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# POINCARE-SOBOLEV AND RELATED INEQUALITIES FOR SUBMANIFOLDS OF $\mathbf{R}^N$

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**We prove Poincaré-Sobolev and related inequalities for rectifiable varifolds in $\mathbf{R}^N$. In particular, all our results apply to properly immersed submanifolds of $\mathbf{R}^N$.**

**Suppose $M \subset B_R = B_R(0) \subset \mathbf{R}^N = \mathbf{R}^{n+k}$ for some $R > 0$, and $V = v(M, \theta)$ is a countably $n$-rectifiable varifold in $B_R$ with generalised mean curvature vector $H$. $\mu$ is the weight measure defined by $\mu = \theta H^n \lfloor M$. $h: M \to R$ is a Lipschitz function.**

**In Theorem 1 we prove a Poincaré-Sobolev result for non-negative $h$ in case $\mu\{\xi: h(\xi) > 0\} < \omega_n R^n$ and $h \in W^{1,p}(\mu)$ for some $p < n$. This generalises a Poincaré result of Leon Simon; but in addition the relevant constant here does not depend on $\mu(B_R)$. Theorem 2 is an Orlicz space result in case $p = n$.**

**The proofs of Theorems 1 and 2 use a covering argument to obtain weak $L^p$ type estimates on $\mu\{\xi: h(\xi) > s\}$.**

**Theorems 3 and 4 are generalisations of Theorems 1 and 2 in case there is no restriction on $\mu\{\xi: h(\xi) \neq 0\}$ (again the constants in the estimates do not depend on $\mu(B_R)$). The conclusion of Theorem 4 is analogous to the conclusion of the John-Nirenberg theorem for functions of bounded mean oscillation.**

We prove Poincaré-Sobolev and related inequalities for rectifiable varifolds in $\mathbf{R}^N$. In particular, all our results apply to properly immersed submanifolds of $\mathbf{R}^N$.

Theorem 1 is a refinement of a result due to Leon Simon. In [**Sc**; p. 70] and [**S**; Theorem 18.4, p. 91] one has a similar Poincaré inequality in case $p = 1$ and $|H|$ is bounded, but with a constant $c$ depending on $\mathbf{M}(V \lfloor B_R)$. In Theorem 1, $c$ depends only on $p$ and the dimension of $V$. This is important in case we have no a priori density bound for $V$ at 0 (as in [**H**], which provided the motivation for the present paper).

We also remark that the Poincaré result in Theorem 1 for $p > 1$ does not seem to follow directly from the case $p = 1$—the usual trick of replacing $h$ by $h^r$ does not work since the integrals in the inequality occur over balls of different radius. Nonetheless, one can use the Sobolev inequality for functions *with* compact support and a cut-off function argument to "bootstrap" up from the $p = 1$ case. However, the proof in Theorem 1 gives the Poincaré result directly for all $p$ and with the constant dependence as noted above. The Sobolev result then follows immediately (as pointed out by Leon Simon) by a simple cut-off function argument from the result in the compact support case (this latter was first established in [**A**; Theorem 7.3] and [**MS**]).

In Theorem 2 we prove an Orlicz space result in case $h \in W^{1,n}(\mu)$, where $n$ is the dimension of $V$ and $\mu$ is the measure in $\mathbf{R}^N$ induced by $V$.

The proofs of Theorems 1 and 2 use a covering argument to obtain weak $L^p$ type estimates on $\mu\{\xi: h(\xi) > s\}$, and were motivated in part by the proof of the Sobolev inequality for functions with compact support in [**S**; Theorem 18.6, p. 93].

Theorems 3 and 4 are generalisations of Theorems 1 and 2 in case there is no restriction on $\mu\{\xi: h(\xi) \neq 0\}$ (again the constants in the estimates do not depend on $\mathbf{M}(V\lfloor B_R)$). They follow directly from Theorems 1 and 2, as was also realised by Leon Simon in the context of his Poincaré inequality discussed previously [private communication]. The conclusion of Theorem 4 is analogous to the conclusion of the John-Nirenberg theorem for functions of bounded mean oscillation.

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NOTATION. Throughout this paper we use the notations and conventions of [**S**].

In each of the following theorems we take the following hypotheses:

**(H):** *$M \subset B_R = B_R(0) \subset \mathbf{R}^N = \mathbf{R}^{n+k}$ for some $R > 0$, and $V = \mathbf{v}(M, \theta)$ is a countably $n$-rectifiable varifold in $B_R$ with generalised mean curvature vector $H$. $\mu$ is the weight measure defined by $\mu = \theta H^n \lfloor M$. $h: M \to \mathbf{R}$ is a Lipschitz function.*

*Convention.* All integrals are taken with respect to $\mu$, unless otherwise clear from context.

THEOREM 1. *Suppose* **(H)**. *Suppose also that $h(\xi) \geq 0$ for all $\xi \in M$ and that $\mu\{\xi: h(\xi) > 0\} \leq \omega_n R^n(1 - \alpha)$ for some $\alpha > 0$.*

*Then there are constants* $c = c(n, p)$ *and* $\beta = \beta(n, \alpha) > 0$ *such that*

$$\left[\int_{B_{\beta R}} h^{np/(n-p)}\right]^{(n-p)/np} \le \frac{c}{\alpha}\left[\int_{B_R} h^p|H|^p + |\nabla^M h|^p\right]^{1/p}$$

*whenever* $1 \le p < n$.

REMARKS. (1) The hypothesis $\mu\{\xi: h(\xi) > 0\} \le \omega_n R^n(1-\alpha)$ for some $\alpha > 0$ is clearly necessary, as one sees by letting $V = \mathbf{v}(M, 1)$ where $M$ consists of two $n$-dimensional affine spaces passing through the origin, and setting $h = 1, 2$ respectively on the two spaces.

The necessity of taking the left integral in the theorem over $B_{\beta R}$, rather than over $B_R$, is clear if one considers a modification of the above example in which one of the affine spaces is displaced slightly from the origin.

(2) From Hölder's inequality one obtains under the same assumptions that

$$\left[\int_{B_{\beta R}} h^q\right]^{1/q} \le cR^{1+n/q-n/p}\left[\int_{B_R} h^p|H|^p + |\nabla^M h|^p\right]^{1/p}$$

in case $1 \le p < n$ and $1 \le q \le np/(n-p)$, or in case $p \ge n$ and $1 \le q < \infty$. In the first case $c = c(n, p)$ and in the second case $c = c(n, q)$.

*Proof of Theorem.* Our main goal is to prove the estimate (11). Without loss of generality assume $R = 1$.

Fix $s > 0$ and define

$$f(\xi) = \min\{h(\xi), s\}. \tag{1}$$

In the following suppose

$$0 < \beta < 1/2. \tag{2}$$

We will later further restrict $\beta$.

Applying the monotonicity formula to $f^p$, we have for each $\xi \in B_\beta$ that

$$\frac{\partial}{\partial \rho}\left[\rho^{-n}\int_{B_\rho(\xi)} f^p\right] \ge -\rho^{-n}\int_{B_\rho(\xi)}[f^p|H| + |\nabla^M f^p|], \tag{3}$$

(in the distributional sense in $r$) provided $0 < \rho < 1-\beta$. (See [S;

18.1, p. 89], where this result is stated for $C^1$ functions. The extension to the Lipschitz case follows by first extending $f$ to a Lipschitz function $\underline{f}$ on $\mathbf{R}^{n+k}$, then mollifying in $\mathbf{R}^{n+k}$, recalling that up to a set of $H^n$ measure zero $M$ is a disjoint union of sets $M_i$, each of which is a subset of a $C^1$ manifold $N_i$, and finally showing that for each $i$ the *integrals* on each side of (3) (over $M_i \cap B_\rho(\xi)$ instead of $M \cap B_\rho(\xi)$) are the limit of corresponding integrals with $f$ replaced by the mollified function $\underline{f}_\varepsilon$. This last step makes essential use of the fact that $\nabla^M$ is a *tangential* derivative.)

For $\mu$ a.e. $\xi$ with $|\xi| < \beta$ and $h(\xi) \geq s$, we see from (2) that

$$
\begin{aligned}
(4) \qquad s^p = f^p(\xi) &\leq \sup_{0<\sigma<1-\beta} \omega_n^{-1}\sigma^{-n} \int_{B_\sigma(\xi)} f^p \\
&\leq \omega_n^{-1}(1-\beta)^{-n} \int_{B_{1-\beta}(\xi)} f^p \\
&\quad + c \int_0^{1-\beta} \tau^{-n} \int_{B_\tau(\xi)} [f^p|H| + |\nabla^M f^p|] \\
&\leq \omega_n^{-1}(1-\beta)^{-n}\omega_n(1-\alpha)s^p \\
&\quad + c \int_0^{1-\beta} \tau^{-n} \int_{B_\tau(\xi)} [f^p|H| + |\nabla^M f^p|] \\
&\leq (1-\alpha/2)s^p + c \int_0^{1-\beta} \tau^{-n} \int_{B_\tau(\xi)} [f^p|H| + |\nabla^M f^p|],
\end{aligned}
$$

for suitable $\beta = \beta(n, \alpha)$, which we now fix.

It follows

$$
\begin{aligned}
&\sup_{0<\sigma<1-\beta} \omega_n^{-1}\sigma^{-n} \int_{B_\sigma(\xi)} f^p \\
&\quad \leq \frac{c}{\alpha} \int_0^{1-\beta} \tau^{-n} \int_{B_\tau(\xi)} [f^p|H| + |\nabla^M f^p|] \\
&\quad \leq \frac{c}{\alpha} \int_0^{1-\beta} \tau^{-n} \int_{B_\tau(\xi)} f^{p-1}[f|H| + |\nabla^M f|] \\
&\quad \leq \frac{c}{\alpha} \left[ \sup_{0<\sigma<1-\beta} \sigma^{-n} \int_{B_\sigma(\xi)} f^p \right]^{1-1/p} \\
&\quad\quad \times \int_0^{1-\beta} \left[ \tau^{-n} \int_{B_\tau(\xi)} f^p|H|^p + |\nabla^M f|^p \right]^{1/p}.
\end{aligned}
$$

Thus for any $0 < \sigma < 1 - \beta$,

$$(5)\quad \left[\sup_{0<\sigma<1-\beta} \omega_n^{-1}\sigma^{-n}\int_{B_\sigma(\xi)} f^p\right]^{1/p}$$
$$\le \frac{c}{\alpha}\int_0^{1-\beta}\left[\tau^{-n}\int_{B_\tau(\xi)} f^p|H|^p + |\nabla^M f|^p\right]^{1/p}$$
$$\le \frac{c}{\alpha}\int_0^{\rho_0}\left[\tau^{-n}\int_{B_\tau(\xi)} f^p|H|^p + |\nabla^M f|^p\right]^{1/p}$$
$$+ \frac{c}{\alpha}\int_{\rho_0}^{1-\beta}\left[\tau^{-n}\int_{B_\tau(\xi)} f^p|H|^p + |\nabla^M f|^p\right]^{1/p}$$
$$\le \frac{c}{\alpha}\int_0^{\rho_0}\left[\tau^{-n}\int_{B_\tau(\xi)} f^p|H|^p + |\nabla^M f|^p\right]^{1/p} + \frac{c_1\Gamma}{\alpha}\rho_0^{1-n/p},$$

where we set

$$(6)\qquad \Gamma = \left[\int_{B_1(0)} f^p|H|^p + |\nabla^M f|^p\right]^{1/p}.$$

Now choose $s_0$ so that

$$(7)\qquad \frac{c_1\Gamma}{\alpha}\left(\frac{1}{10}\right)^{1-n/p} = \frac{1}{2}s_0.$$

For each $s \ge s_0$ choose $\rho_0 = \rho_0(s)$ such that

$$(8)\qquad \frac{c_1\Gamma}{\alpha}(\rho_0^{1-n/p}) = \frac{1}{2}s,$$

i.e.

$$(9)\qquad \rho_0 = c_2\left(\frac{\Gamma}{\alpha s}\right)^{p/(n-p)}.$$

Note that

$$(10)\qquad \rho_0 \le \frac{1}{10}.$$

From (5), (8), (10), (2), (4) we have for $s \ge s_0$ and $\rho_0$ as in (9), that

$$\left[\sup_{0<\sigma<1-\beta} \omega_n^{-1}\sigma^{-n}\int_{B_\sigma(\xi)} f^p\right]^{1/p}$$
$$\le \frac{c}{\alpha}\int_0^{\rho_0}\left[\tau^{-n}\int_{B_\tau(\xi)} f^p|H|^p + |\nabla^M f|^p\right]^{1/p}.$$

Hence

$$\left[\sup_{0<\sigma<(1-\beta)/5} \sigma^{-n} \int_{B_{5\sigma}(\xi)} f^p\right]^{1/p} \leq \frac{c}{\alpha}\rho_0 \left[\tau^{-n} \int_{B_\tau(\xi)} f^p|H|^p + \nabla^M f|^p\right]^{1/p}$$

for some $0 < \tau = \tau(\xi) < \rho_0$.

Since $\rho_0 \leq 1/10 < (1-\beta)/5$ from (10) and (2), it follows from (9) that for this particular $\tau = \tau(\xi) < \rho_0$ we have

$$\int_{B_{5\tau}(\xi)} f^p \leq \frac{c}{\alpha^p}\rho_0^p \int_{B_\tau(\xi)} f^p|H|^p + |\nabla^M f|^p,$$

where $\rho_0$ is as in (9).

Since this is true for $\mu$ a.e. $\xi \in B_\beta \cap \{h \geq s\}$, it follows from (10), (2) and a standard covering argument (see [S: Theorem 3.3, p. 11]) that

$$\int_{B_\beta \cap \{h \geq s\}} f^p \leq \frac{c}{\alpha^p}\rho_0^p \int_{B_1} f^p|H|^p + |\nabla^M f|^p,$$

and so for any $s \geq s_0$ we have (using (9)) that

$$\mu(B_\beta \cap \{h \geq s\}) \leq c\left(\frac{\Gamma\rho_0}{\alpha s}\right)^p \leq c\left(\frac{\Gamma}{\alpha s}\right)^{np/(n-p)}. \tag{11}$$

(Since $\mu(B_\rho \cap \{h > 0\}) < \omega_n$, this last inequality is true for *all* $s > 0$.)

It follows from (11) and the fact $\mu(B_\beta \cap \{h \geq 0\}) \leq \omega_n$ that

$$\begin{aligned} \int_{B_\beta} h^p &= p\int_0^\infty s^{p-1}\mu(B_\beta \cap \{h \geq s\}) \\ &= p\int_0^{\Gamma/\alpha} s^{p-1}\mu(B_\beta \cap \{h \geq s\}) \\ &\quad + p\int_{\Gamma/\alpha}^\infty s^{p-1}\mu(B_\beta \cap \{h \geq s\}) \\ &\leq c\left(\frac{\Gamma}{\alpha}\right)^p + c\int_{\Gamma/\alpha}^\infty s^{p-1}\left(\frac{\Gamma}{\alpha s}\right)^{np/(n-p)} \\ &\leq c\left(\frac{\Gamma}{\alpha}\right)^p + c\left(\frac{\Gamma}{\alpha}\right)^p \int_1^\infty t^{p-1}t^{-np/(n-p)}\,dt \leq c\left(\frac{\Gamma}{\alpha}\right)^p. \end{aligned} \tag{12}$$

(*Remarks.* One can similarly estimate the integral of $h^q$ for any $1 \leq q < np/(n-p)$.)

Finally suppose $\varphi \in C_c^\infty(B_1)$, $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $B_{\beta/2}$, $\varphi \equiv 0$ on $B_1 \sim B_\beta$, and $|D\varphi| \leq c/\beta$. From the appropriate Sobolev inequality for functions with compact support (for example, see [S; Theorem 18.6, p. 93], replace $h$ there with $h^r$ where $r = p(n-1)/(n-p)$, and use Hölder's inequality) it follows

$$\left[\int_{B_1} (\varphi h)^{np/(n-p)}\right]^{(n-p)/n} \le c\int_{B_1} \varphi^p h^p |H|^p + |\nabla^M(\varphi h)|^p$$
$$\le \frac{c}{\alpha^p}\left[\int_{B_1} h^p|H|^p + |\nabla^M h|^p\right],$$

using (12). Hence

$$\left[\int_{B_{\beta/2}} h^{np/(n-p)}\right]^{(n-p)/np} \le \frac{c}{\alpha}\left[\int_{B_1} h^p|H|^p + |\nabla^M h|^p\right]^{1/p}.$$

This establishes the theorem. □

THEOREM 2. *Under the same hypotheses as Theorem* 1, *there exist* $\beta = \beta(n) > 0$, $\gamma_1 = \gamma_1(n) > 0$, *and* $\gamma_2 = \gamma_2(n)$, *such that*

$$\int_{B_{\beta R}} \left(\frac{\alpha h}{\Gamma}\right)^n \exp\left(\frac{\gamma_1 \alpha h}{\Gamma}\right) \le \gamma_2 R^n,$$

*where*

$$\Gamma = \left[\int_{B_R} h^n|H|^n + |\nabla^M h|^n\right]^{1/n}.$$

*Proof*. Choosing $R = 1$ and arguing exactly as in the proof of Theorem 1, with $p = n$, we obtain instead of (5) that

$$\text{(5)}'\qquad \left[\sup_{0<\sigma<1-\beta} \omega_n^{-1}\sigma^{-n}\int_{B_\sigma(\xi)} f^n\right]^{1/n}$$
$$\le \frac{c}{\alpha}\int_0^{\rho_0}\left[\tau^{-n}\int_{B_\tau(\xi)} f^n|H|^n + |\nabla^M f|^n\right]^{1/n}$$
$$+ \frac{\bar{c}_1\Gamma}{\alpha}\log(\rho_0^{-1}).$$

Choose $s_0$ so that

$$\text{(7)}'\qquad \frac{\bar{c}_1\Gamma}{\alpha}\log\left(\frac{1}{10}\right)^{-1} = \frac{1}{2}s_0.$$

For each $s \ge s_0$ choose $\rho_0 = \rho_0(s)$ such that

$$\text{(8)}'\qquad \frac{\bar{c}_1\Gamma}{\alpha}\log\rho_0^{-1} = \frac{1}{2}s,$$

i.e.

$$\rho_0 = \exp\left(-\frac{\overline{c}_2 \alpha s}{\Gamma}\right). \tag{9$'$}$$

Arguing again exactly as before, we obtain for any $s \geq s_0$ that

$$\mu(B_\rho \cap \{h \geq s\}) \leq c\left(\frac{\Gamma \rho_0}{\alpha s}\right)^n \leq c\left(\frac{\Gamma}{\alpha s}\right)^n \exp\left(-\frac{c_3 \alpha s}{\Gamma}\right). \tag{11$'$}$$

(This is then true for any $s > 0$ since $\mu(B_\beta \cap \{h \geq 0\}) < \omega_n$.)

By Fubini's theorem we see that if $\varphi(s)$ is a $C^1$ increasing function of $s$ for $s \geq 0$, and $\varphi(0) = 0$, then (since $h \geq 0$ on $B_\beta \cap M$)

$$\int_{B_\beta} \varphi(u) = \int_0^\infty \varphi'(s) \mu(B_\beta \cap \{h \geq s\})\, ds.$$

If we let

$$\varphi(s) = \left(\frac{\alpha s}{\Gamma}\right)^n \exp\left(\frac{\gamma_1 \alpha s}{\Gamma}\right),$$

where $\gamma_1$ is yet to be chosen, it follows from $(11)'$ and the fact $\mu(B_\beta \cap \{h \geq s\}) < \omega_n$ that

$$\begin{aligned}
&\int_{B_\beta} \left(\frac{\alpha h}{\Gamma}\right)^n \exp\left(\frac{\gamma_1 \alpha h}{\Gamma}\right) \\
&\quad \leq \omega_n \int_0^{\Gamma/\alpha} \left[\frac{\alpha}{\Gamma}\left(\frac{\alpha s}{\Gamma}\right)^{n-1} + \gamma_1 \left(\frac{\alpha s}{\Gamma}\right)^n\right] \exp\left(\frac{\gamma_1 \alpha s}{\Gamma}\right) \\
&\qquad + c \int_{T/\alpha}^\infty \left[\frac{\alpha}{\Gamma}\left(\frac{\alpha s}{\Gamma}\right)^{n-1} + \gamma_1 \frac{\alpha}{\Gamma}\left(\frac{\alpha s}{\Gamma}\right)^n\right] \\
&\qquad \times \exp\left(\frac{\gamma_1 \alpha s}{\Gamma}\right)\left(\frac{\Gamma}{\alpha s}\right)^n \exp\left(-\frac{c_3 \alpha s}{\Gamma}\right) \\
&\quad \leq \gamma_2, \quad \text{say},
\end{aligned}$$

where we choose $\gamma_1 = c_3/2$. □

**Theorem 3.** *Suppose* **(H)**. *Suppose* $\alpha > 0$ *and choose* $N$ *such that* $\mu(M) \leq N\omega_n(1 - \alpha)$.

*Choose any* $\lambda_1 < \cdots < \lambda_M$ *such that*

$$\begin{aligned}
\mu\{h < \lambda_1\} &\leq \omega_n - \alpha, \\
\mu\{\lambda_i < h < \lambda_{i+1}\} &\leq \omega_n - \alpha \qquad \textit{for } i = 1, \ldots, N, \\
\mu\{\lambda_M < h\} &\leq \omega_n - \alpha.
\end{aligned}$$

*This is clearly possible for some* $M \leq N - 1$.

*Then if* $1 \leq p < n$ *and* $p \leq q \leq np/(n-p)$, *there exist constants* $c = c(n, p)$ *and* $\beta = \beta(n, \alpha)$ *such that*

$$\left[\int_{B_{\beta R}} \left(\inf_i |h - \lambda_i|\right)^q\right]^{1/q} \leq \frac{c}{\alpha} R^{1+n/q-n/p} \left[\int_{B_R} \left[\left(\inf_i |h - \lambda_i|\right)^p |H|^p + |\nabla^M h|^p\right]\right]^{1/p}.$$

*The same result holds if* $p \geq n$ *and* $p \leq q < \infty$, *but with* $c = c(n, q)$.

REMARK. The necessity of allowing distinct values for the $\lambda_i$ is clear if one considers examples where $V = \mathbf{v}(M, 1)$, $M$ consists of distinct affine spaces, and $h$ takes a distinct constant value on each affine space.

*Proof of Theorem.* Let

$$\begin{aligned} I_0 &= (-\infty, \lambda_1], \\ I_1 &= [\lambda_i, \lambda_{i+1}] \qquad i = 1, \dots, M-1, \\ I_M &= [\lambda_M, \infty). \end{aligned}$$

Define

$$h_j(\xi) = \begin{cases} \inf_i |h(\xi) - \lambda_i|, & h(\xi) \in I_j, \\ 0, & h(\xi) \notin I_j. \end{cases}$$

Let

$$\underline{h}(\xi) = \inf_i |h(\xi) - \lambda_i| = \sum_j h_j(\xi).$$

Then for each $\xi \in M$ there exists at most one $j$ such that $h_j(\xi) \neq 0$. Moreover, each $h_j(\xi)$ is Lipschitz. Finally, for $H^n$ a.e. $\xi \in M \cap \{h \in I_j\}$ we have $\nabla^M h_j(\xi) = \nabla^M h(\xi)$, and so $\nabla^M \underline{h}(\xi) = \nabla^M h(\xi)$ for $H^n$ a.e. $\xi \in M$.

Taking $\beta$ as in Theorem 1, it follows that

$$\left[\int_{B_{\beta R}} \underline{h}^q\right]^{p/q} = \left[\int_{B_{\beta R}} \left(\sum_j h_j^p\right)^{q/p}\right]^{p/q} \leq \sum_j \left[\int_{B_{\beta R}} (h_j^p)^{q/p}\right]^{p/q}$$

(by Minkowski's inequality, using $q \geq p$)

$$\leq \sum_j \frac{c}{\alpha^p} R^{p+(np/q)-n} \left[ \int_{B_R} h_j^p |H|^p + |\nabla^M h_j|^p \right]$$

(by Theorem 1 and the remark following it)

$$= \frac{c}{\alpha^p} R^{p+(np/q)-n} \left[ \int_{B_R} \underline{h}^p |H|^p + |\nabla^M h|^p \right].$$

REMARK. The restriction $q \geq p$ is required in order that the constant $c$ not depend on $\mu(B_R)$.

THEOREM 4. *Suppose the same hypotheses hold as in the previous theorem.*

*Then there exist* $\beta = \beta(n) > 0$, $\gamma_1 = \gamma_1(n) > 0$, *and* $\gamma_2 = \gamma_2(n)$, *such that*

$$\int_{B_{\beta R}} \left(\frac{\alpha \underline{h}}{\underline{\Gamma}}\right)^n \exp\left(\frac{\gamma_1 \alpha \underline{h}}{\underline{\Gamma}}\right) d\mu \leq \gamma_2 R^n,$$

*where*

$$\underline{h}(\xi) = \inf_i |h(\xi) - \lambda_i|,$$

$$\underline{\Gamma} = \left[ \int_{B_R} \underline{h}^n |H|^n + |\nabla^M h|^n \right]^{1/n}.$$

*Proof*. Define $\lambda_i$ and $h_j$ as in the proof of the previous theorem. Then

$$\int_{B_{\beta R}} (\alpha h_j)^n \exp\left(\frac{\gamma_1 \alpha h_j}{\Gamma_j}\right) \leq \gamma_2 \Gamma_j^n,$$

where $\beta$, $\gamma_1$ and $\gamma_2$ are as in Theorem 2, and where

$$\Gamma_j = \left[ \int_{B_R} h_j^n |H|^n + |\nabla^M h_j|^n \right]^{1/n}.$$

Replacing $\Gamma_j$ by $\underline{\Gamma}$ on the left side (as $\Gamma_j \leq \underline{\Gamma}$), and then summing the inequality over $j$, we obtain the required result. □

## References

[A] W. K. Allard, *On the first variation of a varifold*, Annals of Math., **95** (1972), 417–492.

[H] J. E. Hutchinson, *Some regularity theory for curvature varifolds*, Proc. of the Centre for Mathematical Analysis, **12** (1987), 60–66.

[MS] J. H. Michael and L. M. Simon, *Sobolev and mean-value inequalities on generalized submanifolds of* $R^n$, Comm. Pure and Appl. Math., **26** (1973), 361–379.

[S] L. M. Simon, *Lectures on Geometric Measure Theory*, Proc. of the Centre for Mathematical Analysis, **3** (1983).

[Sc] R. M. Schoen, *Existence and regularity theorems for some geometric variational problems*, Ph.D. thesis, Stanford, 1977.

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