomes

$$(13) \qquad \frac{-4^{j-1}}{n(j-2\alpha)} + \sum_{k=1}^{j-1} \frac{2p_{k-1} p_{j-k-1}}{n^2(k-\alpha)} \left[\frac{1}{j-k-\alpha} + \frac{2(kn-1)}{j-2\alpha} \right].$$

The sum of the terms in (13) corresponding to indices $k = 1$ and $k = j-1$ is

$$\frac{4p_{j-2}[(j-2) + 2(1-\alpha)j]}{n(1-\alpha)(j-1-\alpha)(j-2\alpha)}.$$

If $j \geq 3$, these are the only unbounded terms in (13) as α increases to 1, and so (13) becomes positive. If $j = 2$, the sum in (13) reduces to $2/n(1-\alpha^2)$, which dominates as α approaches 1, and so (13) again becomes positive. One completes the proof as before.

If $n > 1$, the function F of Lemma 3 has the asymptotics $F(w) = w + o(1)$ as $w \to \infty$. We shall need such a function also when $n = 1$.

LEMMA 4. *If $j \geq 4$, then there exists a univalent analytic function F in the complement of the 1-star γ_1 with expansions*

$$F(w) = w + \sum_{k=2}^{\infty} B_k w^{1-k} \quad \textit{and} \quad \log \frac{F(w)}{w} = \sum_{k=2}^{\infty} D_k w^{-k}$$

in a neighborhood of ∞ such that $\operatorname{Re}\{D_j\} > 0$.

Proof. Fix $n = 1$ and replace $F(w, t, \varepsilon)$ by $F(w, t, \varepsilon) - B_1(t, \varepsilon)$. Then the coefficients of

$$\begin{aligned}\log \frac{F(w, 0, \varepsilon) - B_1(0, \varepsilon)}{w} &= \varepsilon \frac{V(w, 0) - V_1(0)}{w} \\ &\quad + \varepsilon^2 \left[\frac{Q(w, 0) - Q_1(0)}{w} - \frac{1}{2}\left(\frac{V(w, 0) - V_1(0)}{w}\right)^2\right] + O(\varepsilon^3) \\ &= \sum_{j=2}^{\infty} D_j(0, \varepsilon) w^{-j}\end{aligned}$$

near infinity satisfy

$$D_j(0, \varepsilon) = \varepsilon V_j(0) + \varepsilon^2 \left[Q_j(0) - \frac{1}{2}\sum_{k=2}^{j-2} V_k(0) V_{j-k}(0)\right] + O(\varepsilon^3)$$

as $\varepsilon \to 0$. The sum is omitted for $j = 2$ and $j = 3$. As before, it is sufficient to find a function θ so that the coefficient of ε^2 is positive. Restrict $j \geq 4$ and choose $\theta(t) = e^{\alpha t}$ for $\alpha < 2$. Then the coefficient of ε^2 becomes

$$\frac{-4^{j-1}}{j - 2\alpha} + \sum_{k=2}^{j-2} \frac{2p_{k-1}p_{j-k-1}}{(k-\alpha)(j-k-\alpha)} + \sum_{k=2}^{j-1} \frac{4(k-1)p_{k-1}p_{j-k-1}}{(k-\alpha)(j-2\alpha)}. \tag{14}$$

If $j \geq 5$, then the only unbounded terms in (14) as α increases to 2 correspond to indices $k = 2$ and $k = j - 2$ from the first sum and $k = 2$ in the second sum. The sum of these terms is

$$\frac{8p_{j-3}[2(j-4) + 3(2-\alpha)]}{(2-\alpha)(j-2-\alpha)(j-2\alpha)},$$

and so (14) becomes positive as α increases to 2. If $j = 4$, the sum of the terms corresponding to $k = 2$ in (14) is $16/(2-\alpha)^2$, which dominates as α approaches 2, and so (14) again becomes positive. One completes the proof as before.

LEMMA 5. *If $n \geq 1$ and $j \geq 2$, then there exists a nonvanishing univalent analytic function F in the complement of the n-star γ_n with expansions*

$$F(w) = w + \sum_{k=1}^{\infty} B_k w^{1-kn} \quad \textit{and} \quad \log \frac{wF'(w)}{F(w)} = \sum_{k=1}^{\infty} E_k w^{-kn}$$

in a neighborhood of ∞ such that $\operatorname{Re}\{E_j\} < 0$.

Proof. Since

$$\log \frac{wF'(w, t, \varepsilon)}{F(w, t, \varepsilon)} = \log F'(w, t, \varepsilon) - \log \frac{F(w, t, \varepsilon)}{w},$$

we may combine several steps in the proofs of Lemmas 2 and 3. It is sufficient to find a function θ so that the difference of (9) and (12) is negative. This difference is

$$\begin{aligned}(15)\quad & 4^{j-1} j \int_0^\infty e^{-jt}\theta(t)^2\,dt \\ & + \sum_{k=1}^{j-1} \frac{2p_{k-1}p_{j-k-1}}{n} \\ & \qquad \times \left[(jkn - j - k^2 n) \int_0^\infty \int_0^\infty e^{-ks-jt+kt}\theta(s)\theta(t)\,ds\,dt \right. \\ & \qquad\qquad \left. -2j(kn-1) \int_0^\infty \int_t^\infty e^{-ks-jt+kt}\theta(s)\theta(t)\,ds\,dt \right].\end{aligned}$$

Substitute $\theta(t) = e^{\alpha t}$ for $\alpha < 1$ into (15). It becomes

$$(16)\qquad \frac{4^{j-1}j}{j-2\alpha} + \sum_{k=1}^{j-1} \frac{2p_{k-1}p_{j-k-1}}{n(k-\alpha)} \left[\frac{jkn - j - k^2 n}{j-k-\alpha} - \frac{2j(kn-1)}{j-2\alpha} \right],$$

and the sum of the terms corresponding to $k = 1$ and $k = j-1$ can be written as

$$\frac{-4p_{j-2}\{(j-2) + (1-\alpha)[j(j-2)+2]\}}{(1-\alpha)(j-1-\alpha)(j-2\alpha)}.$$

If $j \geq 3$, these are the only unbounded terms in (16) as α increases to 1, and so (16) tends to $-\infty$. If $j = 2$, the sum in (16) reduces to $-2/(1-\alpha)^2$ and again dominates, tending to $-\infty$ as α approaches 1. The proof concludes by approximating θ as before.

LEMMA 6. *If $j \geq 4$, then there exists a univalent analytic function F in the complement of the 1-star γ_1 with expansions*

$$F(w) = w + \sum_{k=2}^\infty B_k w^{1-k} \quad \textit{and} \quad \log \frac{wF'(w)}{F(w)} = \sum_{k=2}^\infty E_k w^{-k}$$

in a neighborhood of ∞ such that $\operatorname{Re}\{E_j\} < 0$.

Proof. Fix $n = 1$ and replace $F(w, t, \varepsilon)$ by $F(w, t, \varepsilon) - B_1(t, \varepsilon)$. Since

$$\log \frac{wF'(w, 0, \varepsilon)}{F(w, 0, \varepsilon) - B_1(0, \varepsilon)} = \log F'(w, 0, \varepsilon) - \log \frac{F(w, 0, \varepsilon) - B_1(0, \varepsilon)}{w},$$

we may combine many steps in the proofs of Lemmas 2 and 4. Choose $\theta(t) = e^{\alpha t}$ for $\alpha < 2$. Then it is sufficient to find α so that the difference of (10) and (14) is negative. This difference is

$$\frac{4^{j-1}j}{j-2\alpha} + \sum_{k=2}^{j-2} \frac{2(jk - j - k^2)p_{k-1}p_{j-k-1}}{(k-\alpha)(j-k-\alpha)} - \sum_{k=2}^{j-1} \frac{4j(k-1)p_{k-1}p_{j-k-1}}{(k-\alpha)(j-2\alpha)}. \tag{17}$$

If $j \geq 5$, then the only unbounded terms in (17) as $\alpha \to 2$ correspond to $k = 2$ and $k = j-2$ from the first sum and $k = 2$ from the second sum. The sum of these terms is

$$\frac{-8p_{j-3}[4(j-4) - (2-\alpha)(j-8)]}{(2-\alpha)(j-2-\alpha)(j-2\alpha)},$$

and so (17) becomes negative as α increases to 2. If $j = 4$, the sum of the terms in (17) corresponding to $k = 2$ is $-32/(2-\alpha)^2$, which dominates as $\alpha \to 2$, and so (17) again becomes negative. One completes the proof as before.

3. Coefficient inequalities under constraints. In [4, 5] J. A. Jenkins gave a remarkable extension of a theorem of Teichmüller.

Special Case of Jenkins' General Coefficient Theorem. Let $\mathscr{F}$ be a univalent analytic function in the complement of the n-star γ_n, $n \geq 2$, with expansion $\mathscr{F}(w) = w + \sum_{k=1}^{\infty} \mathscr{B}_k w^{1-k}$ in a neighborhood of ∞. If $\mathscr{B}_k = 0$ for $2 \leq k \leq n/2$, then $\operatorname{Re}\{\mathscr{B}_n\} \leq 0$.

Our Lemma 1 illustrates that his result is in a sense best possible. Jenkins' theorem leads to the following complement to our Lemmas 2 to 6.

Corollary to Jenkins' General Coefficient Theorem. Let $\mathscr{F}$ be a univalent analytic function in the complement of the n-star γ_n, $n \geq 2$, *with expansions*

$$\mathscr{F}(w) = w + \sum_{k=1}^{\infty} \mathscr{B}_k w^{1-k}, \qquad \log \mathscr{F}'(w) = \sum_{k=2}^{\infty} \mathscr{C}_k w^{-k},$$

$$\log \frac{\mathscr{F}(w)}{w} = \sum_{k=1}^{\infty} \mathscr{D}_k w^{-k}, \quad \textit{and} \quad \log \frac{w\mathscr{F}'(w)}{\mathscr{F}(w)} = \sum_{k=1}^{\infty} \mathscr{E}_k w^{-k}$$

in a neighborhood of ∞.

(a) *If* $\mathscr{C}_k = 0$ *for* $k \le n/2$, *then* $\operatorname{Re}\{\mathscr{C}_n\} \ge 0$.

(b) *If* $\mathscr{D}_k = 0$ *for* $k \le n/2$, *then* $\operatorname{Re}\{\mathscr{D}_n\} \le 0$.

(c) *If* $\mathscr{E}_k = 0$ *for* $k \le n/2$, *then* $\operatorname{Re}\{\mathscr{E}_n\} \ge 0$.

Proof. Let $\nu = [n/2] + 1$. Since

$$\begin{aligned} \mathscr{F}'(w) &= \exp\left\{\sum_{k=\nu}^{\infty} \mathscr{C}_k w^{-k}\right\} \\ &= 1 + \mathscr{C}_\nu w^{-\nu} + \cdots + \mathscr{C}_n w^{-n} + O(w^{-n-1}) \end{aligned}$$

as $w \to \infty$, it follows from integration that

$$\mathscr{F}(w) = w + \mathscr{B}_1 - \frac{\mathscr{C}_\nu}{\nu - 1} w^{1-\nu} - \cdots - \frac{\mathscr{C}_n}{n-1} w^{1-n} + O(w^{-n}).$$

In a similar fashion, we obtain

$$\mathscr{F}(w) = w + \mathscr{B}_1 + \mathscr{D}_\nu w^{1-\nu} + \cdots + \mathscr{D}_n w^{1-n} + O(w^{-n})$$

and

$$\mathscr{F}(w) = w + \mathscr{B}_1 - \frac{\mathscr{E}_\nu}{\nu} w^{1-\nu} - \cdots - \frac{\mathscr{E}_n}{n} w^{1-n} + O(w^{-n}).$$

Thus (a), (b), and (c) follow from Jenkins' theorem.

4. Application to coefficient conjectures. In this section we apply Lemmas 2 to 6 in order to provide counterexamples to the conjectures (1) to (3) for those indices promised in §1. The results are just as easily obtained for functions with symmetries, but on first reading one may specialize to the 1-fold symmetric case, which is no symmetry assumption at all. At the end of this section we revive inequalities (1) to (3) under constraints.

First, we shall apply Lemma 1 to the coefficient problem for Σ. Except for the existence of symmetric functions, this result was obtained in **[3]** and in **[14]** by means of second variations of a different form.

THEOREM 1. *Assume that* j, k, n *are positive integers such that* $k \geq 4$, $k = jn$, *and* $k > n$. *Then there exists an n-fold symmetric function* $f \in \Sigma$ *such that the coefficient* b_{k-1} *in the expansion* $f(z) = z + \sum_{m=0}^{\infty} b_m z^{-m}$ *satisfies* $|b_{k-1}| > 2/k$. *In particular, for each* $k \geq 4$, *there exists a function in* Σ *for which* $|b_{k-1}| > 2/k$.

Proof. With j, k, n as in the hypothesis, we choose F from the conclusion of Lemma 1. Then $f = F \circ K_k$ is n-fold symmetric and has coefficient $b_{k-1} = B_j + 2/k$ where $\operatorname{Re} B_j > 0$. This completes the proof. Since $j = k \geq 4$ and $n = 1$ is always an admissible choice of integers, a 1-fold symmetric function always exists, but 1-fold symmetry is no symmetry restriction at all.

THEOREM 2. *Assume that* j, k, n *are positive integers such that* $k \geq 4$, $k = jn$, *and* $k > n$. *Then there exists an n-fold symmetric function* $f \in \Sigma$ *such that the coefficient* c_k *in the expansion* $\log f'(z) = \sum_{m=2}^{\infty} c_m z^{-m}$ *satisfies* $|c_k| > (2(k-1))/k$. *In particular, for each* $k \geq 4$, *there exists a function in* Σ *for which* $|c_k| > 2(k-1)/k$.

Proof. With j, k, n as in the hypothesis, we choose F from the conclusion of Lemma 2. Then $f = F \circ K_k$ is n-fold symmetric and $\log f' = \log(F' \circ K_k) + \log K_k'$ has coefficient $c_k = C_j - (2(k-1)/k)$ where $\operatorname{Re} C_j < 0$. This completes the proof.

THEOREM 3. *Assume that* j, k, n *are positive integers such that* $k \geq 2$, $k = jn$, *and* $k > n$. *Then there exists an n-fold symmetric function* $f \in \Sigma'$ *such that the coefficient* d_k *in the expansion* $\log f(z)/z = \sum_{m=1}^{\infty} d_m z^{-m}$ *satisfies* $|d_k| > 2/k$. *In particular, for each* $k \geq 2$, *there exists a function in* Σ' *for which* $|d_k| > 2/k$.

In addition, for each $k \geq 4$, *there exists a function* $f \in \Sigma_0$ *for which* $|d_k| > 2/k$.

Proof. With j, k, n as in the hypothesis, we choose F from the conclusion of Lemma 3. Then $f = F \circ K_k$ is nonvanishing and n-fold symmetric, and the function

$$\log \frac{f}{z} = \log \frac{F \circ K_k}{K_k} + \log \frac{K_k}{z}$$

has coefficient $d_k = D_j + 2/k$ where $\operatorname{Re} D_j > 0$. To construct $f \in \Sigma_0$, we replace Lemma 3 by Lemma 4. This completes the proof.

THEOREM 4. *Assume that j, k, n are positive integers such that $k \geq 2$, $k = jn$, and $k > n$. Then there exists an n-fold symmetric function $f \in \Sigma'$ such that the coefficient d_k in the expansion $\log zf'(z)/f(z) = \sum_{m=1}^{\infty} e_m z^{-m}$ satisfies $|e_k| > 2$. In particular, for each $k \geq 2$, there exists a function in Σ' for which $|e_k| > 2$.*

In addition, for each $k \geq 4$, there exists a function $f \in \Sigma_0$ for which $|e_k| > 2$.

Proof. Choose F from the conclusion of Lemma 5. Then $f = F \circ K_k$ is nonvanishing and n-fold symmetric, and the function

$$\log \frac{zf'}{f} = \log \frac{K_k(F' \circ K_k)}{F \circ K_k} + \log \frac{zK_k'}{K_k}$$

has coefficient $e_k = E_j - 2$ where $\operatorname{Re} E_j < 0$. To construct $f \in \Sigma_0$, replace Lemma 5 by Lemma 6. This completes the proof.

We conclude this section by verifying inequalities (1) to (3) under constraints.

THEOREM 5. *Let f belong to the class Σ and have expansions*

$$\log f'(z) = \sum_{k=2}^{\infty} c_k z^{-k}, \quad \log \frac{f(z)}{z} = \sum_{k=1}^{\infty} d_k z^{-k}, \quad \textit{and}$$

$$\log \frac{zf'(z)}{f(z)} = \sum_{k=1}^{\infty} e_k z^{-k}$$

in a neighborhood of ∞. Assume that $n \geq 2$.

(a) *If $c_k = 0$ for $k \leq n/2$, then $|c_n| \leq 2(n-1)/n$.*

(b) *If $d_k = 0$ for $k \leq n/2$, then $|d_n| \leq 2/n$.*

(c) *If $e_k = 0$ for $k \leq n/2$, then $|e_n| \leq 2$.*

Proof. Apply the corollary to Jenkins' general coefficient theorem found in §3 to the composition $\mathscr{F} = f \circ K_n^{-1}$. We obtain

$$\operatorname{Re}\{c_n\} \geq -\frac{2(n-1)}{n}, \quad \operatorname{Re}\{d_n\} \leq \frac{2}{n}, \quad \text{and} \quad \operatorname{Re}\{e_n\} \geq -2.$$

The inequalities as stated follow by rotation.

5. General linear problems. Let L denote a continuous linear functional on the space of analytic functions in Δ with the topology of locally uniform convergence. Assume that L is not constant on Σ and that $L(1) = 0$. Then there exists a function f in Σ for which

$\mathrm{Re}\{L\}$ is a maximum. One of the most important tools for solving such extremal problems is Schiffer's boundary variation and fundamental lemma [**11**]. It implies that the omitted set $\Gamma = \mathbb{C}\backslash f(\Delta)$ of an extremal function f consists of finitely many analytic arcs that lie on trajectories of the quadratic differential $L(1/(f-w))\,dw^2$. Trajectories of a quadratic differential $Q(w)\,dw^2$ are arcs $w = w(t)$ on which $Q(w(t))[w'(t)]^2 > 0$, together with their endpoints. As a function of w, the expression $L(1/(f-w))$ has an analytic extension to a neighborhood of Γ. The analytic arcs of the omitted set Γ must be connected, and the points of nonanalyticity can arise only from zeros of $L(1/(f-w))$. Very recently, the authors proved that any zeros of $L(1/(f-w))$ on Γ must be simple zeros. This has the remarkable geometric consequence that the omitted set Γ can fork in at most three equi-angular directions. See [**10**] for details and further information.

The coefficient functional is an example of a continuous linear functional. So, for example, if the function K_n were to provide the maximum of $\mathrm{Re}\{b_{n-1}\}$ over the class Σ, then the n-star γ_n would have to lie on trajectories of the quadratic differential $L(1/(K_n-w))\,dw^2$, where the functional L picks out the $(n-1)$st coefficient of $1/(K_n(z)-w)$ as a function of z near infinity. Since $L(1/(K_n-w)) = w^{n-2}$, this quadratic differential is $w^{n-2}\,dw^2$, and its trajectories from the origin consist of n equally spaced rays. In fact, γ_n does lie on these trajectories for all n. Thus K_n satisfies Schiffer's differential equation for the $(n-1)$st coefficient problem for every n, but is a true extremal function only for $n = 2$ and $n = 3$. Unfortunately, Schiffer's boundary variation does not distinguish the true extremals from "false pretenders". Other tools are needed, as in the proof of Lemma 1. It shows in some sense that $\mathrm{Re}\{b_{n-1}\}$ has a saddle point at K_n for $n \geq 4$.

The purpose of this section is to carry out a similar program for linear problems on the expressions $\log f'$, $\log f/z$, and $\log zf'/f$. If L is a continuous linear functional, then by compactness the expressions $\mathrm{Re}\{L(\log f')\}$ for $f \in \Sigma$, $\mathrm{Re}\{L(\log f/z)$ for Σ' or $f \in \Sigma_0\}$, and $\mathrm{Re}\{L(\log zf'/f)\}$ all assume a maximum. The following theorems describe properties of extremal functions for these problems. Of course, the coefficient problems of §4 are of this type.

THEOREM 6. *Let* L *be a continuous linear functional, and assume that* $\mathrm{Re}\{L(\log f')\}$ *is not constant as* f *varies over* Σ. *If* f *is an* *extremal function for the problem*

$$\max_{f\in\Sigma} \operatorname{Re}\{L(\log f')\},$$

then the omitted set Γ *of* f *consists of finitely many analytic arcs lying on trajectories of the quadratic differential* $-L(1/(f-w)^2)\,dw^2$. *Furthermore,* $L(1/(f-w)^2)$ *can have only simple zeros on* Γ; *hence* Γ *can fork in at most three equi-angular directions.*

Proof. Let f be an extremal function for the problem

$$\max_{f\in\Sigma} \operatorname{Re}\{L(\log f')\}.$$

We subject f to boundary variations (cf. **[11]**) of the form

$$f^* = f + a\rho^2/(f-w) + o(\rho^2), \quad \text{with } w \in \Gamma,\ |a| = 1,\ \text{and } \rho > 0, \tag{18}$$

which operate within the family Σ. Under these variations the change of the functional is of the form

$$\operatorname{Re}\{L(\log f^{*\prime})\} = \operatorname{Re}\{L(\log f')\} + \operatorname{Re}\{a\rho^2 L(-1/(f-w)^2)\} + o(\rho^2)$$

as $\rho \to 0$. Since the functional L can be represented by a measure compactly supported in Δ, the expression $L(1/(f-w)^2)$, defined initially for $w \in \Gamma$, extends to an analytic function in a neighborhood of Γ. It cannot vanish identically for then $\operatorname{Re}\{L(\log f')\}$ would be constant on Σ. Now it follows from Schiffer's fundamental lemma **[11]** that Γ consists of analytic arcs lying on trajectories of the quadratic differential $-L(1/(f-w)^2)\,dw^2$. Since $L(1/(f-w)^2)$ can have only finitely many zeros on Γ, the omitted set Γ can consist of only finitely many analytic arcs joined at these zeros. Next, we show that these zeros can only be simple ones.

After a translation, we may assume that $w = 0$ is a zero on Γ of order k for the function $L(1/(f-w)^2)$. That is, $L(f^{-j}) = 0$ for $j = 2, 3, \dots, k+1$ and $L(f^{-k-2}) \neq 0$. Furthermore, after rotating the functional and the function, we may assume also that $L(f^{-k-2}) < 0$ and that one of the omitted arcs emanates from the origin in the positive horizontal direction. Delete all the arcs of Γ except for that connected part emanating from the origin in the positive horizontal direction that lies inside the disk $|w| \le \delta$, for sufficient small positive δ. Designate this subarc by Γ_δ, and let F_δ be the conformal map from Δ onto the complement of Γ_δ with an expansion of the form $F_\delta(z) = \rho z + O(1)$ around infinity and $\rho > 0$. Then F_δ and the original f are related by a Schwarz function $\omega\colon \Delta \to \Delta$ satisfying $F_\delta \circ \omega = f$ and $\omega'(\infty) = 1/\rho$. As $\delta \to 0$, we have $\rho \to 0$. The function $f_\delta = (1/\rho)F_\delta$ belongs to Σ and maps Δ onto the complement of an arc c_δ which is obtained from dilating Γ_δ by the factor $1/\rho$. Since Γ_δ is an analytic arc emanating from $w = 0$ in the positive horizontal direction, the dilation c_δ approaches the 1-star $c_0 = [0, 4]$ as $\delta \to 0$.

Let F be analytic and univalent in the complement of c_δ and have expansions of the form $F(w) = w + \sum_{j=1}^{\infty} B_j(\delta)w^{1-j}$ and $\log F'(w) = \sum_{j=2}^{\infty} C_j(\delta)w^{-j}$ around infinity. Now $\rho F \circ f_\delta \circ \omega$ belongs to Σ and so $\mathrm{Re}\{L(\log[(\rho F \circ f_\delta \circ \omega)'])\} \le \mathrm{Re}\{L(\log f')\}$. Since

$$\begin{aligned}\log[(\rho F \circ f_\delta \circ \omega)'] &= \log F'(f_\delta \circ \omega) + \log[(\rho f_\delta \circ \omega)'] \\ &= \log f' + \log F'\left(\frac{1}{\rho} f\right),\end{aligned}$$

it follows that $\mathrm{Re}\{L(\log F'(\frac{1}{\rho}f))\} \le 0$. In other words, we have $\mathrm{Re}\{\sum_{j=2}^{\infty} C_j(\delta)\rho^j L(f^{-j})\} \le 0$ when δ is sufficiently small. From the assumption on the order of the zero at $w = 0$, the terms of this series with index $j \le k+1$ are zero. Therefore, we have

$$\mathrm{Re}\{C_{k+2}(\delta)\}\rho^{k+2}L(f^{-k-2}) + O(\rho^{k+3}) \le 0$$

where $L(f^{-k-1})$ is negative. Dividing by ρ^{k+2} and letting $\delta \to 0$ leads us to the conclusion that $\mathrm{Re}\, C_{k+2} \ge 0$ for any univalent function in the complement of the 1-star $c_0 = [0, 4]$ with expansions $F(w) = w + \sum_{j=1}^{\infty} B_j w^{1-j}$ and $\log F'(w) = \sum_{j=2}^{\infty} C_j w^{-j}$ in a neighborhood of ∞. However, in Lemma 2 we obtained such functions with $\mathrm{Re}\{C_{k+2}\} < 0$ for every $k \ge 2$. To avoid a contradiction, the order k of the zero must be 1. This completes the proof.

REMARK. If the functional L picks out the negative of the nth coefficient, then $L(1/(K_n - w)^2) = -(n-1)w^{n-2}$, and the quadratic differential of Theorem 6 becomes $(n-1)w^{n-2}\,dw^2$. That is, the function K_n satisfies Schiffer's differential equation for the problem $\max_{f\in\Sigma} \mathrm{Re}\{-c_n\}$, but is is extremal only for $n = 2$ and $n = 3$. In fact, the proof of Theorem 2 shows in a sense that K_n is a saddle point for this functional whenever $n \ge 4$.

THEOREM 7. *Let* L *be a continuous linear functional, and assume that* $\mathrm{Re}\{L(\log(f/z))\}$ *is not constant as* f *varies over* Σ'. *If* f *is an extremal function for the problem* $\max_{f\in\Sigma'} \mathrm{Re}\{L(\log(f/z))\}$, *then* *the omitted set* Γ *of* f *consists of finitely many analytic arcs lying on trajectories of the quadratic differential* $L(1/(f-w))\,dw^2/w$. *Furthermore, the origin is a simple pole of the quadratic differential; hence* Γ *has a tip at the origin.*

Proof. Let f be an extremal function. Subject f to boundary variations of the form

$$f^* = f + a\rho^2/(f-w) + a\rho^2/w + o(\rho^2), \qquad \text{with } w \in \Gamma,\ |a| = 1,\ \text{and } \rho > 0,$$

which operate within Σ'. The change of the functional is given by

$$\operatorname{Re}\left\{\left(\log\frac{f^*}{z}\right)\right\} = \operatorname{Re}\left\{L\left(\log\frac{f}{z}\right)\right\} + \operatorname{Re}\left\{a\rho^2 L\left(\frac{1}{w(f-w)}\right)\right\} + o(\rho^2)$$

as $\rho \to 0$. This leads as before to the conclusion that the omitted set Γ consists of finitely many analytic arcs lying on trajectories of the quadratic differential $L(1/(f-w))\,dw^2/w$.

To show that the apparent pole of the quadratic differential $L(1/(f-w))\,dw^2/w$ at the origin is not removable, assume for the purpose of contradiction that $L(1/(f-w))$ has a zero of some order $k \geq 1$ at the origin. It follows that $L(f^{-j}) = 0$ for $1 \leq j \leq k$ and that $L(f^{-k-1}) \neq 0$. By rotating the function and functional, we may assume also that $L(f^{-k-1}) > 0$ and that one of the arcs of Γ emanates from the origin in the positive horizontal direction. Construct the functions ω and f_δ and arc c_δ as in the previous proof. Let F be a nonvanishing univalent analytic function in the complement of c_δ and have expansions of the form

$$F(w) = w + \sum_{j=1}^{\infty} B_j(\delta) w^{1-j} \quad \text{and} \quad \log\frac{F(w)}{w} = \sum_{j=1}^{\infty} D_j(\delta) w^{-j}.$$

around infinity. As before, the function $\rho F \circ f_\delta \circ \omega$ belongs to Σ' and so

$$\operatorname{Re}\left\{L\left(\log\frac{\rho F \circ f_\delta \circ \omega}{z}\right)\right\} \leq \operatorname{Re}\left\{L\left(\log\frac{f}{z}\right)\right\}.$$

Since

$$\log\frac{\rho F \circ f_\delta \circ \omega}{z} = \log\frac{f}{z} + \log\frac{F\left(\frac{1}{\rho} f\right)}{\frac{1}{\rho} f},$$

it follows that $\operatorname{Re}\{\sum_{j=k+1}^{\infty} D_j(\delta)\rho^j L(f^{-j})\} \le 0$ when δ is sufficiently small. Dividing by ρ^{k+1}, letting $\delta \to 0$, and using $L(f^{-k-1}) > 0$ leads us to the conclusion that $\operatorname{Re} D_{k+1} \le 0$ for any nonvanishing univalent function in the complement of the 1-star with expansions

$$F(w) = w + \sum_{j=1}^{\infty} B_j w^{1-j} \quad \text{and} \quad \log \frac{F(w)}{w} = \sum_{j=1}^{\infty} D_j w^{-j}$$

in a neighborhood of ∞. However, this contradicts Lemma 3 and completes the proof.

THEOREM 8. *Let L be a continuous linear functional, and assume that $\operatorname{Re}\{L(\log(zf'/f))\}$ is not constant as f varies over Σ'. If f is an extremal function for the problem $\max_{f\in\Sigma'} \operatorname{Re}\{L(\log(zf'/f))\}$, then the omitted set Γ of f consists of finitely many analytic arcs lying on trajectories of the quadratic differential $-L(f/(f-w)^2)\,dw^2/w$. Furthermore, the origin is a simple pole of the quadratic differential; hence Γ has a tip at the origin.*

Proof. One uses the same variations as in the previous proof and at the end substitutes Lemma 5 for Lemma 3. We omit the details.

THEOREM 9. *Let L be a continuous linear functional, and assume that $\operatorname{Re}\{L(\log(f/z)\}$ is not constant as f varies over Σ_0. If f is an extremal function for the problem $\max_{f\in\Sigma_0} \operatorname{Re}\{L(\log(f/z)\}$, then the omitted set Γ of f consists of finitely many analytic arcs lying on trajectories of the quadratic differential $L(1/(f-w)f)\,dw^2$.*

Proof. Under boundary variations of the form (18), which operate also within the family Σ_0, the change in the functional is given by

$$\operatorname{Re}\left\{L\left(\log\frac{f^*}{z}\right)\right\} = \operatorname{Re}\left\{L\left(\log\frac{f}{z}\right)\right\} + \operatorname{Re}\left\{a\rho^2 L\left(\frac{1}{(f-w)f}\right)\right\} + o(\rho^2)$$

as $\rho \to 0$. As before, this leads to the conclusion that the omitted set Γ consists of finitely many analytic arcs lying on trajectories of the quadratic differential $L(1/(f-w)f)\,dw^2$.

THEOREM 10. *Let L be a continuous linear functional, and assume that $\operatorname{Re}\{L(\log(zf'/f))\}$ is not constant as f varies over Σ_0. If f is an extremal function for the problem $\max_{f\in\Sigma_0} \operatorname{Re}\{L(\log(zf'/f))\}$,* *then the omitted set* Γ *of* f *consists of finitely many analytic arcs lying on trajectories of the quadratic differential* $L((w-2f)/(f-w)^2 f)\,dw^2$.

Proof. Under variations of the form (18), the change in the functional is given by

$$\operatorname{Re}\left\{L\left(\log\frac{zf^{*\prime}}{f^*}\right)\right\} = \operatorname{Re}\left\{L\left(\log\frac{zf'}{f}\right)\right\} + \operatorname{Re}\left\{a\rho^2 L\left(\frac{w-2f}{(f-w)^2 f}\right)\right\} + o(\rho^2)$$

as $\rho \to 0$. The conclusion follows as before.

REMARKS. In Theorems 7 to 10 we did not show that the zeros of the quadratic differential on Γ are simple. We do not know whether they are.

For the special case of coefficient problems in Theorems 7 to 10, the functions K_n satisfy the corresponding Schiffer differential equation, but are not extremal as indicated in Theorems 3 and 4. That is, in some sense the functions K_n are again saddle points for these problems.

6. The N **th diameter problem.** Denote by E the continuum omitted by a function $f \in \Sigma$. The Nth diameter of E is defined as $d_N(E) = \max[\prod_{i\neq j}|w_i - w_j|]^{1/[N(N-1)]}$ where the maximum is taken over all sets of N points $w_1, \dots, w_N$ in E. Points $w_1, \dots, w_N$ in E that yield the maximum are called Fekete points of E. Of course, $d_2(E)$ is the Euclidean diameter of E. Various properties of the diameter d_N can be found in texts on function theory. For example, as $N \to \infty$ it is known that $d_N(E)$ decreases monotonically to the capacity of E, which equals one in the present case. Recently, we [**7**, **8**] discussed the geometric character of the extremal set that solves the Nth diameter problem:

Nth Diameter Problem. For each $N \geq 2$, *find the maximum of* $d_N(E)$ *among all the continua* E *omitted by functions in* Σ.

This problem was considered by Schiffer in 1938 [**12**]. By the method of boundary variations, he showed that an extremal configuration Γ consists of a finite number of analytic arcs lying on trajectories of the quadratic differential

$$Q(w)\,dw^2 = -\sum_{i\neq j}\frac{1}{(w-w_i)(w-w_j)}\,dw^2, \tag{19}$$

where the points $w_1, \ldots, w_N$ are the Fekete points of Γ. The Fekete points appear to be simple poles of the quadratic differential. However, it is conceivable that they are removable singularities. It was proved in [**8**] that, indeed, they are simple poles. As a consequence, the Fekete points are all tips of an extremal continuum Γ.

How does Γ connect the Fekete points? The analytic arcs of Γ can join only at zeros of the quadratic differential (19). In [**8**] we conjectured that these zeros are all simple zeros. Geometrically, this means that Γ can fork in at most three equiangular directions. In [**8**] we also eliminated the possibility of zeros of order $k \geq 2$ whenever $k = 3$ or $k + 2$ is nonprime. Later, the first author [**7**] eliminated the possibility of the remaining zeros of order $k \geq 2$. Thus this simple zero conjecture is true. In this section we shall apply the results of §2 to unify and simplify the proof.

THEOREM 11. *Let*

$$Q(w)\,dw^2 = -\sum_{i \neq j} \frac{1}{(w - w_i)(w - w_j)}\,dw^2$$

be the quadratic differential corresponding to an extremal configuration Γ *for the* N*th diameter problem. Each Fekete point* w_j *is a simple pole of* $Q(w)\,dw^2$ *and an endpoint of* Γ. *The zeros of* $Q(w)\,dw^2$ *all lie on* Γ: *they are all simple zeros*; *there are* $N - 2$ *of them*; *and* Γ *forks at each zero in three equiangular directions.*

In particular, an extremal continuum Γ consists of N analytic arcs beginning at the N Fekete points and joining three at a time at the $N - 2$ zeros of $Q(w)\,dw^2$. Thus we have a qualitative picture of the extremal continuum. In general, however, an analytic solution of the problem remains very difficult since we do not know the location of either the zeros or the poles.

For further background and information we refer the reader to [**6, 7, 8**].

Proof of Theorem 11. Assume that the quadratic differential has a zero of order k at a point $w_0 \in \Gamma$. After a translation and rotation we may assume that $w_0 = 0$ and that some arc of Γ emanates from the origin in the positive horizontal direction. Just as in the second paragraph of the proof of Theorem 6, delete from Γ all arcs except for that connected subarc Γ_δ emanating from the origin in the positive horizontal direction that lies inside the disk $|w| \leq \delta$. If the mapping radius of $\mathbb{C}\backslash\Gamma_\delta$ is ρ, then dilate Γ_δ by the factor $1/\rho$ to obtain an arc c_δ with exterior mapping radius one. As $\delta \to 0$, the arc c_δ approaches the 1-star $c_0 = [0, 4]$.

Let F be nonvanishing, analytic, and univalent in the complement of c_δ and have expansions of the form

$$F(w) = w + \sum_{j=1}^{\infty} B_j(\delta) w^{1-j}, \quad \log F'(w) = \sum_{j=2}^{\infty} C_j(\delta) w^{-j} \quad \text{and}$$

$$\log \frac{wF'(w)}{F(w)} = \sum_{j=1}^{\infty} E_j(\delta) w^{-j}$$

around infinity. Now $\rho F(w/\rho)$ is univalent in $\mathbb{C}\backslash\Gamma_\delta$, hence in $\mathbb{C}\backslash\Gamma$. It preserves the mapping radius of $\mathbb{C}\backslash\Gamma$ and serves as a variation of Γ.

Assume now that the Fekete points $w_1, \dots, w_N$ are all nonzero. If ρ is sufficiently small, then the points $\rho F(w_1/\rho), \dots, \rho F(w_N/\rho)$ are not assumed by $\rho F(w/\rho)$ on $\mathbb{C}\backslash\Gamma$. Therefore we have

$$\sum_{i \neq j} \log|\rho F(w_i/\rho) - \rho F(w_j/\rho)| \leq \sum_{i \neq j} \log|w_i - w_j| \tag{20}$$

because of the extremal character of Γ. If we define the Grunsky coefficients G_{ij} of F by the expansion

$$\log \frac{F(w) - F(\omega)}{w - \omega} = \sum_{i,j=1}^{\infty} G_{ij} w^{-i} \omega^{-j}$$

near infinity, then $C_\nu = \sum_{i+j=\nu} G_{ij}$ and (20) becomes

$$\begin{aligned} 0 &\leq \operatorname{Re}\left\{\sum_{j=1}^{N} \log F'(w_j/\rho) - \sum_{i,j=1}^{N} \log \frac{\rho F(w_i/\rho) - \rho F(w_j/\rho)}{w_i - w_j}\right\}. \\ &= \operatorname{Re}\left\{\sum_{\nu=2}^{\infty} C_\nu(\delta)\rho^\nu \mu_\nu - \sum_{i,j=1}^{\infty} G_{ij}(\delta)\rho^{i+j}\mu_i\mu_j\right\} \\ &= \operatorname{Re}\left\{\sum_{\nu=2}^{\infty} \sum_{i+j=\nu} C_\nu(\delta)\rho^\nu(\mu_\nu - \mu_i\mu_j)\right\} \end{aligned} \tag{21}$$

where $\mu_\nu = \sum_{j=1}^{N} 1/w_j^\nu$. By Lemma 4.1 of **[8]**, the factors $\mu_\nu - \mu_i\mu_j = \mu_1^\nu - \mu_1^i\mu_1^j = 0$ for $i+j = \nu \leq k+1$ and $\mu_{k+2} - \mu_i\mu_j = \mu_{k+2} - \mu_1^{k+2} \neq 0$.

Thus the terms in the last expression of (21) are zero for $2 \le \nu \le k+1$. After dividing by ρ^{k+2} and letting $\delta \to 0$, we conclude that

$$0 \le \operatorname{Re}\left\{C_{k+2}(0)(\mu_{k+2} - \mu_1^{k+2})\right\}. \tag{22}$$

We have assumed that some arc of Γ emanates from the origin in the positive horizontal direction. It follows then from the expansion

$$\begin{aligned} Q(w)\,dw^2 &= \left(\frac{Q^{(k)}(0)}{k!}w^k + \cdots\right) dw^2 \\ &= \left((k+1)(\mu_{k+2} - \mu_1^{k+2})w^k + \cdots\right) dw^2 \end{aligned}$$

that $\mu_{k+2} - \mu_1^{k+2}$ is positive. As a consequence, (22) implies that $\operatorname{Re}\{C_{k+2}(0)\} \ge 0$. That is, for all functions F analytic and univalent in the complement of the 1-star c_0, with normalization $F(w) = w + O(1)$ near ∞, the coefficient C_{k+2} in the expansion $\log F'(w) = \sum_{j=2}^{\infty} C_j w^{-j}$ satisfies $\operatorname{Re}\{C_{k+2}\} \ge 0$. For $k \ge 2$ this is in contradiction to Lemma 2, and so we conclude that the zero must have been simple. That is, a zero at a non-Fekete point of Γ must be simple.

Next, assume that one of the Fekete points, say w_1, is at the origin and that the quadratic differential has a removable singularity there. Thus, assume that the $Q(w)$ has a zero of order $k \ge 0$ at the origin ($k = 0$ means that the origin is removable, but not a zero). Then an analysis very similar to the previous paragraph and utilizing Lemma 5.1 of **[8]** leads to the conclusion that all nonvanishing univalent analytic functions F in the complement of the 1-star, with normalization $F(w) = w + O(1)$ near ∞, have the property that the coefficient E_{k+2} in the expansion $\log wF'(w)/F(w) = \sum_{j=1}^{\infty} E_j w^{-j}$ satisfies $\operatorname{Re}\{E_{k+2}\} \ge 0$. For $k \ge 0$ this contradicts Lemma 5. Thus each Fekete point is a simple pole of the quadratic differential, hence an endpoint of Γ.

Now the rational function $Q(w)$ has precisely N simple poles and $\lim_{w\to\infty} w^2 Q(w) = -N(N-1)$. Therefore $Q(w)$ has $N-2$ zeros, counting multiplicities. Each zero on Γ is simple, and Γ can fork in at most three ways at such zeros. A finite induction argument implies that there must be at least $N-2$ such zeros on Γ to permit Γ to have N tips. That is, all $N-2$ zeros must lie on Γ; they must be simple; and Γ must fork in all three equiangular directions possible at these zeros. This completes the proof.

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UNIVERSITY OF DELAWARE
NEWARK, DE 19716

AND

INDIANA UNIVERSITY
BLOOMINGTON, IN 47405