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COMPACTIFICATION OF $M_{\mathbb{P}_3}(0, 2)$ AND PONCELET PAIRS OF CONICS

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Let $M(0, 2)$ denote the quasi-projective variety of isomorphism classes of stable rank 2 vector bundles on $\mathbb{P}_3(C)$ with $c_1 = 0$ and $c_2 = 2$. In this paper we study a natural (irreducible) compactification of $M(0, 2)$ and describe explicitly the sheaves on $\mathbb{P}_3$ which occur in the closure of $M(0, 2)$ in the moduli space of semi-stable sheaves on $\mathbb{P}_3$ with $c_1 = 0$, $c_2 = 2$ and $c_3 = 0$.

Introduction. The space $M(0, 2)$ of stable rank 2 vector bundles on $\mathbb{P}_3$ with $c_1 = 0$, $c_2 = 2$ was investigated in detail by Hartshorne **[Ha2]**. (See also **[Au-Dou]**.) He proved that $M(0, 2)$ has the structure of a fibre space over the 9-dimensional variety R of reguli, the fibre being an open subset of a smooth quadric in $\mathbb{P}_5$. (A regulus is a smooth quadric in $\mathbb{P}_3$ with a distinguished system of generating lines.) If S is the smooth conic in the Grassmannian $\mathbb{G}$ of lines in $\mathbb{P}_3$ given by the generators of a regulus ρ, then the fibre over ρ consists of smooth conics C such that S and C are Poncelet related with S as the inner conic, i.e. a triangle can be inscribed in C which circumscribes S.

To obtain a natural compactification of $M(0, 2)$, we first compactify the fibres over R by taking all conics S, smooth or not, which are Poncelet related to S; the fibre over $\rho = S$ is then a smooth quadric in $\mathbb{P}_5$. We then take as the compactification of the space R of reguli the *Hilbert Scheme* $C(\mathbb{G})$ of all conics contained in the Grassmannian $\mathbb{G}$. The quadric bundle over R extends to a bundle over $C(\mathbb{G})$, namely the Poncelet quadric bundle associated to the tautological conic bundle over $C(\mathbb{G})$; it is constructed by considering also the space of conics which are Poncelet related to singular conics, such that the fibre of this quadric bundle is a pair of hyperplanes in $\mathbb{P}_5$ in the case of a pair of lines and a double hyperplane in $\mathbb{P}_5$ in the case of a double line. This Poncelet quadric bundle Q, which is a normal projective variety, is the compactification of $M(0, 2)$ we study.

The space Q essentially parametrises a family of semi-stable sheaves of rank 2 with $c_1 = c_3 = 0$, $c_2 = 2$. More precisely it is shown that Q is a G.I.T. quotient of a space X^{ss} by SL(2) and that

X^{ss} parametrises a *flat* family of *semi-stable* sheaves with $c_1 = c_3 = 0$, $c_2 = 2$ invariant under the action of SL(2) (see 8.1, 8.2). The smooth points of Q correspond exactly to stable sheaves. We describe in §§9, 10 explicitly the sheaves occurring in the family parametrised by X^{ss}.

Let $\overline{M(0, 2)}$ by the (schematic) closure of $M(0, 2)$ in the Maruyama scheme of semi-stable sheaves on $\mathbb{P}_3$ with $c_1 = 0$, $c_2 = 2$, $c_3 = 0$. We investigate the canonical morphism $Q \to \overline{M(0, 2)}$ defined by the family parametrised by X^{ss} and prove (Theorem 4.4) that the normalisation $M(0, 2)$ of $\overline{M(0, 2)}$ is isomorphic to the variety obtained by blowing down Q along the fibres of a $\mathbb{P}_1$-fibration (see 4.2) on a codimension 5 subvariety contained in the singular locus of Q. Moreover the canonical map $M(0, 2) \to \overline{M(0, 2)}$ is bijective and the smooth points of $M(0, 2)$ are precisely the stable sheaves.

We now briefly describe the contents of the different sections of the paper.

In §2 we mainly review the theory of $M(0, 2)$ from the point of view of monads, jumping lines and Poncelet conics. It is in particular shown that the set of second order jumping lines of a bundle $\mathscr{E} \in M(0, 2)$ is the conic $S^0 \subset \mathbb{G}$ "conjugate" to the conic S defined by the regulus associated with $\mathscr{E}$. This result will be generalized in 7.6 to the case of sheaves which are limits of elements in $M(0, 2)$.

We deal with the Hilbert scheme $C(\mathbb{G})$ of conics in $\mathbb{G}$ and the associated Poncelet quadric bundle $Q \to C(\mathbb{G})$ in §3. It is shown that $C(\mathbb{G})$ is smooth (3.8) and that Q is a normal variety (3.13). We determine the singularities of Q in terms of Poncelet pairs $(S, C^\vee)$ (3.12).

In §4 we define 4 irreducible Weil divisors Q_0, Q_α, Q_β, Q_e on Q and the complement M of the union of these divisors consists of Poncelet related pairs $(S, C^\vee)$ where S is a regular cut of $\mathbb{G}$ by a plane in $\mathbb{P}_5$ and $C^\vee$ is smooth (i.e. corresponds to $M(0, 2)$). Let $\mathrm{Sing}(Q)$ be the singular set of Q and let Q_{exc} be the elements of $\mathrm{Sing}(Q)$ lying over the space of double lines in $C(\mathbb{G})$. It is shown in 4.2 that Q_{exc} is fibred naturally into a $\mathbb{P}_1$-bundle, the fibres $\mathbb{P}_1$ being the spaces of double structures on a line contained in $\mathbb{G}$.

The main theorem comparing Q and $\overline{M(0, 2)}$ is stated in 4.4. Assuming certain results that are proved in the later sections, it is proved in 4.5 that Q can be blown down to a (normal) variety along the $\mathbb{P}_1$-fibration of Q_{exc} and that the canonical map $Q \to \overline{M(0, 2)}$ induces an isomorphism of this blown down variety onto the normalisation of $\overline{M(0, 2)}$.

A geometric invariant theoretic (G.I.T.) description of $C(\mathbb{G})$ is given in §5: $C(\mathbb{G}) = Y^{ss}//\mathrm{SL}(2)$ where Y^{ss} is the space of semistable points for a linearised action of $\mathrm{SL}(2)$ on a space Y. In this section we also give a criterion for a point of $\mathrm{Gr}(U \otimes W)$ to be stable (resp. semi-stable) for the action of $\mathrm{SL}(U)$, where U and W are finite dimensional spaces and Gr_q denotes the Grassmannian of q-dimensional subspaces, Prop. 5.1.1.

In §6 a similar G.I.T. parametrisation of $Q = X^{ss}//\mathrm{SL}(2)$ is given for the Poncelet bundle Q.

We construct in §7 a *flat* family $\{\mathscr{N}_y\}$ of sheaves (of rank 4 on $\mathbb{P}_3$) parametrised by $y \in Y^{ss}$. These will correspond to kernel sheaves in the monad description of sheaves which are limits of elements in $M(0, 2)$. The proof of the flatness of the family, which involves, among other things, the use of the Eagon-Northcott complex, is given in Proposition 7.1. If $y \in Y$ and S is the corresponding conic in $C(\mathbb{G})$, it is shown in 7.6 that the space of "second-order" jumping lines of $\mathscr{N}_y$ (defined as the support of the sheaf $R^1\mathscr{N}_y$ on $\mathbb{G}$) is the "conjugate" conic S^0. This result is of importance in the investigation of the map $Q \to \overline{M(0, 2)}$.

In §8 we construct a flat family $\{\mathscr{F}_x\}$, $x \in X^{ss}$, of rank 2 sheaves on $\mathbb{P}_3$ with $c_1 = c_3 = 0$, $c_2 = 2$ parametrised by X^{ss}. In fact a family of monads parametrised by X^{ss} is constructed; these monads are not necessarily self-dual as the sheaves are not self-dual. We calculate some cohomology groups of $\mathscr{F}_x$.

In the last two sections we give explicit descriptions of the sheaves $\mathscr{F}_x$, essentially in terms of the configuration in $\mathbb{P}_3$ defined by a Poncelet pair $(S, C^\vee)$. For instance sheaves in $Q_e \backslash Q_\alpha \cup Q_\beta \cup Q_0$ are given by suitable elementary transformations of a null-correlation bundle or of the trivial bundle of rank 2 (9.1). A detailed study of all these sheaves is carried out to prove their stability (resp. semi-stability).

0. Notation and conventions. All vector spaces and varieties will be over a fixed algebraically closed field k of characteristic 0.

$G_m V$ denotes the Grassmannian of m-dimensional subspaces of the vector space V, $\mathbb{P}_n = \mathbb{P}V = G_1 V$ the projective space, $\dim V = n + 1$.

The invertible sheaf of degree d on $\mathbb{P}V$ is $\mathscr{O}(d)$, s.t. $V^\vee = \Gamma(\mathbb{P}V, \mathscr{O}(1))$. For an $\mathscr{O}_{\mathbb{P}V}$-module $\mathscr{F}$ we use the abbreviations $\mathscr{F}(d) = \mathscr{F} \otimes \mathscr{O}(d)$ and $h^i\mathscr{F}(d)$ for the dimension of $H^i\mathscr{F}(d) = H^i(\mathbb{P}V, \mathscr{F}(d))$. The sheaf of the trivial vector bundle with fibre F is denoted by $F \otimes \mathscr{O}$.

0.1. The evaluation map $V^\vee \otimes \mathscr{O} \to \mathscr{O}(1)$ gives rise to the Koszul complex homomorphisms $\bigwedge^{p+1} V^\vee \otimes \mathscr{O}(-1) \to \bigwedge^p V^\vee \otimes \mathscr{O}$ defined as the composition $\bigwedge^{p+1} V^\vee \otimes \mathscr{O}(-1) \to \bigwedge^p V^\vee \otimes V^\vee \otimes \mathscr{O}(-1) \to \bigwedge^p V^\vee \otimes \mathscr{O}$. The image is identified with $\Omega^p(p)$, the sheaf of p-differentials in twist p. In particular $\Omega^n(n) = \bigwedge^{n+1} \otimes \mathscr{O}(-1)$ and $\Gamma\Omega^p(p+1) = \bigwedge^{p+1} V^\vee$. The Koszul homomorphism with respect to the fibres over $\langle x \rangle \in \mathbb{P}V$ is contraction with x, $\bigwedge^{p+1} V^\vee \otimes \langle x \rangle \to \bigwedge^p V^\vee$.

We frequently use isomorphisms $\bigwedge^{n-p} V \simeq \bigwedge^{p+1} V^\vee$ based on a fixed isomorphism $\bigwedge^{n+1} V \simeq \mathscr{k}$. Then the Koszul homomorphism for the fibres is $\wedge x$ (up to sign) and we have the commutative diagram

$$\begin{array}{ccccc}
\bigwedge^{p+1} \otimes \langle x \rangle & \longrightarrow & \Omega^p(p)(\langle x \rangle) & \subset & \bigwedge^p V^\vee \\
\wr\| & & \wr\| & & \wr\| \\
\bigwedge^{n-p} V \otimes \langle x \rangle & \longrightarrow & \bigwedge^{n-p} V \wedge x & \subset & \bigwedge^{n-p+1} V
\end{array}$$

Here $\mathscr{F}(\mathrm{pt})$ denotes the fibre $\mathscr{F}_{\mathrm{pt}}/\mathscr{m}_{\mathrm{pt}}\mathscr{F}_{\mathrm{pt}}$.

Using the Koszul complex it is standard to verify that there are natural isomorphisms

$$\bigwedge^k V \underset{\approx}{\longrightarrow} \mathrm{Hom}(\mathbb{P}V, \Omega^{p+k}(p+k), \Omega^p(p))$$

for any $k, p \geq 0$. The homomorphism corresponding to $a \in \bigwedge^k V$ is contraction on the fibres or wedging:

$$\bigwedge^{n-p-k} V \wedge x \underset{a\wedge}{\longrightarrow} \bigwedge^{n-p} V \wedge x$$

and it extends to the Koszul complex. Under these isomorphisms composition of homomorphisms corresponds to the wedge product up to signs. More generally, if E and F are vector spaces, we have canonical isomorphisms

$$\mathrm{Hom}\left(E, F \otimes \bigwedge^k V\right) \simeq \mathrm{Hom}(E \otimes \Omega^{p+k}(p+k), F \otimes \Omega^p(p))$$

for any $p, k \geq 0$. Given an operator of the left side the homomorphism of the sheaves is uniquely induced by the diagram

$$\begin{array}{ccc}
E \otimes \bigwedge^{n-p-k} V \otimes \mathscr{O}(-1) & \longrightarrow & F \otimes \bigwedge^{n-p} V \otimes \mathscr{O}(-1) \\
\downarrow & & \downarrow \\
E \otimes \Omega^{p+k}(p+k) & \longrightarrow & F \otimes \Omega^p(p)
\end{array}$$

where we use $\bigwedge^{n-p} V \simeq \bigwedge^{q+1} V^\vee$. Finally, if we choose bases of the vector spaces, a homomorphism

$$k^m \otimes \Omega^{p+k}(p+k) \to k^n \otimes \Omega^p(p)$$

is considered as an $m \times n$-matrix (a_{ij}) of elements $a_{ij} \in \bigwedge^k V$ in such a way that $k^m \to k^m \otimes \bigwedge^k V$ is described by $(c_1, \ldots, c_m) \to (c_1, \ldots, c_m) \cdot (a_{ij})$. It is sometimes convenient, to consider $k^m \otimes \bigwedge^k V^\vee \to k^n$ instead.

As a special case we mention:

0.2. LEMMA. *Let $B \subset k^m \otimes V$ and let $k^m \otimes \Omega^1(1) \xrightarrow{b} B^\vee \otimes \mathcal{O}$ be the homomorphism induced by $k^m \otimes V^\vee \to B^\vee$. Then b is an epimorphism iff $(k^m \otimes v) \cap B = 0$ for any $v \in V$.*

Proof. Consider B as a matrix $k^p \to k^m \otimes V$. Then b is an epimorphism iff $b^\vee$ is a subbundle, i.e. $k^p \otimes x \xrightarrow[B]{} k^m \otimes V \wedge x$ is injective for any $x \in V$. Since $\lambda \circ B \wedge x = 0$ is equivalent to $\lambda \circ B = c \otimes x$ for some $c \in k^m$, the lemma follows.

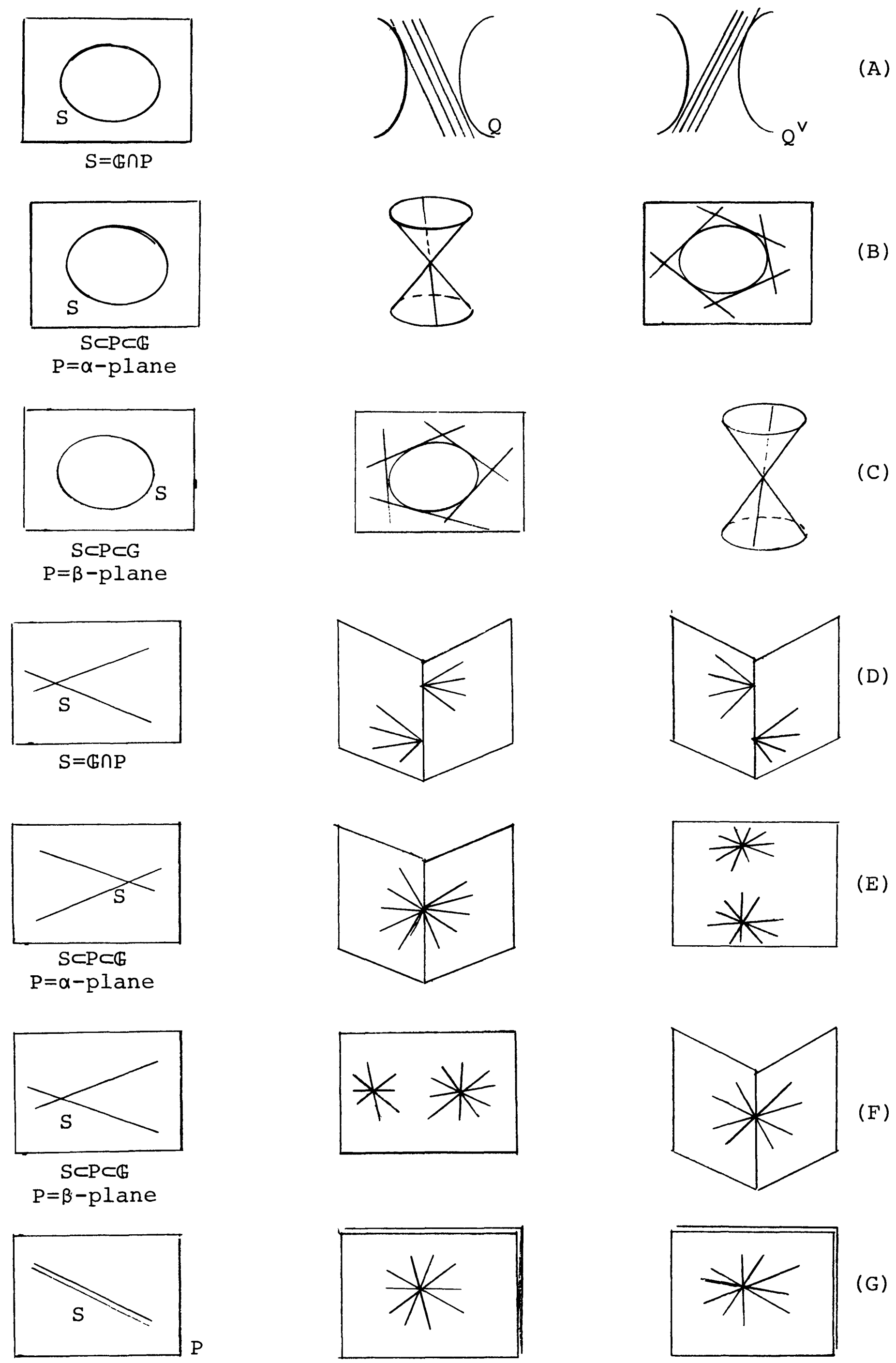
0.3. Incidence transformation. From now on $\dim V = 4$, $\mathbb{P}_3 = \mathbb{P}V$, and $\mathbb{G} = G_2 V \subset \mathbb{P}\bigwedge^2 V$. We consider the flag manifold $\mathbb{F} \subset \mathbb{P}_3 \times \mathbb{G}$ of pairs $(x, 1)$ with $x \in 1$ and let $\mathbb{P}_3 \xleftarrow[p]{} \mathbb{F} \xrightarrow[q]{} \mathbb{G}$ denote the projections, which is a $\mathbb{P}_2$ (resp. $\mathbb{P}_1$) bundle. Since p^* is exact, the functor $R^\cdot = R^\cdot q_* p_*$ is a cohomology functor. Some of the standard direct images are:

$$R^0\mathcal{O}_{\mathbb{P}_3} = \mathcal{O}_{\mathbb{G}}, \quad R^1\Omega^1 = \mathcal{O}_{\mathbb{G}}, \quad R^0\Omega^1(1) = Q^\vee,$$

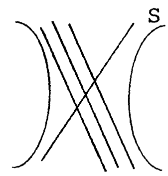
$$R^1\mathcal{O}_{\mathbb{P}_3}(-m-2) = S^m S \otimes \bigwedge^2 S = S^m S \otimes \mathcal{O}_{\mathbb{G}}(-1),$$

where S, Q denote the universal sub-, quotient bundles on $\mathbb{G}$, and S^m denotes the symmetric power, $m \geq 0$.

0.4. Conics in $\mathbb{G}$ and reguli. We denote by $C(\mathbb{G})$ the Hilbert scheme of conics in $\mathbb{G}$. This is a smooth variety of dimension 9, see 3.8. Each conic $S \subset \mathbb{G}$ defines a plane $P \subset \mathbb{P}\bigwedge^2 V$, such that $S \subset \mathbb{G} \cap P$. If P is not contained in $\mathbb{G}$ (as an α-plane, i.e. a plane consisting of all lines through a point in $\mathbb{P}_3$, or as a β-plane, i.e. a plane consisting of all lines in a plane in $\mathbb{P}_3$) then $S = \mathbb{G} \cap P$. The system of lines in $\mathbb{P}_3$ parametrised by a given conic $S \subset \mathbb{G}$ can be visualised as a "complete" regulus. This is a quadric $Q \subset \mathbb{P}_3$ with $pq^{-1}(S)$ as its underlying set together with the system of lines on it given by S. We give below a list of all types of complete reguli in $\mathbb{P}_3$, which arise in this way from conics in $\mathbb{G}$. The complete reguli obtained by the configuration of the dual lines in $\mathbb{P}_3^\vee$ are also given and denoted by $Q^\vee$.



Note that in the last case (G) the double regulus cannot remember the plane P of the conic S. In other words, while the reduced line S can be recovered from the configuration of this regulus in $\mathbb{P}_3$, the double structure on the line is not determined by it, see 3.9. If $S = G \cap P$ is a regular conic section the quadric Q spanned by the lines is regular and has two systems of lines. The conic S is isomorphic to any line of the second system.



Since $Q \simeq \mathbb{P}_1 \times \mathbb{P}_1$ we can identify S with the first factor, s.t.

$$\Gamma\mathscr{O}_S(3) = \Gamma\mathscr{O}_Q(3, 0).$$

The second factor parametrises a second (the conjugate) conic $S^0 \subset \mathbb{G}$ which is the intersection $S^0 = \mathbb{G} \cap P^0$, where P^0 is the plane orthogonal to P with respect to the quadratic form of $\mathbb{G}$.

We denote by $C^0(\mathbb{G})$ the open part of the Hilbert scheme of regular plane sections.

1. Conics and kernel bundles.

1.1. *Standard resolution of* $\mathscr{O}_Q(3, 0)$. Let $\mathscr{O}_Q(3, 0)$ be defined by the conic $S \in C^0(\mathbb{G})$. We can choose a basis $e_0, \dots, e_3 \in V$ with dual basis $z_0, \dots, z_3 \in V^\vee$, s.t. Q has the equation $z_0z_3 - z_1z_2 = 0$ and is the image of the standard Segre imbedding $z_0 = s_0t_0$, $z_1 = s_0t_1$, $z_2 = s_1t_0$, $z_3 = s_1t_1$. Then $\mathscr{O}_Q(3, 0)$ is generated by the liftings of s_0^3, $s_0^2s_1$, $s_0s_1^2$, $s_1^3 \in \Gamma\mathscr{O}_{\mathbb{P}_1}(3)$. It is then straightforward to verify that the sequence

$$0 \to k^2 \otimes \mathscr{O}(-2) \xrightarrow[B]{} k^6 \otimes \mathscr{O}(-1) \xrightarrow[A]{} k^4 \otimes \mathscr{O} \to \mathscr{O}_Q(3, 0) \to 0$$

with

$$B = \left[\begin{array}{ccc|ccc} -z_3 & z_1 & & z_2 & -z_0 & \\ & -z_3 & z_1 & & z_2 & -z_0 \end{array}\right], \qquad A = \left[\begin{array}{cccc} -z_2 & z_0 & & \\ & -z_2 & z_0 & \\ & & -z_2 & z_0 \\ \hline -z_3 & z_1 & & \\ & -z_3 & z_1 & \\ & & -z_3 & z_1 \end{array}\right],$$

is a resolution in $\mathbb{P}_3$.

We also have the exact sequence

$$0 \to k^2 \xrightarrow[N^*]{} k^2 \otimes V \xrightarrow[B]{} k^6 \to 0$$

where N^* is the matrix

$$N^* = \boxed{\begin{matrix} e_0 & e_2 \\ e_1 & e_3 \end{matrix}}\,.$$

REMARK. $\det N^* = e_0e_3 - e_1e_2$ is the equation of the dual quadric $Q^\vee \subset \mathbb{P}V^\vee$ as can be easily verified.

1.2. If $\mathscr{K}$ denotes the kernel of $B^\vee(-1)$ we obtain the exact diagram

$$\text{(1)}\quad \begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & \mathscr{K} \arrow[r] \arrow[d] & k^6 \otimes \mathscr{O} \arrow[r, "B^\vee"] \arrow[d, "B^\vee"] & k^2 \otimes \mathscr{O}(1) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & k^2 \otimes \Omega^1(1) \arrow[r] \arrow[d, "N^\vee"] & k^2 \otimes V^\vee \otimes \mathscr{O} \arrow[r] \arrow[d, "N^\vee"] & k^2 \otimes \mathscr{O}(1) \arrow[r] & 0 \\ & k^2 \otimes \mathscr{O} \arrow[r, equal] \arrow[d] & k^2 \otimes \mathscr{O} \arrow[d] & & \\ & 0 & 0 & & \end{tikzcd}$$

By 0.2 $N^\vee$ is an epimorphism and thus $\mathscr{K}$ is locally free. Of course by the resolution above we also have the exact sequence

$$\text{(2)}\qquad 0 \to \mathscr{K}^\vee(-1) \to \Gamma^\vee \otimes \mathscr{O} \to \mathscr{O}_Q(3, 0) \to 0,$$

where $\Gamma^\vee = \Gamma\mathscr{O}_Q(3, 0)$, and we obtain dually

$$\text{(2}^\vee\text{)}\qquad 0 \to \Gamma \otimes \Omega^3(4) \to \mathscr{K}(1) \to \mathscr{O}_Q(-1, 2) \to 0,$$

since $\mathrm{Ext}^1_{\mathscr{O}}(\mathscr{O}_Q(a, b), \mathscr{O}) = \mathscr{O}_Q(2-a, 2-b)$, which follows since the dualizing sheaf $\omega_Q = \mathscr{O}_Q(-2, -2)$. We are going to investigate the sections of $\mathscr{K}(1)$. By the first column of (1) we are given the diagram

$$\begin{tikzcd} 0 \arrow[r] & \Gamma\mathscr{K}(1) \arrow[r] \arrow[d, equal] & k^2 \otimes \Gamma\Omega^1(2) \arrow[r] \arrow[d, equal] & k^2 \otimes \Gamma\mathscr{O}(1) \arrow[r] \arrow[d, equal] & 0 \\ & \Gamma \otimes \Lambda^4 V^\vee \arrow[d, "\wr\|"] & k^2 \otimes \Lambda^2 V^\vee \arrow[d, "\wr\|"] & k^2 \otimes V^\vee \arrow[d, "\wr\|"] & \\ 0 \arrow[r] & \Gamma \arrow[r] & k^2 \otimes \Lambda^2 V \arrow[r, "\wedge N^\vee"'] & k^2 \otimes \Lambda^3 V \arrow[r] & 0 \end{tikzcd}$$

A direct calculation shows that $\dim \Gamma = 4$ and that Γ is presented by the matrix $\Gamma^* : k^4 \to k^2 \otimes \bigwedge^2 V$

$$\Gamma^* = \begin{bmatrix} \xi & 0 \\ \omega & \xi \\ \eta & \omega \\ 0 & \eta \end{bmatrix}$$

with $\xi = e_0 \wedge e_1$, $\omega = e_0 \wedge e_3 - e_1 \wedge e_2$, $\eta = e_2 \wedge e_3$. In particular $H^1 \mathscr{K}(1) = 0$.

1.3. LEMMA. (1) *The conic S is parametrised by $s^2\xi + st\omega + t^2\eta$.*

(2) *If the zero scheme $Z(\gamma)$ of a section $\gamma \in \Gamma \simeq \Gamma\mathscr{K}(1)$ is not empty, it is a line $l \in S$ and*

(3) $\qquad \gamma = (s, t) \otimes (s^2\xi + st\omega + t^2\eta) = (s^3, s^2t, st^2, t^3) \circ \Gamma^*.$

Proof. (1) is immediate from 1.1 by looking at the embedding of the first factor of $\mathbb{P}_1 \times \mathbb{P}_1$.

(2) If $\gamma \in \Gamma$ then γ vanishes in $\langle x \rangle$ iff $\gamma \wedge x = 0$, see 0.1. If $\gamma = (\alpha_0, \dots, \alpha_3) \circ \Gamma^*$ this means that

$$\alpha_0 \xi \wedge x + \alpha_1 \omega \wedge x + \alpha_2 \eta \wedge x = 0,$$
$$\alpha_1 \xi \wedge x + \alpha_2 \omega \wedge x + \alpha_3 \eta \wedge x = 0.$$

However by the definition of ξ, ω, η the vectors $\xi \wedge x$, $\eta \wedge x$ are linearly independent, and there is at most one relation of the vectors, i.e.

$$\operatorname{rank} \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} = 1.$$

But it is well known that then

$$(\alpha_0, \dots, \alpha_3) = (s^3, s^2t, st^2, t^3)$$

which proves (3), and thus $Z(\gamma) = s^2\xi + st\omega^t + t^2\eta \in S$.

1.3.1. COROLLARY. *The correspondence $\mathscr{K} \leftrightarrow S$ is* 1:1 *between the kernels $\mathscr{K}$ of regular N's (with four independent entries) and the regular conics $S \in C^0(\mathbb{G})$.*

1.3.2. COROLLARY. *If $\mathscr{K}$ is defined by a regular N, we have the exact sequence*

$$0 \to \Gamma(\mathscr{K}(1)) \otimes \mathscr{O} \to \mathscr{K}(1) \to \mathscr{O}_Q(-1, 2) \to 0.$$

Proof. Clearly $\mathscr{K}$ is defined by $S \in C^0(\mathbb{G})$. By the lemma conversely S is defined by $\mathscr{K}$. Given an arbitrary regular N and $\mathscr{K}$, a conic S is defined by $s^2\xi + st\omega + t^2\eta$ which gives $\mathscr{K}$. Then the corollary follows from $(2^\vee)$ or the lemma.

1.4. Remark. Let $G_2^0(k^2 \otimes V)$ denote the open set of the Grassmannian of all 2-dimensional subspaces $N \subset k^2 \otimes V$ which are presented by matrices with 4 independent vectors. The map $N \mapsto S$ is a morphism

$$G_2^0(k^2 \otimes V) \to C^0(\mathbb{G})$$

onto $C^0(\mathbb{G})$. It is invariant under the action of SL(2) given by $(g, N) \to (f \otimes \mathrm{id})(N)$, and thus factorizes into an isomorphism

$$G_2^0(k^2 \otimes V)/\,\mathrm{SL}(2) \simeq C^0(\mathbb{G}).$$

The transposition map $\left(\begin{smallmatrix} x & x' \\ y & y' \end{smallmatrix}\right) \to \left(\begin{smallmatrix} x' & y \\ x' & y' \end{smallmatrix}\right)$ induces the involution $S \to S^0$.

We finally state two further beautiful geometric properties of a bundle $\mathscr{K}$.

1.5. Proposition. *Let $\mathscr{K}$ be defined as above and let S resp. Q be the associated conic resp. quadric. Then*

(i) *the dual quadric $Q^\vee \subset \mathbb{P}_3^\vee$ is the set of jumping planes of $\mathscr{K}$, i.e. of all planes $P \subset \mathbb{P}_3$ with $h^0(\mathscr{K}|P) \neq 0$.*

(ii) $R^1\mathscr{K} = \mathscr{O}_{S^0}(1)$, *where R^1 is the first incidence transform,* 0.3.

Proof. (i) Let $H = \{f = 0\}$ with $f \in V^\vee$. There is a splitting of

$$0 \to f \otimes \mathscr{O}_H \to \Omega^1(1)|H \to \Omega^1_H(1) \to 0$$

which is induced from the Koszul-complex. Therefore we obtain the exact sequence

$$\begin{array}{ccccccc} 0 & \to & \Gamma(\mathscr{K}|H) & \to & k^2 \otimes \Gamma(\Omega^1(1)|H) & \to & k^2 \otimes \Gamma\mathscr{O}_H \\ & & & & \wr\| & & \wr\| \\ & & & & k^2 \otimes f & \xrightarrow[N^{*\vee}]{} & k^2 \end{array}$$

Hence $h^0(\mathscr{K}|H) \neq 0$ iff $\det = f(e_0)f(e_3) - f(e_1)f(e_2) = 0$, i.e. iff $f \in Q^\vee$.

(ii) If we apply R^1 to $(2^\vee)$ we obtain $R^1\mathscr{K} = R^1\mathscr{O}_Q(-2, 1)$. Since S is the first factor, we find $H^1\mathscr{O}_Q(-2, 1)\otimes\mathscr{O}_l = 0$ except for $l \in S^0$. Now $R^1\mathscr{O}_Q(-2, 1)$ being supported on S^0, we can obtain it as the simple direct image under $Q \to S^0$, which is the second projection. Therefore $R^1\mathscr{O}_Q(-2, 1) = \mathscr{O}_{S^0}(1)$.

2. Review of $M(0, 2)$. The bundles $\mathscr{E} \in M(0, 2)$ can be constructed in two different ways: from a linear system on a conic $S \subset \mathbb{G}$ as mentioned in the introduction and from monads, see **[Ha2]**. We summarize both in the following

2.1. THEOREM. *A rank-2 bundle $\mathscr{E}$ on $\mathbb{P}_3$ belongs to $M(0, 2)$ if and only if it is a member of one of the following exact diagrams (displays). These can be derived from each other.*

$$(\mathrm{D}_1)\qquad \begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathscr{E}^\vee(-1) \arrow[r] \arrow[d] & L\otimes\mathscr{O} \arrow[r] \arrow[d] & \mathscr{O}_Q(3, 0) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{K}^\vee(-1) \arrow[r] \arrow[d] & \Gamma^\vee\otimes\mathscr{O} \arrow[r] \arrow[d] & \mathscr{O}_Q(3, 0) \arrow[r] & 0 \\
 & M^\vee\otimes\mathscr{O} \arrow[r, equal] \arrow[d] & M^\vee\otimes\mathscr{O} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

$$(\mathrm{D}_1^\vee)\qquad \begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & L^\vee\otimes\Omega^3(4) \arrow[r] \arrow[u] & \mathscr{E}(1) \arrow[r] \arrow[u] & \mathscr{O}_Q(-1, 2) \arrow[r] & 0 \\
0 \arrow[r] & \Gamma\otimes\Omega^3(4) \arrow[r] \arrow[u] & \mathscr{K}(1) \arrow[r] \arrow[u] & \mathscr{O}_Q(-1, 2) \arrow[r] \arrow[u, equal] & 0 \\
 & M\otimes\Omega^3(4) \arrow[r, equal] \arrow[u] & M\otimes\Omega^3(4) \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

$$
(\mathrm{D}_2)\qquad
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M\otimes\Omega^3(3) \arrow[r] \arrow[d, equal] & \mathscr{K} \arrow[r] \arrow[d] & \mathscr{E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M\otimes\Omega^3(3) \arrow[r] & \mathscr{k}^2\otimes\Omega^1(1) \arrow[r] \arrow[d] & \mathscr{M} \arrow[r] \arrow[d] & 0 \\
 & & N^\vee\otimes\mathscr{O} \arrow[r, equal] \arrow[d] & N^\vee\otimes\mathscr{O} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$

Explanation of (D_1). Q is the regulus of a regular conic section $S\subset\mathbb{G}$, s.t. $\Gamma\mathscr{O}_S(3)=\Gamma\mathscr{O}_Q(3,0)=\Gamma^\vee$ and $L\subset\Gamma^\vee$ is a 2-dimensional subspace without base points. $L\otimes\mathscr{O}\to\mathscr{O}_Q(3,0)$ and $\Gamma^\vee\otimes\mathscr{O}\to\mathscr{O}_Q(3,0)$ are the induced epimorphisms, see 1.2.

Explanation of $(\mathrm{D}_1^\vee)$. This is obtained by applying $\mathscr{H}om(\cdot,\mathscr{O})$ to (D_1), where we use $\mathscr{O}^\vee=\Omega^3(4)$ by formal reasons and

$$\mathscr{E}xt^1_{\mathscr{O}}(\mathscr{O}_Q(a,b),\mathscr{O})\simeq\mathscr{O}_Q(2-a,2-b).$$

The latter follows by using the dualizing sheaf $\omega_Q=\mathscr{O}_Q(-2,-2)$.

Explanation of (D_2). $M\subset\mathscr{k}^2\otimes\bigwedge^2V$ and $N\subset\mathscr{k}^2\otimes V$ are 2-dimensional subspaces such that M is contained in the kernel of the composed operator $\mathscr{k}^2\otimes\bigwedge^2V\to N^\vee\otimes V\otimes\bigwedge^2V\to N^\vee\otimes\bigwedge^3V$. By 0.1 we obtain a complex

$$M\otimes\Omega^3(3)\underset{\mu}{\longrightarrow}\mathscr{k}^2\otimes\Omega^1(1)\underset{\nu}{\longrightarrow}N^\vee\otimes\mathscr{O},$$

and we suppose that μ is a subbundle and ν an epimorphism. Such a complex is called *monad* and the display (D_2) is called the display of the monad.

Proof. (1) If $\mathscr{E}$ is defined by (D_1), it must be a rank-2 bundle since $\mathscr{K}^\vee(-1)$ is locally free by 1.2 and $\mathscr{E}=\mathscr{E}^\vee(-1)^\vee(-1)$. Furthermore its Chern classes must be $c_1=0$, $c_2=2$, and $h^0\mathscr{E}=0$ since $h^0\mathscr{K}=0$. Hence it is stable and a member of $M(0,2)$. The same can be proved if $\mathscr{E}$ is defined by (D_2).

(2) It was shown in **[Ha2]** that $h^1\mathscr{E}(-2) = 0$ for any $\mathscr{E} \in M(0, 2)$. Then the Beilinson spectral sequence, see **[OSS]**, of $\mathscr{E}$ degenerates, and its E_2 level yields the monad in (D_2):

$$H^2(\mathscr{E}(-3)) \otimes \Omega^3(3) \to H^1(\mathscr{E}(-1)) \otimes \Omega^1(1) \to H^1(\mathscr{E}) \otimes \mathscr{O}$$

so that $M = H^2\mathscr{E}(-3)$, $k^2 = H^1\mathscr{E}(-1)$, $H^1\mathscr{E} = N^\vee$. Then automatically μ and ν are sub- resp. quotient bundles.

(3) Clearly the displays (D_1) and $(\mathrm{D}_1^\vee)$ are dual to each other. If (D_1) is given we get (D_2) from the results of 1.2, for there it was shown that $\mathscr{K}$ is a kernel as in the column in the middle of (D_2). We also can derive $(\mathrm{D}_1^\vee)$ from (D_2) as follows. By 0.2, ν is an epimorphism iff $(k^2 \otimes v) \cap N = 0$ for any $v \in V$. Let now $k^2 \otimes V^\vee \to N^\vee$ be given by the matrix $N^*\colon k^2 \to k^2 \otimes V$. It is elementary to derive that the condition for N is satisfied iff N^* is one of the matrices

$$\begin{bmatrix} e_0 & e_1 \\ e_2 & e_3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} e_0 & e_1 \\ e_2 & e_0 \end{bmatrix}$$

where $e_0, \dots, e_3 \in V$ is a basis. In the first case $\mathscr{K}$ is an extension as in $(\mathrm{D}_1^\vee)$ by 1.2, and hence $(\mathrm{D}_1^\vee)$ follows from $(\mathrm{D}_2^\vee)$. We show now that the second case cannot occur: As in 1.2 we find that the kernel Γ of $k^2 \otimes \bigwedge^2 V \xrightarrow[\wedge N^*]{} k^2 \otimes \bigwedge^3 V$ is generated by the matrix $(e_{ij} = e_i \wedge e_j)$

$$\Gamma^* = \begin{bmatrix} e_{01} & 0 \\ e_{21} & e_{01} \\ e_{20} & e_{21} \\ 0 & e_{20} \end{bmatrix}.$$

Since $M \subset \Gamma$, the matrix $M^*\colon k^2 \to k^2 \otimes \bigwedge^2 V$ representing M must be a product $M^* = A \circ \Gamma^*$ with a usual 2×4 matrix A. It follows that the entries a_{ij} of M^* are contained in the span of $e_0 \wedge e_1$, $e_0 \wedge e_2$, $e_1 \wedge e_2$. Therefore for any $z \in V$

$$a_{ij} \wedge z = \alpha_{ij}(z) e_0 \wedge e_1 \wedge e_2 + z_3 \tilde{a}_{ij}$$

where α_{ij} are linear in the coordinates z_0, z_1, z_2 only and $\tilde{a}_{ij} \in \bigwedge^3 V$. Hence, if $z_3 = 0$,

$$M^* \wedge z = (a_{ij} \wedge z) = (\alpha_{ij}(z) e_0 \wedge e_1 \wedge e_2)$$

and we see that this matrix is degenerate on the conic $z_0 = 0$, $\det(\alpha_{ij}(z)) = 0$. This shows that $k^2 \otimes \Omega^3(3) \xrightarrow[M^*]{} k^2 \otimes \Omega^1(1)$ is not a subbundle along this conic.

2.1.1. REMARK. Assume that ν is regular in (D_2) and μ injective but not necessarily a subbundle. Then $L = (\Gamma/M)^\vee \subset \Gamma^\vee$ is base point free iff μ is a subbundle.

Proof. We still obtain diagram $(D_1^\vee)$, but in (D_1) there might occur the cokernel $\mathrm{Ext}^1(\mathcal{E}(1), \mathcal{O})$ in the top row and left column. Now both conditions are satisfied iff $\mathrm{Ext}^1(\mathcal{E}(1), \mathcal{O}) = 0$.

2.1.2. REMARK. If ν is regular in D_2, i.e. coming from a conic $S \in C^0(G)$, and μ injective, the sheaf $\mathcal{E}$ is still stable, of rank 2, $h^0\mathcal{E} = 0$, $c_1 = 0$, $c_2 = 2$, $c_3 = 0$. These sheaves occur as kernels in sequences

$$0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{O}_l(1) \to 0$$

where $\mathcal{E}' \in \overline{M(0,1)}$ and $l \subset \mathbb{P}_3$ is a line, see 9.1.

2.1.3. REMARK. The monad in (D_2) is determined by $\mathcal{E}$ up to equivalence. This means that if $\mathcal{E}$ and $\mathcal{E}'$ are given by (M, N) and (M', N') then $\mathcal{E} \simeq \mathcal{E}'$ iff there exists $g \in \mathrm{GL}(2, k)$ s.t.

$$\begin{array}{ccc} M \subset k^2 \otimes \bigwedge^2 V & & k^2 \otimes V \supset N \\ \downarrow \qquad \Big\downarrow{\scriptstyle g\otimes \mathrm{id}} & \text{and} & \Big\uparrow{\scriptstyle g^\vee \otimes \mathrm{id}} \qquad \uparrow \\ M' \subset k^2 \otimes \bigwedge^2 V & & k^2 \otimes V \supset N' \end{array}$$

The proof follows easily from the identifications of M, k^2, $N^\vee$ with $H^2\mathcal{E}(-3)$, $H^1\mathcal{E}(-1)$, $H^1\mathcal{E}$ respectively.

2.2. *Sections of* $\mathcal{E}(1)$. If $\lambda \in L^\vee \simeq \Gamma\mathcal{E}(1)$ is a section of $\mathcal{E}(1)$ we obtain the exact diagram

$$\begin{tikzcd}
 & 0 & 0 & \\
\mathcal{E}^\vee(-1) \arrow[r] & \langle\lambda\rangle^\vee \otimes \mathcal{O} \arrow[r] \arrow[u] & \mathcal{O}_Z \arrow[r] \arrow[u] & 0 \\
\mathcal{E}^\vee(-1) \arrow[r] \arrow[u, equal] & L \otimes \mathcal{O} \arrow[r] \arrow[u] & \mathcal{O}_Q(3,0) \arrow[r] \arrow[u] & 0 \\
 & \langle f\rangle \otimes \mathcal{O} \arrow[u] \arrow[r, phantom, "\simeq"] & \mathcal{O} \arrow[u] & \\
 & 0 \arrow[u] & &
\end{tikzcd}$$

where Z denotes the zero scheme of the section, and f spans the kernel. Then f as an element of $\Gamma^\vee = \Gamma\mathscr{O}_S(3)$ has three zeros on S with the sequence

$$0 \to \mathscr{O}_S \to \mathscr{O}_S(3) \to \mathscr{O}_y \to 0.$$

If $Q \xrightarrow[\pi]{} S$ is the projection to the first factor, π^* yields the right-hand column and thus $Z = \pi^{-1}(y)$ consists of three lines of the system S.

Note that we have an isomorphism $\mathbb{P}L^\vee \simeq \mathbb{P}L$ since L is 2-dimensional.

2.2.1. LEMMA. *Let $\gamma \in \Gamma \simeq \Gamma\mathscr{H}(1)$ and $\lambda \in L^\vee \simeq \Gamma\mathscr{E}(1)$ be two sections with $\varnothing \neq Z(\gamma) \subset Z(\lambda)$. Then γ maps into $\langle\lambda\rangle \subset L^\vee$ under $\Gamma \to L^\vee$.*

Proof. Let λ' be the image of γ. Obviously also $Z(\gamma) \subset Z(\lambda')$. If f', f are the polynomials in $L \subset \Gamma^\vee = \Gamma\mathscr{O}_S(3)$ corresponding to λ', λ they must have a common zero. If λ', λ were independent, also f', f would be independent, contradicting the assumption that $L^\vee$ is base point free.

2.3. *Jumping lines of $\mathscr{E}$*. Let (M, N) be a monad defining a bundle $\mathscr{E} \in M(0, 2)$. Using the isomorphism $\bigwedge^2 V \simeq \bigwedge^2 V^\vee$ we can consider a representing matrix M^* of M as a matrix of linear forms on $\bigwedge^2 V$. If $\xi \in \bigwedge^2 V$ we write

$$M^*(\xi)\colon k^2 \xrightarrow[*]{} k^2 \otimes \bigwedge^2 V \xrightarrow[\wedge\xi]{} k^2 \otimes \bigwedge^4 V \simeq k^2$$

or $M^*(\xi) = M^* \wedge \xi$. The equation $\det M^*(\xi) = 0$ is then uniquely determined by $\mathscr{E}$ up to a scalar. If we apply the incidence transformation R^1 to the monad (D_2) we obtain $R^1\mathscr{E}(-1) = R^1\mathscr{M}(-1)$ and

$$0 \to k^2 \otimes \mathscr{O}_{\mathbb{G}}(-1) \xrightarrow[R^1\mu]{} k^2 \otimes \mathscr{O}_{\mathbb{G}} \to R^1\mathscr{E}(-1) \to 0$$

such that $R^1\mu$ is induced by the matrix $M^*(\xi)$. Similarly if we apply $\otimes\mathscr{O}_l(-1)$ to the monad we get

$$\begin{tikzcd} k^2 \otimes H^1\Omega^3(2) \otimes \mathscr{O}_l \arrow[r] \arrow[d, equal] & k^2 \otimes H^1\Omega^1 \otimes \mathscr{O}_l \arrow[r] \arrow[d, equal] & H^1\mathscr{M}|_l(-1) \arrow[r] & 0 \\ k^2 \arrow[r, "M^*(l)"'] & k^2 & & \end{tikzcd}$$

and we obtain, that for a line l

(4) $$\mathscr{E}|_l \simeq \mathscr{O}_l(-i) \otimes \mathscr{O}_l(i) \quad \text{iff} \quad \operatorname{rk} M^*(l) = 2 - i.$$

As a consequence, if $\tilde{J} = \{\det M^* = 0\}$ is the hypersurface in $\mathbb{P}\bigwedge^2 V$, the hypersurface

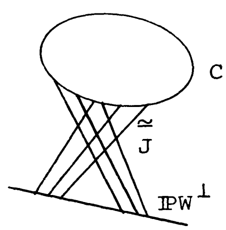
$$J = \tilde{J} \cap \mathbb{G} = \operatorname{Supp} R^1\mathscr{E}(-1)$$

is the set of jumping lines of $\mathscr{E}$. If $W^\perp \subset \bigwedge^2 V$ is the orthogonal of $W = \langle \xi, \omega, \eta \rangle$, we have $\xi|W^\perp = \omega|W^\perp = \eta|W^\perp = 0$ and thus $M^* = 0$ on $\mathbb{P}W^\perp$. Since $S^0 = \mathbb{G} \cap PW^\perp$ we find that J is singular along S^0. We even have

$$S^0 = \operatorname{Sing} J,$$

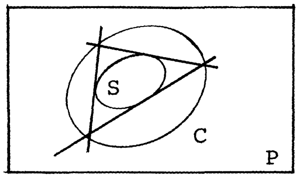
since $M^* \neq 0$ away from $W^\perp$. By (4) this is the set of jumping lines of order 2.

Let finally $C = \tilde{J} \cap \mathbb{P}W$ be the conic cut out by $\mathbb{P}W$. Since $\mathbb{P}W^\perp \cap \mathbb{P}W \neq \emptyset$ and $\mathbb{P}W^\perp$ is exactly the singular locus of $\det M^*$, $\tilde{J}$ is the cone over C with vertex $\mathbb{P}W^\perp$. Note that C must be smooth, since otherwise M^* could be given the form $\left(\begin{smallmatrix} \alpha & 0 \\ \beta & \gamma \end{smallmatrix}\right)$ with $\alpha \in \mathbb{G}$, and then μ would be degenerate in α.



2.4. *The associated Poncelet pair.* Let $\mathscr{E} \in M(0, 2)$ and let S, C be the conics associated with $\mathscr{E}$, see 2.2, 2.3. These are conics in the same plane $P \subset \mathbb{P}\bigwedge^2 V$ with $S = P \cap \mathbb{G}$, $C = P \cap \tilde{J}$.

2.4.1. PROPOSITION (*Hartshorne* **[Ha2]**). *The conic C is Poncelet related to S with respect to the pencil $\mathbb{P}L \subset |\mathscr{O}_S(3)|$, i.e. the tangents to S in the points of any divisor of the pencil meet on C.*



Proof. $L \subset \Gamma^\vee = \Gamma\mathscr{O}_S(3)$ and let $\Gamma \subset k^2 \otimes \bigwedge^2 V$ be presented by a matrix $\Gamma^*\colon k^4 \to k^2 \otimes \bigwedge^2 V$ as in 1.2, see also display (D_1). Then S is the conic parametrised by $s^2\xi + st\omega + t^2\eta$, and $P = \mathbb{P}W$, $W = \langle \xi, \omega, \eta \rangle$. By Lemma 1.3 a section $\gamma \in \Gamma \simeq \Gamma\mathscr{K}(1)$ with the zero line $l = s^2\xi + st\omega + t^2\eta$ is given by

$$\gamma = (s^3, s^2t, st^2, t^3) \circ \Gamma^* = (s, t) \otimes l.$$

Let now $\lambda \in L^\vee$ and $f \in L \subset \Gamma^\vee$ the corresponding polynomial having $Z(\lambda)$ as its zeros, 2.2. If

$$l_i = s_i^2\xi + s_it_i\omega + t_i^2\eta$$

are two zeros of f, $i = 0, 1$, let $\gamma_i = (s_i, t_i) \otimes l_i$ be the corresponding sections of $\mathscr{K}(1)$. By Lemma 2.2.1 both γ_0 and γ_1 map into $\langle \lambda \rangle \subset L^\vee$ under $\Gamma \simeq \Gamma\mathscr{K}(1) \to \Gamma\mathscr{E}(1) \simeq L^\vee$. Therefore M, γ_1, γ_2 are in a 3-dimensional subspace of Γ. If M is given by $M^* = A \circ \Gamma^*\colon k^2 \xrightarrow[A]{} k^4 \xrightarrow[\Gamma^*]{} k^2 \otimes \bigwedge^2 V$ there must be a nontrivial relation

$$bA + a_0(s_0^3, s_0^2t_0, s_0t_0^2, t_0^3) + a_1(s_1^3, \dots, t_1^3) = 0$$

(with $b \neq 0$ since the two rows are independent).

Now it is elementary to verify that the equations of the tangents to S in l_i in the plane $\mathbb{P}W$ are

$$s_i^2\xi(p) + s_it_i\omega(p) + t_i^2\eta(p) = 0.$$

If $\langle p \rangle \in \mathbb{P}W$ is the intersection point, we therefore find

$$(s_i^3, s_i^2t_i, s_it_i^2, t_i^3) \circ \Gamma^*(p) = 0$$

by the shape of Γ^*. But then the above relation implies $b \circ M^*(p) = b \circ A \circ \Gamma^*(p) = 0$, i.e. $\det M^*(p) = 0$.

2.4.2. COROLLARY. *There is a bijection* $[\mathscr{E}] \leftrightarrow (S, C)$ *between* $M(0, 2)$ *and the set of Poncelet pairs with* $S \in C^0(\mathbb{G})$ *and* C *smooth.*

Proof. If (S, C) is given, the conic C determines a pencil on S, since by Poncelet's closure theorem of **[Gr]**, **[Gr-Ha]**, through any point of C there is a triangle tangent to S. By (D_1), S and the pencil determine a bundle in $M(0, 2)$.

2.5. REMARKS. (1) This corollary has been generalized to arbitrary instanton bundles with $h^0\mathscr{E}(1) = 2$ in **[Bö-Tr]**, where C becomes a curve of $\deg = c_2$.

(2) The pair (C, S) of conics reflects the two components of the monad (M, N). The Poncelet relation between S, C is the geometric expression for the monad to form a complex.

2.6. Summarizing the results of 1.5, 2.2, 2.3, 2.4, we have: If $\mathscr{E} \in M(0, 2)$ there is a conic $S \in C^0(\mathbb{G})$ with quadric Q and conjugate conic S^0, and a smooth Poncelet conic C in the plane P of S, s.t.

(1) The pencil describing the Poncelet relation is the pencil of zero lines of sections of $\mathscr{E}(1)$.

(2) The conic S^0 is the set of jumping lines of $\mathscr{E}$ of order 2 and $S^0 = \operatorname{Supp} R^1\mathscr{E}$ $(= \operatorname{Supp} R^1\mathscr{K})$.

(3) If $\tilde{J}$ is the cone over C with vertex $P^\perp$ then $J = \tilde{J} \cap \mathbb{G}$ is the hypersurface of all jumping lines of $\mathscr{E}$ and $J = \operatorname{Supp} R^1\mathscr{E}(-1)$ $(= \operatorname{Supp} R^1\mathscr{M}(-1))$.

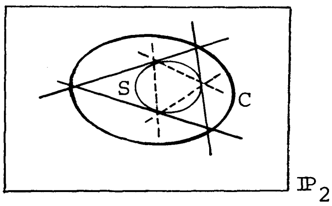
(4) $S^0 = \operatorname{Sing} J$.

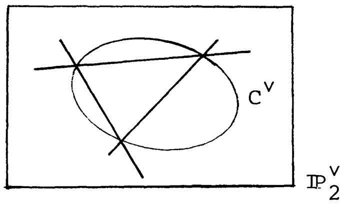
(5) The dual quadric $Q^\vee$ is the set of all jumping planes of $\mathscr{E}$, i.e. of all planes H with $H^0(\mathscr{E}|H) \neq 0$.

3. Quadric bundles of Poncelet conics.

3.1. Given two conics S and C in the projective plane $\mathbb{P}_2$ one can try to inscribe a triangle in C which is circumscribed about S. Poncelet's theorem states that, if there is one such triangle, one can start from any point on C to construct such a triangle, see also **[Gr-Ha]**, **[Gr]**. If $C^\vee$ is the polar dual of C in the dual plane with respect to S, the Poncelet condition simply says that there are three points on S, such that the dual triangle in $\mathbb{P}_2^\vee$ has its vertices on $C^\vee$. This condition is now symmetric in S, $C^\vee$. If it is satisfied we also call S, $C^\vee$ a Poncelet pair of conics and $C^\vee$ a Poncelet conic with respect to S. For a given regular S the set of all regular Poncelet conics $C^\vee$

with respect to S is open in a quadric in the $\mathbb{P}_5$ of all conics with $\mathbb{P}_2^\vee$. We need an explicit description of this quadric, which doesn't seem to exist in the literature. There is a formula for S, C to be a Poncelet pair, first derived by Cayley, see **[Sa]**, p. 342, and **[Gr-Ha]**. This however is not symmetric in S, C and one has to transform it for a pair S, $C^\vee$. In 3.2 we give both a functorial and an explicit description of it.





3.2. Let W be a 3-dimensional vectorspace and the conic $S \subset \mathbb{P}W$ be given by $\sigma \in (S^2W)^\vee$, i.e. by a symmetric bilinear form $W \times W \xrightarrow[\sigma]{} k$ or $W \xrightarrow[\sigma]{} W^\vee$. To σ we associate the two canonical forms:

$$s^2\sigma \colon s^2w \to S^2W^\vee \xrightarrow[\approx]{} (S^2W)^\vee$$
$$\sigma \cdot \sigma \colon \ \to (S^2W)^\vee.$$

The first is the functorial map $S^2\sigma$ followed by the canonical isomorphism, the second is defined by

$$x \cdot y \mapsto \sigma(x, y)\sigma.$$

Now we define

$$Q\sigma = S^2\sigma - \tfrac{1}{2}\sigma \cdot \sigma.$$

Thus $Q\sigma \colon S^2W \to (S^2W)^\vee$ is a symmetric bilinear form on S^2W and defines a quadric in $\mathbb{P}S^2W$. We can define $Q\sigma$ as well by

$$\begin{aligned} Q\sigma(xy, x'y') = \tfrac{1}{2}[&\sigma(x, x')\sigma(y, y') \\ &+\sigma(x, y')\sigma(y, x') - \sigma(x, y)\sigma(x', y')]. \end{aligned}$$

3.3. *The matrix representation of* $Q\sigma$. Let e_0, e_1, e_2 be a basis of W and z_0, z_1, z_2 the dual basis of $W^\vee$. The given form σ will be expressed by

$$(\sigma(e_i, e_j)) = \begin{pmatrix} 2s_{00} & 2_{01} & s_{02} \\ s_{01} & 2s_{11} & s_{12} \\ s_{02} & s_{12} & 2s_{22} \end{pmatrix}$$

such that the conic S is given by the equation $\sum_{i\leq j} s_{ij}z_iz_j = 0$. Then the symmetric matrix $Q(\sigma) = (Q\sigma(e_ie_j, e_ke_l))$ is

$$
Q(\sigma) = \begin{array}{c|c|c|c|c|c|c|}
 & 00 & 01 & 02 & 11 & 12 & 22 \\
\hline
00 & 2s_{00}^2 & s_{00}s_{01} & s_{00}s_{02} & s_{01}^2 - 2s_{00}s_{11} & s_{01}x_{02} - s_{00}s_{12} & s_{01}s_{02} - s_{00}s_{22} \\
\hline
01 & & 2s_{00}a_{11} & s_{00}s_{12} & s_{01}s_{11} & s_{02}s_{11} & s_{02}s_{12} - s_{01}s_{22} \\
\hline
02 & & & 2s_{00}s_{22} & s_{01}s_{12} - s_{02}s_{11} & s_{01}s_{22} & s_{02}s_{22} \\
\hline
11 & & & & 2s_{11}^2 & s_{11}s_{12} & s_{12}^2 - 2s_{11}s_{22} \\
\hline
12 & & & & & 2s_{11}s_{22} & s_{12}s_{22} \\
\hline
22 & & & & & & 2s_{22}^2 \\
\hline
\end{array}
$$

When the conic $C^\vee$ is given by the equation $\sum_{i\leq j} c_{ij}e_ie_j = 0$ in $\mathbb{P}_2^\vee$ and c denotes the column of the coefficients ordered as above, we get

$$Q\sigma(c, c) = c^t \circ Q(s) \circ c =: 2Q(s, c).$$

Explicitly we have for $Q(s, c)$ the expression

$$
\begin{aligned}
s_{00}^2c_{00}^2 &+ s_{00}s_{01}c_{00}c_{01} + s_{00}s_{02}c_{00}c_{02} + (s_{01}^2 - 2s_{00}s_{11})s_{00}s_{11} \\
&+ (s_{01}s_{02} - s_{00}s_{12})c_{00}c_{12} + (s_{02}^2 - 2s_{00}s_{22})c_{00}c_{22} \\
&+ s_{00}s_{11}c_{01}^2 + \cdots .
\end{aligned}
$$

REMARK. When ordered by the products $s_{ij}s_{kl}$ the coefficients as functions in the c_{ij} are the same as in the s_{ij}. This proves that $Q(s, c) = Q(c, s)$ and that the condition $Q(s, c) = 0$ is symmetric in s and c.

3.4. PROPOSITION. *Let $S \subset \mathbb{P}W$ and $C^\vee \subset \mathbb{P}W^\vee$ be the regular conics with the equations*

$$\sum_{i\leq j} s_{ij}z_iz_j = 0 \quad \textit{resp.} \sum_{\imath\leq j} c_{ij}e_ie_j = 0.$$

Then $(S, C^\vee)$ is a Poncelet pair if and only if $Q(s, c) = 0$.

Proof. Let A, B be two 3×3 matrices, not necessarily symmetric and define $\theta(A, B)$, $\theta'(A, B)$ by

$$\det(\lambda A + B) = \lambda^3 \det A + \lambda^2\theta(A, B) + \lambda\theta'(A, B) + \det B.$$

Then for any matrix M we have $\theta(M\circ A, M\circ B) = \det(M)\cdot\theta(A, B)$.

The formula of *Cayley*, **[Sa]**, says that if S, T are symmetric matrices representing two conics S, $T \subset \mathbb{P}U$ then S, T is a Poncelet pair (with S as the "inner" conic) if and only if

$$\theta(S, T)^2 = 4\det(S)\theta(T, S).$$

Now let

$$S = \begin{pmatrix} 2s_{00} & s_{01} & s_{02} \\ s_{01} & 2s_{11} & s_{12} \\ s_{02} & s_{12}2 & s_{22} \end{pmatrix}, \quad \text{resp. } C = \begin{pmatrix} 2c_{00} & c_{01} & c_{02} \\ c_{01} & 2c_{11} & c_{12} \\ c_{02} & c_{12} & 2c_{21} \end{pmatrix},$$

be the matrices of the given conics. The polar dual C of $C^\vee$ with respect to S then has the matrix $T = S \circ C \circ S$. Applying the formula of Cayley we get

$$\theta(S, S \circ C \circ S)^2 = 4\det(S)\theta(S \circ C \circ S, S),$$

which is equivalent to

$$\theta(I, C \circ S)^2 = 4\theta(C \circ S, I).$$

Now by a rather lengthy calculation this condition is equivalent to $Q(s, c) = 0$.

3.5. The quadratic form $Q(s, c)$ determines a *quadric bundle* $Q \subset \mathbb{P}S^2W^\vee \times \mathbb{P}S^2W$ over $\mathbb{P}S^2W^\vee$ with fibres

$$Q_s = \{\langle c\rangle \in \mathbb{P}S^2W | Q(s, c) = 0\}.$$

By using the homogeneous coordinates $s_{00}, \dots, s_{22}$ and $c_{00}, \dots, c_{22}$ we can easily determine the singular locus of Q. If the conic $S \subset \mathbb{P}W$ given by $\langle s\rangle \in \mathbb{P}S^2W^\vee$ is non-degenerate, then obviously Q_s is smooth, and therefore Q is smooth over the open set of regular conics in $\mathbb{P}S^2W^\vee$.

3.6. *Singularities of Q over* $\mathbb{P}S^2W^\vee$. Since $\mathrm{Sing}(Q)$ is contained in the inverse image of the discriminant locus of $\mathbb{P}S^2W^\vee$ of degenerate conics, it is enough to describe $Q_s \cap \mathrm{Sing}(Q)$ for s degenerate. We consider the two cases, where S is a pair of distinct lines or a double line.

Case 1. If the conic S given by s consists of a *pair* of *distinct lines,* we can choose a basis of W in such a way that S is given by $z_1z_2 = 0$, i.e. $s_{ij} = 0$ for $(i, j) \neq (1, 2)$. Then $Q(s, c) = c_{11}c_{22}$ and Q_s is a pair of distinct 4-planes in $\mathbb{P}S^2W$.

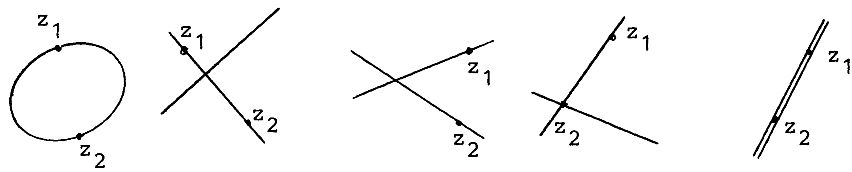
Calculating in addition all the partial derivatives

$$\frac{\partial Q}{\partial s_{ij}}, \quad \frac{\partial Q}{\partial c_{ij}}$$

in (s, c), we find that

$$Q_s \cap \operatorname{Sing}(Q) = \{\langle c\rangle \in \mathbb{P}S^2W \mid c_{11} = c_{22} = 0,\ c_{01}c_{02} - c_{00}c_{12} = 0\}.$$

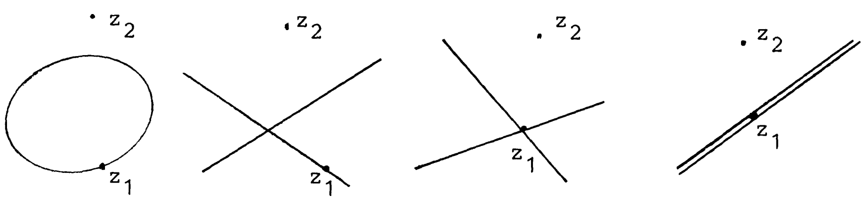
This is a regular quadric in the 3-dimensional intersection $c_{11} = c_{22} = 0$ of the two components of Q_s. In order to illustrate the points of $Q_s \cap \operatorname{Sing}(Q)$ as conics in $\mathbb{P}W^\vee$ let as before $C^\vee$ denote the conic $\sum_{i\leq j} c_{ij}e_ie_j = 0$. The condition $c_{11} = c_{22} = 0$ means that $C^\vee$ is a conic through both $z_1, z_2 \in \mathbb{P}W^\vee$. The possible cases of $C^\vee$ are:



c is a regular point | c is a regular point | c is a singular point | c is a singular point | c is a singular point.

These cases can be checked easily by the above description of $Q_s \cap \operatorname{Sing}(Q)$. Note that in the second case $C^\vee$ is singular whereas (s, c) is a regular point of Q.

If $c_{11} = 0$ but $c_{22} \neq 0$ the pair (s, c) is in one component of the fibre Q_s which consists of all conics $C^\vee$ passing through z_1 but not through z_2:



In each of these cases (s, c) is a regular point.

Case 2. If the conic S given by s consists of a *double line*, we can choose a basis of W in such a way that S is given by $z_2^2 = 0$, i.e. $s_{ij} = 0$ for $(i, j) \neq (2, 2)$. In this case $Q(s, c) = c_{22}^2$ and Q_s is a double 4-plane. Again by looking at the partial derivatives we find that

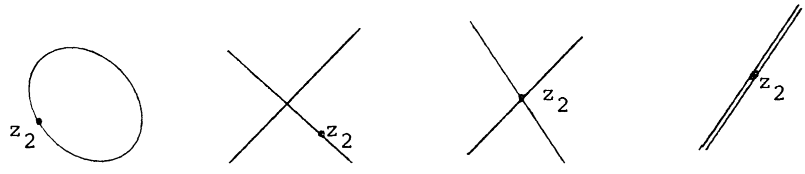
$$Q_s \cap \operatorname{Sing}(Q) = \{\langle c\rangle \in \mathbb{P}S^2W \mid c_{22} = c_{02} = c_{12} = 0\},$$

which is a 2-plane in the 4-plane Q_s.

The condition $c_{22} = 0$ means that any $C^\vee \in Q_s$ passes through z_2. The additional conditions $c_{02} = c_{12} = 0$ say that $C^\vee$ is a conic

$$c_{00}e_0^2 + c_{01}e_0e_1 + c_{11}e_1^2 = 0.$$

Any such is degenerate with vertex z_2 and conversely. Thus the list of possible conics $C^\vee$ in the fibre Q_s is:



c is a regular point | c is a regular point | c is a singular point | c is a singular point.

3.7. *Associated quadric bundles.* The functor Q of 3.2 can be applied to any scheme of quadrics. In our case we need it only for schemes of conics. Let $E \to T$ be a rank 3 vectorbundle over a scheme together with a quadratic form σ as a morphism $E \times E \xrightarrow[\sigma]{} L$ to a line bundle over T. Then σ defines the scheme of zeros $C \subset \mathbb{P}E$ in the projectified bundle, which we call a scheme of conics:

$$\begin{tikzcd}[column sep=small] E \times E \arrow[rr] \arrow[dr] & & L \arrow[dl] \\ & T & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] C \arrow[r, phantom, "\subset"] \arrow[dr] & \mathbb{P}E \arrow[d] \\ & T \end{tikzcd}$$

It is constructed in such a way that for geometric points $t \in T$ we have a conic $C_t \subset \mathbb{P}E_t$. We need this only in the case where T is a reduced variety.

REMARK. In **[Na-Ra]** Narasimhan-Ramanan give a more abstract definition and prove that any conic bundle is of the above form.

Now we can apply the functor Q to obtain the quadric bundle QC as the zero scheme of $Q\sigma$:

$$\begin{tikzcd}[column sep=small] S^2E \times S^2E \arrow[rr, "Q\sigma"] \arrow[dr] & & L^2 \arrow[dl] \\ & T & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] QC \arrow[r, phantom, "\subset"] \arrow[dr] & \mathbb{P}S^2E \arrow[d] \\ & T \end{tikzcd}$$

We call QC the *Poncelet quadric bundle* associated to C. We shall need this construction only in the case of the universal conic over the Hilbert scheme of conics contained in the Grassmannian $\mathbb{G} \subset \mathbb{P}\bigwedge^2 V$.

3.8. *The Hilbert scheme of conics in* $\mathbb{G} \subset \mathbb{P}\bigwedge^2 V$. Let $C(\mathbb{G})$, resp. $C(\mathbb{P}\bigwedge^2 V)$, denote the Hilbert schemes of conics in $\mathbb{G}$, resp. $\mathbb{P}\bigwedge^2 V$. There is a natural embedding $C(\mathbb{G}) \subset C(\mathbb{P}\bigwedge^2 V)$ as the subscheme of conics $S \subset \mathbb{P}\bigwedge^2 V$ contained in $\mathbb{G}$ schematically. Any conic $S \in C(\mathbb{G})$ defines a plane $P \subset \mathbb{P}\bigwedge^2 V$ such that $S \subset \mathbb{G} \cap P$ as a subscheme. If $P \not\subset \mathbb{G}$ then $S = \mathbb{G} \cap P$ is a plane section. We distinguish the following exceptional sets of $C(\mathbb{G})$:

$$\begin{aligned} \Sigma_0 &= \text{set of singular conics in } C(\mathbb{G}), \\ \Sigma_\alpha &= \text{set of conics contained in an } \alpha\text{-plane } P \subset \mathbb{G}, \\ \Sigma_\beta &= \text{set of conics contained in a } \beta\text{-plane } P \subset \mathbb{G}. \end{aligned}$$

Then $C^0(\mathbb{G}) = C(\mathbb{G}) \setminus \Sigma_0 \cup \Sigma_\alpha \cup \Sigma_\beta$ is the open part of regular plane conic sections.

3.8.1. PROPOSITION. *$C(\mathbb{G})$ is a smooth, irreducible variety of dimension 9 and Σ_0, Σ_α, Σ_β are irreducible divisors in this manifold.*

For the proof we will make use of the following lemma, see SGA, VII, Prop. 1.7.

LEMMA. *Let $X_1 \subset X_2 \subset X_3$ be schemes with X_2, X_3 smooth and X_1 a locally complete intersection. Then there is an exact sequence (on X_1)*

$$0 \to N_{X_1, X_2} \to N_{X_1, X_3} \to N_{X_2, X_3}|X_1 \to 0$$

where N_{X_i, X_j} denotes the normal bundle of X_i in X_j.

Proof. (1) Using the differential criterion for the smoothness for Hilbert schemes, the smoothness of $C(\mathbb{G})$ and its dimension will follow if we have proved $h^1(S, N_{S,\mathbb{G}}) = 0$, $h^0(S, N_{S,\mathbb{G}}) = 9$ for any conic $S \subset \mathbb{G}$.

(2) If $S \subset P \subset \mathbb{P}_5 = \mathbb{P}\bigwedge^2 V$ is a conic in a plane P in $\mathbb{P}_5$ it is immediate to see that $h^1(S, N_{S,P}) = 0$, $h^0(S, N_{S,P}) = 5$, and using the lemma, that also $h^1(S, N_{S,\mathbb{P}_5}) = 0$, $h^0(S, N_{S,\mathbb{P}_5}) = 14$, where one can use that $N_{P,\mathbb{P}_5} = 3\mathcal{O}_P(1)$.

(3) Let $P \subset \mathbb{G}$. Then $h^1(P, N_{P,\mathbb{G}}) = 0$, $h^1(P, N_{P,\mathbb{G}}(-2)) = 1$ and $h^0(P, N_{P,\mathbb{G}}(-2)) = h^2(P, N_{P,\mathbb{G}}(-2)) = 0$, whereas $h^0(P, N_{P,\mathbb{G}}) = 3$. This can be proved by applying the lemma to $P \subset \mathbb{G} \subset \mathbb{P}_5$, so that

we have the exact sequence

$$0 \to N_{P,\mathbb{G}} \to 3\mathscr{O}_P(1) \to \mathscr{O}_P(2) \to 0,$$

since $N_{\mathbb{G},\mathbb{P}_5} = \mathscr{O}_{\mathbb{G}}(2)$. Dualizing this and then tensoring by $\mathscr{O}_P(1)$ we obtain

$$0 \to \mathscr{O}_P(-1) \to 3\mathscr{O}_P \to N^{\vee}_{P,\mathbb{G}} \otimes \mathscr{O}_P(1) \to 0.$$

But $N_{P,\mathbb{G}}$ is of rank 2 and its determinant bundle is $\mathscr{O}_P(1)$, so that $N^{\vee}_{P,\mathbb{G}} \otimes \mathscr{O}_P(1) \cong N_{P,\mathbb{G}}$. Thus we have the exact sequence

$$0 \to \mathscr{O}_P(1) \to 3\mathscr{O}_P \to N_{P,\mathbb{G}} \to 0,$$

and the statements in (3) follow immediately.

(4) To prove (1) we distinguish *case* 1: $P \subset \mathbb{G}$ and *case* 2: $P \not\subset \mathbb{G}$. In *case* 1 we have an exact sequence for $S \subset P$

$$0 \to N_{S,P} \to N_{S,\mathbb{G}} \to N_{P,\mathbb{G}}|S \to 0.$$

From the exact sequence

$$0 \to N_{P,\mathbb{G}}(-2) \to N_{P,\mathbb{G}} \to N_{P,\mathbb{G}}|S \to 0$$

we see, using (3), that

$$h^1(S, N_{P,\mathbb{G}}|S) = 0 \quad \text{and} \quad h^0(S, N_{P,\mathbb{G}}|S) = 4.$$

Using (2) we now conclude that indeed $h^1(S, N_{S,\mathbb{G}}) = 0$ and $h^0(S, N_{S,\mathbb{G}}) = 9$. In *case* 2 the conic S is the (schematic) intersection $S = \mathbb{G} \cap P$. Let us consider the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & N_{S,P} \arrow[d] \arrow[dr, dashed] & & \\
0 \arrow[r] & N_{S,\mathbb{G}} \arrow[r] & N_{S,\mathbb{P}_5} \arrow[r] \arrow[d] & N_{\mathbb{G},\mathbb{P}_5}|S \arrow[r] & 0 \\
 & & N_{P,\mathbb{P}_5}|S \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

We claim that the vertical sequence is split; in fact the natural map $N_{S,P} \to N_{\mathbb{G},\mathbb{P}_5}|S$ is an isomorphism. This follows from the fact that the line bundle on P determined by the ideal sheaf of S in P is the pull back of the line bundle determined by the ideal sheaf of $\mathbb{G}$ in $\mathbb{P}_5$, by the inclusion $P \to \mathbb{P}_5$. Since the vertical sequence is split, the map

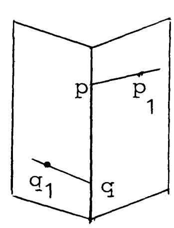
$$H^1(S, N_{S,\mathbb{G}}) \to H^1(S, N_{S,\mathbb{P}_5})$$

is an injection while $H^1(S, N_{S,\mathbb{P}_5}) = 0$ by (2). Thus $H^1(S, N_{S,\mathbb{G}}) = 0$. We also have

$$h^0(S, N_{S,\mathbb{G}}) = h^0(S, N_{S,\mathbb{P}_5}) - h^{-}(S, N_{\mathbb{G},\mathbb{P}_5}|S) = 14 - 5 = 9.$$

This completes the proof of the smoothness.

(5) By their definition Σ_0, Σ_α, Σ_β are subvarieties of $C(\mathbb{G})$. Clearly Σ_0 is 1-codimensional and Σ_α, Σ_β are $\mathbb{P}_5$-bundles over a $\mathbb{P}_3$ (as space of all α- resp. β-planes) and hence also 1-codimensional. Being bundles Σ_α, Σ_β are already irreducible. So we are left to show that also Σ_0 is irreducible. To do this let $\Omega \subset \Sigma_0$ be the open set of conics consisting of two different lines and which are not in $\Sigma_\alpha \cup \Sigma_\beta$. It suffices to show that Ω is irreducible (in fact $\dim(\Sigma_0 \setminus \Omega) < 8$ since $\dim \Sigma_0 \cap \Sigma_\alpha$, $\dim \Sigma_0 \cap \Sigma_\beta = 7$ and the subvariety $\Sigma_0' \subset \Sigma_0$ of double lines is of dimension 6, see 3.9.1). Now $\mathrm{PGL}(V)$ acts transitively on Ω as can be seen from the configuration (D) in 0.4: A conic $S \in \Omega$ corresponds to a pair of planes and two points in their intersection in $\mathbb{P}V$ and thus is determined by a pair (p, p_1), (q, q_1) of pairs of points in $\mathbb{P}_3$ such that the four points are independent. It is now immediate that, given two such configurations in $\mathbb{P}_3$, one can be taken into another by a linear transformation in $\mathrm{PGL}(V)$.



(6) Finally to prove the irreducibility of $C(\mathbb{G})$, it is sufficient to prove that the open dense subset $C^0(\mathbb{G}) = C(\mathbb{G}) \setminus \Sigma_0 \cup \Sigma_\alpha \cup \Sigma_\beta$ is irreducible. But this is isomorphic to the open set $G_3^0 \bigwedge^2 V$ in the Grassmannian of 2-planes in $\mathbb{P} \bigwedge^2 V$ cutting $\mathbb{G}$ in a regular conic,

see the morphism $C(\mathbb{G}) \to G_3 \bigwedge^2 V$ in 3.10 below. But the latter is irreducible.

3.9. *The double lines in* $C(\mathbb{G})$. If $l \subset \mathbb{G}$ is a line in $\mathbb{P}\bigwedge^2 V$ contained in $\mathbb{G}$, there is a unique α-plane P_α and a unique β-plane P_β with $l \subset P_\alpha, P_\beta$. The pencil $\mathrm{Pen}(l)$ of planes spanned by these is the unique pencil such that $l = \mathbb{G} \cap P$ for $P \in \mathrm{Pen}(l)$, $P \neq P_\alpha, P_\beta$. This can be proved easily by choosing a basis of $\bigwedge^2 V$ containing a basis of l.

If now $S \in C(\mathbb{G})$ is a double line, the plane P with $S \subset \mathbb{G} \cap P$, which is determined by S, must be a member of $\mathrm{Pen}(S_{\mathrm{red}})$, and conversely any $P \in \mathrm{Pen}(S_{\mathrm{red}})$ determines a conic structure S on S_{red} with $S \subset \mathbb{G} \cap P$ schematically. Therefore the conics $S \in C(\mathbb{G})$ supported by a line $l \subset \mathbb{G}$ are in 1:1 correspondence with the planes $P \in \mathrm{Pen}(l)$. We even have

3.9.1. LEMMA. *Let* $\Sigma_0' \subset \Sigma_0 \subset C(\mathbb{G})$ *be the subvariety of double lines in* $C(G)$. *Then* Σ_0' *is a* $\mathbb{P}_1$*-bundle over the Hilbert scheme* $\mathscr{L}(\mathbb{G})$ *of lines in* $\mathbb{G}$, *and* $\dim \Sigma_0' = 6$.

Proof. $S \to S_{\mathrm{red}}$ is a morphism $\Sigma_0' \to \mathscr{L}(\mathbb{G})$ whose fibres are the pencils $\mathrm{Pen}(l)$. It is left to the reader to verify that this is a $\mathbb{P}_1$-bundle. It follows that $\dim \Sigma_0' = 6$.

3.10. *The modification* $C(\mathbb{G}) \to G_3 \bigwedge^2 V$. Let $Z = G_3 \bigwedge^2 V$ be the Grassmannian of 2-planes in $\mathbb{P}\bigwedge^2 V$. We denote by $\mathbb{W} \to Z$ the tautological 3-bundle and by W_z its fibre over z. The projective bundle $\mathbb{P}S^2\mathbb{W}^\vee$ of quadratic forms in the fibres of $\mathbb{W}$ can be considered as the Hilbert scheme of conics in $\mathbb{P}\bigwedge^2 V$. Therefore we have an embedding

$$\begin{tikzcd} C(\mathbb{G}) \arrow[rr, hook] \arrow[dr, "\mu"'] & & \mathbb{P}S^2\mathbb{W}^\vee \arrow[dl, "\pi"] \\ & Z & \end{tikzcd}$$

and the composed morphism is a modification of the Grassmannian $G_3 \bigwedge^2 V$ with exceptional divisors Σ_α, Σ_β. Thus the modification consists in putting in all conics in α- or β-planes.

3.11. *The quadric bundle* $Q \to C(\mathbb{G})$. The universal conic over $\mathbb{P}S^2\mathbb{W}^\vee$ can be constructed as follows. Let $\pi^*\mathbb{W}$ be the pull back of the tautological bundle to $\mathbb{P}S^2\mathbb{W}^\vee$. There is a universal quadratic form σ on $\pi^*\mathbb{W}$ which has values in the relative hyperplane bundle $\mathscr{L} = \mathscr{O}_{\mathbb{P}S^2\mathbb{W}^\vee}(1)$ such that we have

$$(5)\qquad \begin{tikzcd} \pi^*\mathbb{W} \otimes \pi^*\mathbb{W} \arrow[rr, "\sigma"] \arrow[dr] & & \mathscr{L} \arrow[dl] \\ & \mathbb{P}S^2\mathbb{W}^\vee & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr] \arrow[dr] & & \mathbb{P}\pi^*\mathbb{W} \arrow[dl] \\ & \mathbb{P}S^2\mathbb{W}^\vee & \end{tikzcd}$$

The universal conic C over $\mathbb{P}S^2\mathbb{W}^\vee$ is the zero locus of the form σ. Clearly the restriction of the conic bundle C to $C(\mathbb{G}) \subset \mathbb{P}S^2\mathbb{W}^\vee$ is the universal conic bundle over the Hilbert scheme $C(\mathbb{G})$.

Now we apply the Poncelet functor Q to the universal conic bundle over $C(\mathbb{G}) \subset \mathbb{P}S^2\mathbb{W}^\vee$. Thus from (5) we obtain the quadric bundle QC and its restriction Q to $C(\mathbb{G})$:

$$(6)\qquad \begin{tikzcd} Q \arrow[r, hook] \arrow[d] & QC \arrow[d] & \subset \pi^*\mathbb{P}S^2\mathbb{W} \\ C(\mathbb{G}) \arrow[r, hook] & \mathbb{P}S^2\mathbb{W}^\vee. & \end{tikzcd}$$

If we consider $\pi^*\mathbb{P}S^2\mathbb{W}$ as a fibre product we have

$$(7)\qquad \begin{tikzcd} Q \arrow[r, hook] \arrow[drr] & QC \arrow[r, hook] \arrow[dr] & \mathbb{P}S^2\mathbb{W}^\vee \times_Z \mathbb{P}S^2\mathbb{W} \arrow[d] \\ & & Z \end{tikzcd}$$

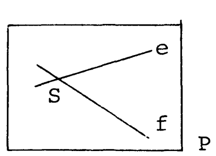
Then for any $z \in Z$ the fibre

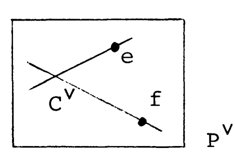
$$(QC)_z \subset \mathbb{P}S^2 W_z^\vee \times \mathbb{P}S^2 W_z$$

is the Poncelet hypersurface of bidegree 2 considered in 3.5. It is at the same time the quadric bundle over $\mathbb{P}S^2 W_z^\vee$, or $QC|\mathbb{P}S^2 W_z^\vee$.

3.12. *Singularities of Q*. If the conic $S \in C(\mathbb{G})$ is regular then the fibre Q_S of Q over S is non-degenerate and therefore Q is smooth over the open part of regular conics. If S is singular, then $Q_S \cap \operatorname{Sing} Q$ has the same description as in 3.6. The proof consists in using local coordinates (derived from $C(\mathbb{G}) \subset \mathbb{P}S^2\mathbb{W}^\vee$ for example) and then in calculating partial derivative as in 3.6. Thus

(i) If $S \in C(\mathbb{G})$ consists of two different lines e, f, the pair $(S, C^\vee)$ is a singular point iff $C^\vee$ is singular and passes through both e and f.





(ii) If $S \in C()$ is a double line with $S_{\text{red}} = e$, then $(S, C^\vee)$ is a singular point iff $C^\vee$ is singular and $e \in \operatorname{Sing} C^\vee$.

3.12.1. Corollary. *The codimension of* $\operatorname{Sing} Q$ *is* 3.

3.13. Proposition. *The quadric bundle* Q *is a normal irreducible variety.*

Proof. Let $C(\mathbb{G}) \underset{j}{\hookrightarrow} \mathbb{P}S\mathbb{W}^\vee$ be the embedding of 3.9. From diagram (6) we obtain the diagram

$$\begin{tikzcd} Q \arrow[rr, hook] \arrow[dr] & & j^*\pi^*\mathbb{P}S^2\mathbb{W} \arrow[dl] \\ & C(\mathbb{G}) & \end{tikzcd}$$

where $j^*\pi^*\mathbb{S}P^2\mathbb{W}$ is the bundle of all conics in the dual plane defined by the universal conic over $C(\mathbb{G})$. The embedding of Q into this bundle is regular, since any fibre of Q consists of the Poncelet quadric in the corresponding fibre $\mathbb{P}S^2W$ of $j^*\pi^*\mathbb{P}S^2\mathbb{W}$. Since $C(\mathbb{G})$ is smooth by 3.8.1, it follows that Q is a local complete intersection, see SGA 6, VIII, Prop. 1.5. On the other hand, the codimension of the singular set of Q is ≥ 2 by 3.12.1. It follows from **[Hal**, Prop. 8.23] that Q is normal. Since Q is equidimensional of dimension 13, $Q|C^0(\mathbb{G})$ is irreducible and $\dim(Q \setminus Q|C^0(\mathbb{G})) < 13$, Q must also be irreducible.

4. Boundary components of Q and Main Theorem. We define 4 (positive) divisors on Q as follows. Let Q_0, Q_α, Q_β be respectively the inverse images of Σ_0, Σ_α, Σ_β (3.8) by the canonical projection $Q \to C(\mathbb{G})$, and let Q_e be the subvariety of pairs $(S, C^\vee)$ with $C^\vee$ singular.

4.1. Proposition. *The divisors* Q_0, Q_α, Q_β *and* Q_e *are irreducible.*

Proof. Since the Poncelet bundle associated to the space of conics in $\mathbb{P}_2$ is irreducible, we see that Q_α and Q_β are irreducible. To show that Q_e is irreducible, let Ω' be the open subset of Q_e consisting of $(S, C^\vee)$ where S is a regular cut of $\mathbb{G}$ by a plane in $\mathbb{P}_5$. Then Ω' is irreducible and Ω' is dense in Q_e and Q_e is of pure dimension 12 and $\dim(Q_3 - \Omega') \leq 11$.

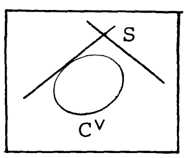
To prove that Q_0 is irreducible consider the diagram, see (7) in 3.11,

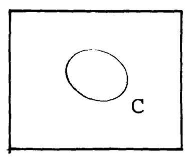
$$\begin{tikzcd}
Q_0 \arrow[r, hook] \arrow[d] & Q \arrow[r, hook] \arrow[d] & PS^2\mathbb{W}^\vee \times_Z PS^2\mathbb{W} \arrow[r] \arrow[d] & PS^2\mathbb{W} \arrow[dl] \\
\Sigma_0 \arrow[r, hook, "j"] & C(\mathbb{G}) \arrow[r, "\mu"] & Z &
\end{tikzcd}$$

and the induced map φ

$$\begin{tikzcd}
Q_0 \arrow[rr, "\varphi"] \arrow[dr] & & j^*\mu^*\mathbb{P}S^2(\mathbb{W}) \arrow[dl] \\
& \Sigma_0 &
\end{tikzcd}$$

Let $R \subset j^*\mu^*\mathbb{P}S^2(\mathbb{W})$ be the space of smooth conics. Since Σ_0 is irreducible we see that R is irreducible. For $C \in R$, $\varphi^{-1}(C)$ consists of pairs (S, C) where S is a singular conic in the dual plane one of whose components touches the dual conic $C^\vee$.





Thus $\varphi^{-1}(C)$ is irreducible of dimension 3, being the image of $\mathbb{P}_1 \times \mathbb{P}_2$ by a finite map. Hence $\varphi^{-1}(R)$ is irreducible. Now $\varphi^{-1}(R)$ is open and dense in Q_0, since, in the space of conics in $\mathbb{P}_2$ through a point in $\mathbb{P}_2$, the subspace consisting of smooth conics is dense.

4.2. *$\mathbb{P}_1$-fibration on the exceptional set Q_{exc}.* Recall, 3.12, that the singular set, $\operatorname{Sing} Q$, of Q is contained in $Q_0 \cap Q_0$ and consists of pairs $(S, C^\vee)$ where S and $C^\vee$ are both singular having a position as in (i), (ii), 3.12. Let $\Sigma_0' \subset \Sigma_0$ be the space of double lines and let

$$Q_{\mathrm{exc}} = \operatorname{Sing} Q \cap \pi^{-1}(\Sigma_0'), \quad \pi \colon Q \to C(\mathbb{G}).$$

We claim that Q_{exc} has a natural structure of a $\mathbb{P}_1$-bundle, a fibre being the pencil $\mathbb{P}_1$ of double structures on a line contained in $\mathbb{G}$; see 3.9.1. Let $L(\mathbb{G})$ be the Hilbert scheme of lines contained in $\mathbb{G}$ and $D \to L(\mathbb{G})$ the tautological $\mathbb{P}_1$-bundle. Let $S^2(D) \to L(\mathbb{G})$ be the $\mathbb{P}_2$-bundle which is the relative Hilbert scheme of pairs of points on the fibres of $D \to L(\mathbb{G})$.

A point of Q_{exc} consists of a pair $(S, C^\vee)$ where S is a line with a double structure in $\mathbb{G}$ and $C^\vee$ is a singular conic in the dual plane having a singularity at the point in the dual plane determined by S. Let l be the reduced line associated to S. We may view $C^\vee$ as a pair of points on l. Thus we obtain a map $Q_{\text{exc}} \to S^2(D)$, whose fibre at $(l, p, q) \in S^2(D)$, where l is a line in $\mathbb{G}$ and $p, q \in l$, is the pencil $\mathrm{Pen}(l)$ of double structures on l contained in $\mathbb{G}$.

Note that we have the diagram of $\mathbb{P}_1$-fibrations

$$\begin{array}{ccc} Q_{\text{exc}} & \longrightarrow & S^2(D) \\ \big\downarrow & & \big\downarrow \\ \Sigma'_0 & \longrightarrow & L(\mathbb{G}) \end{array} .$$

4.3. *The component $\overline{M(0, 2)}$ of the Maruyama scheme containing $M(0, 2)$.* Let $\overline{M}(2; 0, 2, 0)$ be the Maruyama scheme of all semistable coherent rank 2 sheaves on $\mathbb{P}_3$ with Chern classes $c_1 = 0$, $c_2 = 2$, $c_3 = 0$. The moduli space of vector bundles $M(0, 2)$ is a smooth connected open subset of $\overline{M}(2; 0, 2, 0)$; we denote by $\overline{M(0, 2)}$ the (reduced) schematic closure of $M(0, 2)$ in $\overline{M}(0; 0, 2, 0)$, and by $\widetilde{M(0, 2)} \xrightarrow[\nu]{} \overline{M(0, 2)}$ its normalisation.

4.4. THEOREM. (1) *The variety Q (see 3.11) can be blown down along the $\mathbb{P}_1$-fibration $Q_{\text{exc}} \to S^2(D)$ (defined in 4.2) to a normal variety $\underset{\sim}{Q}$, i.e. the push-out $\underset{\sim}{Q}$ of the diagram*

$$\begin{array}{ccc} Q_{\text{exc}} & \longrightarrow & Q \\ \big\downarrow & & \\ S^2(D) & & \end{array}$$

exists in the category of varieties over k.

(2) *There exists a canonical morphism*

$$Q \xrightarrow[\varphi]{} \overline{M(0, 2)}$$

which induces an isomorphism of the blown down variety $\underset{\sim}{Q}$ onto the normalisation $\widetilde{M(0, 2)}$ of $\overline{M(0, 2)}$.

(3) *Let* Q^0 *denote* $Q \setminus Q_e \cup Q_\alpha \cup Q_\beta \cup Q_0$ (*see* 4.1). *The restriction of* φ *to* Q^0 *maps* Q^0 *isomorphically onto* $M(0, 2)$. *The inverse of this isomorphism is the map of Corollary* 2.4.2 *which associates to a bundle the corresponding Poncelet pair* (S, C) *of smooth conics. Moreover the "boundary"* $\overline{M(0, 2)} \setminus M(0, 2)$ *is the union of the four Weil divisors which are the images by* φ *of* Q_e, Q_α, Q_β *and* Q_0.

(4) *The normalisation map* $\widetilde{M(0, 2)} \xrightarrow{\nu} \overline{M(0, 2)}$ *is bijective and the smooth points of* $\overline{M(0, 2)}$ *correspond precisely to the stable sheaves in* $\overline{M(0, 2)}$.

REMARK 1. In the formulation of the theorem in **[Na-Tr]** the blow down of the $\mathbb{P}_1$-fibration of Q_{exc} had been overlooked.

REMARK 2. Under $X^{ss}//\mathrm{SL}(2) \cong Q$ the stable points of X^{ss} under the SL(2)-action correspond precisely to the smooth points of Q, see 6.7.1.

REMARK 3. The sheaves or their equivalence classes in the 4 boundary components can be characterised geometrically by the Poncelet pairs $(S, C^\vee)$ by which they are defined. This can be found in §§9, 10.

In particular the generic points of the divisor Q_e are the sheaves $\mathcal{F}$, which are obtained by the elementary transformations

$$0 \to \mathcal{F} \to \mathcal{E}' \to \mathcal{O}_L(1) \to 0,$$

where $\mathcal{E}'$ is a bundle in $M(0, 1)$ and L is a line in $\mathbb{P}_3$, see 9.1.

REMARK 4. The semi-stable but non-stable sheaves in the boundary are characterised as extensions

$$0 \to \mathcal{J}_{L\cup q} \to \mathcal{F} \to \mathcal{J}_{K\cup p} \to 0,$$

where L, K are lines and p, q points in $\mathbb{P}_3$, $\mathcal{J}_{L\cup q}$ is the ideal sheaf of the union $L \cup \{q\}$ (with a simple multiple structure in q if $q \in L$), see Theorem 10.5. The blow down of the $\mathbb{P}_1$-fibration of Q_{exc} is explained in terms of the sheaves in 10.6.

REMARK 5. For any $[\mathcal{F}] \in \overline{M(0, 2)}$ let $S^0 = \operatorname{Supp} R^1\mathcal{F}$, $J = \operatorname{Supp} R^1\mathcal{F}(-1)$, where R^1 is the incidence transformation. These sets are the generalised sets of jumping lines of order $\geqq 2$ resp. of all jumping lines of $\mathcal{F}$. In the proof of 8.3 it is shown that S^0 and J only depend on the equivalence class $[\mathcal{F}]$. However, which is more important, we have the

COROLLARY. *Any* $[\mathscr{F}] \in \overline{M(0,2)}$ *is already determined by the pair* (S^0, J).

Proof. By 8.3 (d) the pair (S^0, J) determines a quadric hypersurface $\tilde{J} \subset \mathbb{P}\bigwedge^2 V$ such that $J = \mathbb{G} \cap \tilde{J}$ and $\tilde{J}$ is singular along $S^0 \subset \operatorname{Sing} J$. If $[\mathscr{F}] \notin \varphi(Q_{\mathrm{exc}})$ then $(S^0, \tilde{J})$ determines the plane $\mathbb{P}W$ or $\mathbb{P}W^{\perp}$ by 8.3 (e), (f). If $\mathbb{P}W^{\perp} \subset \mathbb{P}\bigwedge^2 V$ is any splitting of $\mathbb{P}\bigwedge^2 V \cong \mathbb{P}\bigwedge^2 V^{\vee} \to \mathbb{P}W^{\vee}$, see 8.3 (b), then $C^{\vee} = \mathbb{P}W^{\vee} \cap \tilde{J}$, 8.3 (d), and hence $\mathscr{F}$ is determined by (S^0, J) through $(S, C^{\vee})$. If however $[\mathscr{F}] \in \varphi(Q_{\mathrm{exc}})$, then we can choose any plane $\mathbb{P}W$ in the pencil $\operatorname{Pen}(S_{\mathrm{red}})$ and define $C^{\vee} = \mathbb{P}W^{\vee} \cap \tilde{J}$. Then $(S, C^{\vee})$ determines the class $[\mathscr{F}]$ independently of the choice of $\mathbb{P}W$ by (1) of the theorem, i.e. (S^0, J) determines $[\mathscr{F}]$ in this case, too.

4.5. *Proof of Theorem* 4.4. (a) We first prove the existence of the canonical map φ and part (3) of the theorem. In §6 we will construct a projective variety X with an SL(2)-action such that, if X^{ss} is the open subset of semi-stable points for this action, then the good quotient $X^{ss}//\mathrm{SL}(2)$ is isomorphic to Q. Moreover we construct in §8 a flat family $\{\mathscr{F}_x\}$, $x \in X^{ss}$, of rank 2 coherent sheaves on $\mathbb{P}_3$ with $c_1 = 0$, $c_2 = 2$, $c_3 = 0$. It is proved in §§9, 10 that the sheaves $\mathscr{F}_x$ are semi-stable. Moreover if x and x' are on the same SL(2)-orbit then $\mathscr{F}_x \simeq \mathscr{F}_{x'}$. Hence there is a canonical morphism from X^{ss} to the Maruyama scheme $\overline{M}(2; 0, 2, 0)$, and this induces a morphism $Q \xrightarrow[\Phi]{} \overline{M}(2; 0, 2, 0)$.

(b) We will now prove that Φ maps Q onto $\overline{M(0,2)}$, so that we will obtain a morphism $Q \xrightarrow[\varphi]{} \overline{M(0,2)}$. We first show that Φ maps Q^0 isomorphically onto $M(0,2) \subset \overline{M}(2; 0, 2, 0)$. Let X^0 be the inverse image of Q^0 in X^{ss}. Now each sheaf $\mathscr{F}_x$, $x \in X^{ss}$, comes with a monad display (22) in §8. If $x \in X^0$ this is a monad of a bundle by 6.8 with $\mathscr{N}_y = \mathscr{K}_N$ (see 7.1.2) and $\mathscr{A}_y = N^{\vee} \otimes \mathscr{O}$. So, X^0 and hence its quotient Q^0 are mapped into $M(0,2)$ under Φ. The Poncelet pair $(S, C^{\vee})$ of smooth conics associated to x is the Poncelet pair associated to the bundle in 2.4.2 with $C^{\vee}$ the polar dual of C, see 6.8. Now by Corollary 2.4.2 $\Phi|Q^0$ is a bijection from Q^0 onto $M(0,2)$. Since both varieties are smooth $\Phi|Q^0$ is an isomorphism. Since Q is irreducible by Proposition 3.13 and Q^0 is dense in Q, we now see that Φ maps Q onto $\overline{M(0,2)}$. We thus have the canonical morphism $Q \xrightarrow[\varphi]{} \overline{M(0,2)}$. Observe that we have proved already the first parts of (2) and (3) of the theorem.

(c) We now proceed to prove (1) and (2). It is proved in 8.4 with preparations in §7 that $Q \xrightarrow[\varphi]{} \overline{M(0,2)}$ is injective on $Q \setminus Q_{\text{exc}}$, constant on the fibres of the $\mathbb{P}_1$-fibration on Q_{exc} and induces an injective map $S^2(D) \to \overline{M(0,2)}$. Since Q is normal (Proposition 3.13) and φ is onto, φ lifts to a surjective morphism $Q \xrightarrow{\tilde{\varphi}} \widetilde{M(0,2)}$. Since the fibres of φ are connected, the normalization map $\widetilde{M(0,2)} \xrightarrow[\nu]{} \overline{M(0,2)}$ is bijective. Using the Stein factorization of $\tilde{\varphi}$ and Zariski's main theorem, we see that

$$\tilde{\varphi}_* \mathscr{O}_Q = \mathscr{O}_{\widetilde{M(0,2)}}.$$

If $Q_{\text{exc}} \xrightarrow[\eta]{} S^2(D)$ is the $\mathbb{P}_1$-fibration we also have $\eta_* \mathscr{O}_{Q_{\text{exc}}} = \mathscr{O}_{S^2(D)}$, so that the map $S^2(D) \xrightarrow[\psi]{} \widetilde{M(0,2)}$ induced by $S^2(D) \to \overline{M(0,2)}$ is a morphism. We thus have a commutative diagram of morphisms

$$\begin{array}{ccc} Q_{\text{exc}} & \longrightarrow & Q \\ \big\downarrow{\scriptstyle \eta} & & \big\downarrow{\scriptstyle \tilde{\varphi}} \\ S^2(D) & \longrightarrow & \widetilde{M(0,2)} \end{array}$$

with $\tilde{\varphi}_* \mathscr{O}_Q = \mathscr{O}_{\widetilde{M(0,2)}}$. It is easy to verify from this that $\widetilde{M(0,2)}$ is the required push out. Thus we have proved (1) and (2) of the theorem, and also that the normalization map ν is bijective.

(d) From the above we know that

$$Q \setminus Q_{\text{exc}} \to \widetilde{M(0,2)} \setminus \tilde{\varphi}(Q_{\text{exc}})$$

is an isomorphism. Under this the divisors Q_e, Q_α, Q_β, Q_0 are mapped to divisors, since $Q_{\text{exc}} \subset Q_0$ and $\dim Q_{\text{exc}} = 8$ (as a $\mathbb{P}_2$-bundle over $\Sigma_0' \subset C(\mathbb{C})$). Hence they are also mapped to divisors in $\widetilde{M(0,2)}$ or $\overline{M(0,2)}$. This proves the second part of (3).

(e) To complete the proof of (4), observe that Q_{exc} is contained in the singular locus $Q_{\text{Sing}} \subset Q_0$ of Q. Now $Q_{\text{Sing}} \setminus Q_{\text{exc}}$ is dense in Q_{Sing}. In fact by 3.12, a Poncelet pair $(S, C^\vee)$ in Q_{Sing} corresponds to a singular conic S with a pair of points one on each component. We can approximate a double line with a pair of (eventually coincident) points on it by a singular conic consisting of distinct lines along with a point on each component. Since $\tilde{\varphi}$ is an isomorphism on $Q \setminus Q_{\text{exc}}$, we see that $\tilde{\varphi}(Q_{\text{Sing}}) = \widetilde{M(0,2)}_{\text{Sing}}$. On the other hand by Theorem

10.5, a point in Q corresponds to a stable sheaf precisely when it is a non-singular point of Q.

5. Geometric invariant parametrisation of $C(\mathbb{G})$ **.** In order to define a universal family of sheaves we have to construct a suitable parametrisation of the quadric bundle $Q \to C(\mathbb{G})$, since a universal family does not even exist over $M(0, 2)$ **[Hi-Na]**. This will be done in such a way that we construct a morphism $X \to Y$ of projective varieties, acted on by $\mathrm{SL}(2, \mathscr{k})$, which is equivariant, and such that the induced morphism $X^{\mathrm{ss}}// \mathrm{SL}(2) \to Y^{\mathrm{ss}}// \mathrm{SL}(2)$ on the good quotients is the quadric bundle.

As a first step we construct Y in this section. Recall that $C^0(\mathbb{G}) = G_2^0(\mathscr{k}^2 \otimes V)/\mathrm{SL}(2)$, where $G_2^0(\mathscr{k}^2 \otimes V)$ is the open part of the Grassmannian consisting of regular subspaces $N \subset \mathscr{k}^2 \otimes V$ defining the right parts $\mathscr{k}^2 \otimes \Omega^1(1) \xrightarrow[\nu]{} N^\vee \otimes \mathscr{O}$ of bundle monads, 1.4. The space $G_2^{\mathrm{ss}}(\mathscr{k}^2 \otimes V)$ of semi-stable points of the Grassmannian however does not parametrise the complete Hilbert-scheme, see Remark 5.9. The parameter space Y is constructed in such a way that its semi-stable points form a modification of $G_2^{\mathrm{ss}}(\mathscr{k}^2 \otimes V)$. It is essentially the set of $\Gamma \in G_4(\mathscr{k}^2 \otimes \bigwedge^2 V)$ such that

$$\Gamma \subset \mathrm{Ker}\left(\mathscr{k}^2 \otimes \bigwedge^2 V \to N^\vee \otimes \bigwedge^3 V\right)$$

for some $N \in G_2(\mathscr{k}^2 \otimes V)$, and only if N is regular we have an exact sequence

$$0 \to \Gamma \to \mathscr{k}^2 \otimes \bigwedge^2 V \to N^\vee \otimes \bigwedge^3 V \to 0. \tag{8}$$

It turned out that in the degenerate cases of N the subspaces Γ provide us with the necessary information in the limit cases: They determine the degenerate conics in $C(\mathbb{G})$ and we have $\Gamma \simeq \Gamma\mathscr{N}(1)$, where $\mathscr{N}$ is the corresponding kernel sheaf, thereby generalizing the results of §1, whereas the degenerate spaces N do not determine them. The result is the space $Y \subset G_4(\mathscr{k}^2 \otimes W)$ constructed in 5.7. In 5.8 we show that N is determined by Γ for semi-stable Γ.

As a preparation we prove a stability criterion for points in Grassmannians, of the above type, which is essential for our constructions.

5.1. Let U, W be finite dimensional vector spaces and let $\mathrm{SL}(U)$ act on $G_q(U \otimes W)$ by $L \mapsto (g \otimes \mathrm{id})(L)$. This action is induced from the linear action of $\mathrm{SL}(U)$ on $\bigwedge^q(U\otimes W)$ via the Plücker embedding $G_q(U\otimes W)\subset \mathbb{P}\bigwedge^q(U\otimes W)$. Therefore it makes sense to characterize the stable and semi-stable points of this action in the sense of Mumford, **[Mu-Fo]**, see also **[Ne1]**. The result is

5.1.1. Proposition (Stability criterion). *Let* $\mathrm{SL}(U)$ *act on the Grassmannian* $G_q(U\otimes W)$ *as above. Then a point* $L\in G_q(U\otimes W)$ *is stable (semi-stable) if and only if*

$$\dim L\cap(U'\otimes W) \underset{(\leq)}{<} \frac{\dim U'}{\dim U}\dim L$$

for any proper subspace $0\neq U'\subsetneqq U$.

Proof. Let λ be a (non-trivial) 1-parameter subgroup of $\mathrm{SL}(U)$. In a basis $(e_0,\dots,e_l)$ of U, λ is given by the diagonal matrix $\lambda(t)=[t^{\alpha_0},\dots,t^{\alpha_l}]$, $\alpha_0\geq\alpha_1\geq\cdots\geq\alpha_l$, $\sum_k\alpha_k=0$.

Let $v_0,\dots,v_n$ be any basis of W. Consider the basis $\{e_i\otimes v_j\}$ of $U\otimes W$. Since the action of $\mathrm{SL}(U)$ on $U\otimes W$ is given by $g(u\otimes w)=gu\otimes w$ for $u\in U$, $w\in W$, the action of λ on $U\otimes W$ is given by the diagonal matrix

$$[t^{\alpha_0},\dots,t^{\alpha_0},t^{\alpha_1},\dots,t^{\alpha_1},\dots,t^{\alpha_l},\dots,t^{\alpha_l}]$$

(each t^{α_j} occurring $(n+1)$ times), with respect to the basis $f_0=e_0\otimes v_0,\dots,f_n=e_0\otimes v_n$; $f_{n+1}=e_1\otimes v_0,\dots,f_{2n+1}=e_1\otimes v_n$; …; $f_{ln+1}=e_l\otimes v_0,\dots,f_{(l+1)n+l}=e_l\otimes v_n$.

We now use **[Mu-Fo**, p. 88, $(***)_N$ or **Ne1**, p. 121] with $N=1$ to get

$$\mu(L,\lambda)=-q\cdot r_{(l+1)n+l}+\sum_{i=0}^{(l+1)n+1}\dim(L\cap L_i)(r_{i+1}-r_i)$$

where L_i is the subspace spanned by $(f_0,\dots,f_i)$. In our case

$$r_0=\cdots=r_n=\alpha_0;\quad r_{n+1}=\cdots=r_{2n+1}=\alpha_1;$$
$$r_{ln+1}=\cdots=r_{(l+1)n+l}=\alpha_l.$$

Hence

$$(*)\qquad \mu(L,\lambda)=-q\alpha_l+\sum_{k=0}^{l-1}\dim(L\cap L_{(k+1)n_k})(\alpha_{k+1}-\alpha_k).$$

As in [**Ne1**, p. 121] we consider the cases

$\alpha_0 = \cdots = \alpha_p = l-p$; $\alpha_{p+1} = \cdots = \alpha_l = -(p+1)$ for $0 \le p < l-1$.

Here $\mu(L, \lambda) = q(p+1) - \dim(L \cap L_{(p+1)n+p})(l+1)$. But $L_{(p+1)n+p} = U'_p \otimes W$, where U'_p is the span (in U) of $e_0, \dots, e_p$.

Thus $\mu(L, \lambda) = q(p+1) - \dim(L \cap (U'_p \otimes W))(l+1)$.

Thus $\mu(L, \lambda) > 0$ (resp. ≥ 0) if and only if

$$\dim(L \cap (U'_p \otimes W)) < \frac{l+1}{p+1} \cdot q \qquad (\text{resp. } \le)$$

for $0 \le p < l$. Since every $(p+1)$-dimensional subspace of U is conjugate under $\mathrm{SL}(U)$ to U'_p the result follows. In 6.3 we need

5.1.2. PROPOSITION. *Let $U = k^2$, $\dim W = 3$ and let $M \in G_2(k^2 \otimes W)$ and $\Gamma \in G_4(k^2 \otimes W)$ with $M \subset \Gamma$. Then*

(1) *The pair $(M, \Gamma) \in G_2 \times G_4$ is semi-stable if and only if M and Γ are semi-stable in G_2 and G_4 respectively.*

(2) *If (M, Γ) is semi-stable and one of the components is stable, then (M, Γ) is stable.*

Proof. (1) Let $\lambda(t) = \begin{pmatrix} t^{\alpha_0} & \\ & t^{\alpha_1} \end{pmatrix}$ be a 1-parameter subgroup with $\alpha_0 + \alpha_1 = 0$, $\alpha_1 < 0$. As in the proof of 5.1.1 we have

$$\mu(M, \lambda) = \alpha_1(-\dim M + 2\dim M \cap (e_0 \otimes W)),$$
$$\mu(\Gamma, \lambda) = \alpha_1(-\dim \Gamma + 2\dim \Gamma \cap (e_0 \otimes W)).$$

Since the action on the product is linearised via the tensor-product $\bigwedge^2(k^2 \otimes W) \otimes \bigwedge^4(k^2 \otimes W)$ we must have

$$(*) \qquad \mu(M, \Gamma, \lambda) = \mu(M, \lambda) + \mu(\Gamma, \lambda).$$

From this it is clear that (M, Γ) is semi-stable if both components are semi-stable. To prove the converse, let (M, Γ) be semi-stable. Then we have

$$\dim M \cap (e_0 \otimes W) \dim \Gamma \cap (e_0 \otimes W) \le \tfrac{1}{2}(\dim M + \dim \Gamma) \le 3.$$

If M were not semi-stable, $\dim M \cap e_0 \otimes W = 2$ or $M \subset e_0 \otimes W$. By the inclusion $M \subset \Gamma$ also $\dim \Gamma \cap (e - 0 \otimes W) \ge 2$, but by the previous inequality this should be ≤ 1. Similarly if Γ were not semi-stable, $\dim \Gamma \cap (e_0 \otimes W) \ge 3$ and then $M \cap (e_0 \otimes W) = 0$. However this is not possible since $\dim M = 2$.

(2) Follows directly from the formula $(*)$.

From now on W will be 3-dimensional and $\mathbb{P}_2 = \mathbb{P}W$.

5.2. *The conic* $S(\Gamma)$. Let $\Gamma \subset k^2 \otimes W$ be a 4-dimensional subspace, $\dim W = 3$. The description of the conic $S(\Gamma)$ in 1.3 motivates the following definition

$$S(\Gamma) := \{\langle w \rangle \in \mathbb{P}W \mid u \otimes w \in \Gamma \text{ for some } u \neq 0 \text{ in } k^2\}.$$

For arbitrary Γ this set could be the whole plane, for example if $u \otimes W \subset \Gamma$ for some u.

5.2.1. Proposition. (1) *For* $\Gamma \in G_4(k^2 \otimes W)$ *the following conditions are equivalent*:

(a) Γ *is semi-stable.*
(b) $S(\Gamma)$ *is a conic.*
(c) $S(\Gamma) \neq \mathbb{P}W$.

(2) Γ *is stable iff the conic* $S(\Gamma)$ *is regular.*

Proof. (1) If Γ is not semi-stable, by 5.1.1 there is some $0 \neq u \in k^2$ with $u \otimes W \subset \Gamma$ $(\dim(u \otimes W) \cap \Gamma \geq 3)$ and hence $S(\Gamma) = \mathbb{P}W$. This proves (c) $\Rightarrow$ (a). It remains to prove (a) $\Rightarrow$ (b). We consider the 2-dimensional kernel Σ in

$$0 \to \Sigma \to k^2 \otimes W^\vee \to \Gamma^\vee \to 0.$$

Since $G_4(k^2 \otimes W) \to G_2(k^2 \otimes W^\vee)$ is an SL(2)-equivariant isomorphism Σ is stable (semi-stable) iff Γ is stable (semi-stable). By the criterion 5.1.1 this means $\dim \Sigma \cap (u \otimes W^\vee) < 1$ (≤ 1) for any $u \neq 0$. If Σ is the image of the matrix $\Sigma^* = \left(\begin{smallmatrix} z & w \\ z' & w' \end{smallmatrix}\right) : k^2 \to k^2 \otimes W^\vee$, we find that Σ is semi-stable iff the determinant $zw' - z'w \neq 0$ in $S^2 W^\vee$, and this then defines a conic. To see that this is $S(\Gamma)$, we consider the diagram on $\mathbb{P}W$

$$\begin{tikzcd}
 & & 0 \arrow[d] & & & \\
 & 0 \arrow[r] \arrow[d] & k^2 \otimes \Omega^1(1) \arrow[r] \arrow[d] & \Gamma^\vee \otimes \mathcal{O} \arrow[r] \arrow[d, equal] & \mathscr{E} \arrow[r] & \mathcal{O} \\
0 \arrow[r] & \Sigma \otimes \mathcal{O} \arrow[r] \arrow[d] & k^2 \otimes W^\vee \otimes \mathcal{O} \arrow[r] \arrow[d] & \Gamma^\vee \otimes \mathcal{O} \arrow[r] & 0 & \\
 & k^2 \otimes \mathcal{O}(1) \arrow[r, equal] \arrow[d] & k^2 \otimes \mathcal{O}(1) \arrow[d] & & & \\
 & \mathscr{E} \arrow[d] & 0 & & & \\
 & {} & & & &
\end{tikzcd} \tag{9}$$

in which the two cokernels identify. The homomorphism $\Sigma \otimes \mathscr{O} \to \mathscr{k}^2 \otimes \mathscr{O}(1)$ is injective in the semi-stable case. By the fibre description of $\mathscr{k}^2 \otimes \Omega^1(1) \to \Gamma^\vee \otimes \mathscr{O}$, see 0.1, we find that $\operatorname{supp} \mathscr{C} = S(\Gamma)$, and the left column shows that this is a conic given by $\det \Sigma^* = 0$.

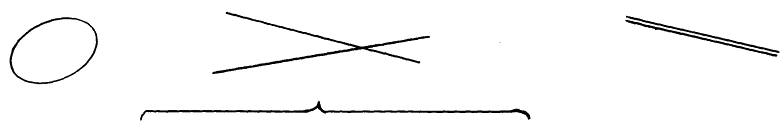
(2) Now it is easy to see that $S(\Gamma)$ is regular iff Σ^* has no zero as any entry in any equivalent representation. This is equivalent to $\Sigma \cap (u \otimes W^\vee) = 0$ for any $u \neq 0$, i.e. Σ stable.

5.2.2. Remark. (1) If $S(\Gamma)$ is regular then $\mathscr{C} \simeq \mathscr{O}_S(3)$ of degree 3 on S. In the following $S(\Gamma)$ shall always be given the structure of the equation $\det \Sigma^* = 0$.

(2) $S(\Gamma)$ is nothing but the determinantal variety of the homomorphism of the top row.

5.3. *Normal forms.* In the following tableau the matrices are Σ^*, Γ^* defining Σ, Γ as images of $\mathscr{k}^2 \to \mathscr{k}^2 \otimes W^\vee$, $\mathscr{k}^4 \to \mathscr{k}^2 \otimes W$ respectively, and $e_0, e_1, e_2 \in W$, $z_0, z_1, z_2 \in W^\vee$ are dual bases.

S(Γ)



	(I)	(II')	(II)	(II")	(III)
Σ^*	$\begin{bmatrix} z_2 & -z_1 \\ -z_1 & z_0 \end{bmatrix}$	$\begin{bmatrix} z_2 & -z_1 \\ 0 & z_0 \end{bmatrix}$	$\begin{bmatrix} z_2 & 0 \\ 0 & z_0 \end{bmatrix}$	$\begin{bmatrix} z_2 & 0 \\ -z_1 & z_0 \end{bmatrix}$	$\begin{bmatrix} z_2 & -\alpha z_1 - \beta z_0 \\ 0 & z_0 \end{bmatrix}$
Γ^*	$\begin{bmatrix} e_0 & 0 \\ e_1 & e_0 \\ e_2 & e_1 \\ 0 & e_2 \end{bmatrix}$	$\begin{bmatrix} e_0 & 0 \\ e_1 & 0 \\ e_2 & e_1 \\ 0 & e_2 \end{bmatrix}$	$\begin{bmatrix} e_0 & 0 \\ e_1 & 0 \\ 0 & e_1 \\ 0 & e_2 \end{bmatrix}$	$\begin{bmatrix} e_0 & 0 \\ e_1 & e_0 \\ 0 & e_1 \\ 0 & e_2 \end{bmatrix}$	$\begin{bmatrix} e_0 & 0 \\ e_1 & 0 \\ \alpha e_2 & e_1 \\ \beta e_2 & e_0 \end{bmatrix}$
	stable case	semi-stable cases			

5.3.1. Lemma. *If $\Gamma \in G^4(\mathscr{k}^2 \otimes W)$ is semi-stable it can be presented by one of the normal forms in the above tableau.*

The proof follows immediately if we choose the bases of $W^\vee$ so that $S(\Gamma)$ is defined by the matrices Σ^* of the form given.

5.4. Remark. Note that the normal forms of type II$'$, II, II$''$ give the same conic although they are not on the same orbits under SL(2).

However the orbit of Γ of type II is in the closure of that of type II$'$ or II$''$ in the tableau. For if we consider the 1-parameter subgroup $\left(\begin{smallmatrix}\alpha & \\ & \alpha^{-1}\end{smallmatrix}\right)$, its action on the Γ of type II$'$ is given by

$$\operatorname{Im}\begin{bmatrix} e_0 & \\ e_1 & \\ e_2 & e_1 \\ & e_2 \end{bmatrix}\begin{bmatrix}\alpha & \\ & \alpha^{-1}\end{bmatrix} = \operatorname{Im}\begin{bmatrix} \alpha e_0 & \\ \alpha e_1 & \\ \alpha e_2 & \alpha^{-1}e_1 \\ & \alpha^{-1}e_2 \end{bmatrix} = \operatorname{Im}\begin{bmatrix} e_0 & \\ e_1 & \\ \alpha^2 e_2 & e_1 \\ & e_2 \end{bmatrix}$$

and the latter tends to the direct sum as $\alpha \to 0$.

5.5. *An unusual parametrisation of* $\mathbb{P}_5$. By the previous results we have a morphism $G_4(\not{k}^2 \otimes W)^{ss} \to \mathbb{P}S^2W^\vee$ given by $\Gamma \to \langle \det \Sigma^* \rangle$, where Σ^* is any matrix defining Σ. This morphism factors through the good quotient, **[Mu-Fo]**, **[Ne1]**, $G_4(\not{k}^2 \otimes W)^{ss} // \mathrm{SL}(2) \to \mathbb{P}S^2W^\vee$ and we have the

5.5.1. PROPOSITION. $G_4(\not{k}^2 \otimes W)^{ss} // \mathrm{SL}(2) \simeq \mathbb{P}S^2W^\vee$ *is an isomorphism.*

Proof. Indeed this is a bijective morphism, which follows from the listing of the normal forms above. Since $\mathbb{P}DS^2W^\vee$ is smooth and the quotient is irreducible and reduced, it must be an isomorphism by Zariski's main theorem.

5.6. In order to obtain a similar *parametrisation of the Hilbert scheme* of all conics in $\mathbb{P}_5 = \mathbb{P}\bigwedge^2 V$ and later in $\mathbb{G} = G_2V \subset \mathbb{P}\bigwedge^2 V$ we use the above parametrisation for each plane $P \subset \mathbb{P}_5$ and let the planes vary in the Grassmannian $G_3 \bigwedge^2 V$. So we consider the tautological bundle

$$\mathbb{W} \to G_3 \bigwedge^2 V = Z.$$

We denote by $W = W_z \subset \bigwedge^2 V$ the 3-dimensional subspace given by $z \in Z$, see 3.10. As in the absolute case there is the induced action of $\mathrm{SL}(2)$ on the Grassmann bundle

$$G_4(\not{k}^2 \otimes \mathbb{W}) \to Z.$$

This can be linearised as follows. Since $\mathbb{W} \subset Z \times \bigwedge^2 V$ we get the embedding

$$G_4(\not{k}^2 \otimes W) \subset Z \times G_4\left(\not{k}^2 \otimes \bigwedge^2 V\right) \subset Z \times \mathbb{P}\bigwedge^4\left(\not{k}^2 \otimes \bigwedge^2 V\right)$$

as the subvariety of pairs (z, Γ) satisfying $\Gamma \subset k^2 \otimes \bigwedge^2 V$. Since Z is not affected by the action, it is enough to consider the linear action on $\bigwedge^4(\mathscr{k}^2 \otimes \bigwedge^2 V)$. Therefore the relative statement on stability reads the same way as that in 5.1.1, 5.1.2.

5.6.1. PROPOSITION. *For $(z, \Gamma) \in G_4(\mathscr{k}^2 \otimes \mathbb{W})$ the following conditions are equivalent*:

(i) (z, Γ) *is stable (semi-stable).*
(ii) $\dim \Gamma \cap (u \otimes W_z) \leq 1$ (≤ 2) *for any* $0 \neq u \in \mathscr{k}^2$.
(iii) $S(\Gamma) \subset \mathbb{P}W_z$ *is a regular conic (conic).*

Also we obtain analogously

5.6.2. PROPOSITION. *The map $(z, \Gamma) \to S(\Gamma)$ induces an isomorphism $G_4(\mathscr{k}^2 \otimes \mathbb{W})^{ss}//\mathrm{SL}(2) \simeq \mathbb{P}S^2\mathbb{W}^\vee$ of the good quotient with the Hilbert scheme $C(\mathbb{P}\bigwedge^2 V) = \mathbb{P}S^2\mathbb{W}^\vee$ of conics in $\mathbb{P}\bigwedge^2 V$.*

5.7. *Parametrisation of the Hilbert scheme $C(\mathbb{G})$.* Since we are only interested in conics contained in the Plücker quadric $\mathbb{G} = G_2V \subset \mathbb{P}\bigwedge^2 V$, we have to characterise those (z, Γ) for which the conic $S(\Gamma) \subset \mathbb{G} \cap \mathbb{P}W_z$ scheme-theoretically. Note that by this, the case where $\mathbb{G} \cap \mathbb{P}W_z$ is a pair of lines and $S(\Gamma)$ is a double line with $S(\Gamma)_{\mathrm{red}}$ being one of the lines, is excluded. To do this we consider the quadratic forms:

$$\sigma(\Gamma) = \det \Sigma^* \in S^2 W_z \text{ determined up to a scalar},$$
$$\rho(z)k = \text{quadratic form } \in S^2 W_z \text{ of } \mathbb{G} \cap \mathbb{P}W_z.$$

Note that $\sigma(\Gamma) = 0$ if Γ is not semi-stable, and $\rho(z) = 0$ if $\mathbb{P}W_z \subset \mathbb{G}$. The condition $S(\Gamma) \subset \mathbb{G} \cap \mathbb{P}W_z$ is now expressed by $\rho(z) \wedge \sigma(\Gamma) = 0$, which is well defined. Next we consider the open subset

$$G_2^0(\mathscr{k}^2 \otimes V)$$

of the Grassmannian of right monads N as in §1, s.t. the morphism $\mathscr{k}^2 \otimes \Omega^1(1) \to N^\vee \otimes \mathscr{O}$ is surjective, see 1.2. Recall that then N defines a 4-dimensional kernel Γ by (8) and a regular conic section $S(\Gamma) = \mathbb{G} \cap \mathbb{P}W_z$ by 1.3. Therefore we have an equivariant morphism

$$G_2^0(\mathscr{k}^2 \otimes V) \xrightarrow{\varepsilon} G_4(\mathscr{k}^2 \otimes \mathbb{W})^s.$$

Now we define

$$Y \subset G_4(\mathbb{A}^2 \otimes \mathbb{W})$$

as the closure of the image of ε, which is an imbedding. Then Y is 12-dimensional and irreducible. Also Y is invariant under SL(2) since the image of ε is. Therefore we have the induced action and linearisation and

$$Y^{\mathrm{ss}} = Y \cap G_4(\mathbb{A}^2 \otimes \mathbb{W})^{\mathrm{ss}}.$$

Since Y is defined as the closure, the condition $\sigma(\Gamma) \wedge \rho(z) = 0$ is satisfied for any $(z, \Gamma) \in Y^{\mathrm{ss}}$ and therefore the conic $S(\Gamma)$ is contained in $\mathbb{G} \cap \mathbb{P}W_z$ as a subscheme. Therefore there is the morphism

$$Y^{\mathrm{ss}} \to C(\mathbb{G}) \subset \mathbb{P}S^2\mathbb{W}^\vee$$

given by $\sigma(\Gamma)$. This factorizes through the good quotient $Y^{\mathrm{ss}}//\,\mathrm{SL}(2)$.

5.7.1. Proposition. $Y^{\mathrm{ss}}//\,\mathrm{SL}(2) \to C(\mathbb{G})$ *is an isomorphism.*

Proof. Clearly the morphism is surjective, since its image contains that of $G_2^0(\mathbb{A}^2 \otimes V)$, which is dense, and since the quotient is projective. It is also injective: $\mathbb{P}W_z$ is determined by $S(\Gamma)$ and also the equivalence class of $\Gamma \subset \mathbb{A}^2 \otimes W_z$ by the normal forms, see 5.4. Since the quotient is also integral, **[Mu-Fo]**, and $C(\mathbb{G})$ is smooth, it must be an isomorphism by Zariski's main theorem.

5.7.2. Remark. One can even show that Y^{ss} is smooth, whereas Y is singular. To do this, consider the subvariety $Y' \subset G_4(\mathbb{A}^2 \otimes \mathbb{W})$ defined by $\sigma(\Gamma) \wedge \rho(z) = 0$. We have $Y \subset Y'$ and $Y^{\mathrm{ss}} \subset Y'^{\mathrm{ss}}$, Y' being also invariant. Now one can show that $\dim T_p Y'^{\mathrm{ss}} = 12$ for any $p \in Y'^{\mathrm{ss}}$. One has to consider the different types of points, choose appropriate bases of Γ and the bundle W, and to use local coordinates of the Grassmannian $G_3 \bigwedge^2 V$. It turns out that the Jacobian matrix of the equations of Y' always has rank 5 in the 17-dimensional manifold $G_4(\mathbb{A}^2 \otimes \mathbb{W})^{\mathrm{ss}}$. Therefore for any $p \in Y^{\mathrm{ss}}$

$$12 = \dim_p Y^{\mathrm{ss}} \leq \dim_p Y'^{\mathrm{ss}} \leq \dim T_p Y'^{\mathrm{ss}} = 12.$$

Hence Y^{ss} is smooth and defines a component of Y'. However we don't need this result.

5.8. *The morphism* $Y^{ss} \to G_2^{ss}(k^2 \otimes V)$. If $\Gamma \in G_4(k^2 \otimes \bigwedge^2 V)$ we can consider the induced homomorphism

$$\begin{array}{ccccc} k^2 \otimes \bigwedge^2 V^\vee & \to k^2 \otimes \bigwedge^2 V^\vee \otimes V^\vee \to & \Gamma^\vee \otimes V^\vee \\ \wr\| & & \wr\| \\ k^2 \otimes V & \xrightarrow[h(\Gamma)]{\qquad\qquad} & \Gamma^\vee \otimes \bigwedge^3 V \end{array} \tag{10}$$

5.8.1. LEMMA. *If* $(z, \Gamma) \in Y^{ss}$ *then* $N = \operatorname{Ker} h(\Gamma)$ *is 2-dimensional and semi-stable.*

By this we obtain a morphism $Y^{ss} \xrightarrow{\nu} G_2^{ss}(k^2 \otimes V)$. It is now easy to see that the morphism ε of 5.7 is a *section* of ν over $G_2^0(k^2 \otimes V)$, since for regular N the space Γ defined through (8) is stable with $N \subset \operatorname{Ker} h(\Gamma)$ and thus $N = \operatorname{Ker} h(\Gamma)$.

Proof of the Lemma. (a) If the conic $S(\Gamma)$ is regular, then Γ is presented by a matrix

$$k^4 \xrightarrow[\begin{bmatrix} \xi & 0 \\ \omega & \xi \\ \eta & \omega \\ 0 & \eta \end{bmatrix}]{\qquad\qquad} k^2 \otimes W_z, \quad W_z \subset \bigwedge^2 V,$$

such that each vector $s^2\xi + st\omega + t^2\eta \in \bigwedge^2 V$ is decomposable, since $S(\Gamma) \subset \mathbb{G}$. This is equivalent to $\xi \wedge \xi = 0$, $\eta \wedge \eta = 0$, $\xi \wedge \omega = 0$, $\eta \wedge \omega = 0$ and $\omega \wedge \omega + 2\xi \wedge \eta = 0$.

If $S(\Gamma) = \mathbb{G} \cap \mathbb{P}W_z$ is a regular conic section, there is nothing to prove, because then N must be given as in 1.2. If $\mathbb{P}W_z \subset \mathbb{G}$ is an α-plane, then there is a vector $x \in V$ with $\xi = x \wedge x'$, $\eta = x \wedge y$, $\omega = x \wedge y'$ for some $x', y, y' \in V$. These vectors must form a basis, since ξ, ω, η are independent. Now we see that N must be presented by

$$k^2 \xrightarrow[\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}]{\qquad\qquad} k^2 \otimes V,$$

which is semi-stable.

(b) If $\mathbb{P}W_z \subset \mathbb{G}$ is a β-plane, we must have $\xi, \omega, \eta \in \bigwedge^2 H$, $H \subset V$, s.t. also $\omega \wedge \omega = 0$ and hence all products are zero. Since ξ, ω, η are independent, we find a basis x, x', y of H s.t. $\xi = x \wedge x'$, $\omega = -x' \wedge y$, $\eta = y \wedge x$. In this case the kernel is represented by

$$k^2 \xrightarrow[\begin{bmatrix} x & y \\ x' & x \end{bmatrix}]{} k^2 \otimes V$$

which is stable in this case.

(c) If $S(\Gamma) = \mathbb{G} \cap \mathbb{P}W_z$ is a pair of lines, Γ must be presented in normal form, see 5.3,

$$\begin{bmatrix} \xi & 0 \\ \omega & 0 \\ \eta & \omega \\ 0 & \eta \end{bmatrix},$$

where the lines in $\mathbb{G}$ are parametrised by $\langle s\xi + t\omega \rangle$ and $\langle s\omega + t\eta \rangle$. Then $\xi = x \wedge x'$, $\omega = x \wedge y'$, $\eta = y \wedge y'$ for some vectors x, y, y', which are independent. Then the kernel is represented by

$$k^2 \xrightarrow[\begin{bmatrix} x & 0 \\ y & y' \end{bmatrix}]{} k^2 \otimes V$$

which again is semi-stable.

(d) All other cases are treated analogously.

5.9. REMARK. One can consider $Y^{ss} \to G_2^{ss}(k^2 \otimes V)$ as a Schubert-type blow up by filling in the 4-dimensional subspaces

$$\Gamma \subset \operatorname{Ker}\left(k^2 \otimes \bigwedge^2 V \to N^\vee \otimes \bigwedge^3 V\right).$$

If we go to the quotients we get a modification

$$\begin{array}{ccc} Y^{ss}//\mathrm{SL}(2) & \longrightarrow & G_2^{ss}(k^2 \otimes V)//\mathrm{SL}(2) \\ \| & & \| \\ C(\mathbb{G}) & \longrightarrow & R \end{array}$$

It is not difficult to see that R is a ramified cover of the space $\mathbb{P}S^2V$ of all quadrics in $\mathbb{P}V^\vee$ by looking at $[N] \to \langle \det N^* \rangle$. Away from the singular quadrics this is the 2-sheeted cover of regular reguli by distinguishing a system of lines in a quadric, see Remark 1.4. The transposition

$$\begin{bmatrix} x & x' \\ y & y' \end{bmatrix} \longrightarrow \begin{bmatrix} x & y \\ x' & y' \end{bmatrix}$$

of the parametrising matrices defines the decktransformation. The space R is the minimal completion of this covering. By the above modification $C(\mathbb{G}) \to R$ it is possible to extend the quadric bundle of Poncelet conics.

6. Geometric invariant parametrisation of the quadric bundle $Q \to C(\mathbb{G})$.

6.1. As in 5.6 the Grassmann bundle $G_2(\mathscr{k}^2 \otimes \mathbb{W}) \to Z = G_3 \bigwedge^2 V$ can be described as the flag variety of pairs $(z, M) \in Z \times G_2(\mathscr{k}^2 \otimes \bigwedge^2 V)$ with $M \subset \mathscr{k}^2 \otimes W_z$. Analogously the induced group action of SL(2) can be linearised through the Plücker embedding by the action of $\bigwedge^2(\mathscr{k}^2 \otimes \bigwedge^2 V)$. Since Z is not affected, (z, M) is (semi-)stable iff $M \in G_2(\mathscr{k}^2 \otimes \bigwedge^2 V)$ is (semi-)stable. To each M we can also associate a quadratic form $\det M^* \in S^2 W_z$ where $M^*: \mathscr{k}^2 \to \mathscr{k} \otimes W_z$ represents M. As in 4.6 we obtain the

6.1.1. PROPOSITION. (1) *For each* $(z, M) \in G_2(\mathscr{k}^2 \otimes \mathbb{W})$ *the following are equivalent*:

(i) (z, M) *is semi-stable* (*stable*).

(ii) $\dim M \cap (u \otimes W_z) \leq 1$ $(= 0)$ *for any* $u \neq 0$.

(iii) $\det M^* \neq 0$ *in* $S^2 W_z$ ($\det M^* = 0$ *is the equation of a regular conic*).

6.2. By this result we obtain a morphism $(z, M) \to \langle \det M^* \rangle$

$$G_2^{ss}(\mathscr{k}^2 \otimes \mathbb{W}) // \mathrm{SL}(2) \to \mathbb{P}S^2\mathbb{W}.$$

This morphism is bijective and hence an isomorphism.

6.3. Let now $X \subset G_2(\mathscr{k}^2 \otimes \mathbb{W}) \times_Z G_4(\mathscr{k}^2 \otimes \mathbb{W})$ be the flag subvariety of the product bundle, defined as the set of all (z, M, Γ) with $(z, \Gamma) \in Y$ and $M \subset \Gamma$. If $Y \underset{\gamma}{\longrightarrow} G_4(\mathscr{k}^2 \otimes \bigwedge^2 V)$ is the canonical composition

$$Y \hookrightarrow G_4(\mathscr{k}^2 \otimes \mathbb{W}) \hookrightarrow Z \times G_4\left(\mathscr{k}^2 \otimes \bigwedge^2 V\right) \to G_4\left(\mathscr{k}^2 \otimes \bigwedge^2 V\right),$$

and if $\mathbb{T}$ is the tautological subbundle on the Grassmannian G_4, we obviously have $X = G_2\gamma^*\mathbb{T}$. Thus $X \to Y$ is a Grassmann bundle, $\dim X = 16$. The induced SL(2)-action on the product bundle leaves X invariant, and the action is again linearised via the embeddings

$$\begin{aligned} X &\subset G_2(\not{k}^2 \otimes \mathbb{W}) \times_Z G_4(\not{k}^2 \otimes \mathbb{W}) \\ &\subset Z \times G_2\left(\not{k}^2 \otimes \bigwedge^2 V\right) \times G_4\left(\not{k}^2 \otimes \bigwedge^2 V\right) \\ &\subset Z \times \mathbb{P}\bigwedge^2\left(\not{k}^2 \otimes \bigwedge^2 V\right) \times \mathbb{P}\bigwedge^4\left(\not{k}^2 \otimes \bigwedge^2 V\right) \\ &\subset Z \times \mathbb{P}\left(\bigwedge^2\left(\not{k}^2 \otimes \bigwedge^2 V\right) \otimes \bigwedge^4\left(\not{k}^2 \otimes \bigwedge^2 V\right)\right). \end{aligned}$$

Thus $(z, M, \Gamma) \in X$ is (semi-)stable iff $(M, \Gamma) \in G_2(\not{k}^2 \otimes W_z) \times G_4(\not{k}^2 \otimes W_z)$ is (semi-)stable. Fortunately we have the

6.3.1. LEMMA. (1) $(z, M, \Gamma) \in X$ *is semi-stable iff each component is semi-stable in* $G_2(\not{k}^2 \otimes W_z)$, $G_4(\not{k}^2 \otimes W_z)$ *respectively.*

(2) $(z, M, \Gamma) \in X^{ss}$ *is stable if one of the components is stable.*

(3) *There are stable pairs* $(z, M, \Gamma) \in X^s$ *without* M *and* Γ *being stable, see Remark* 6.7.2.

Proof. (1) and (2) had been proved in 5.1.2. In this proof it is possible that $\mu(M, \lambda)$ or $\mu(\Gamma, \lambda')$ are zero for different λ's but never simultaneously. In such a case (M, Γ) is stable but not M and $\mathbb{T}$. An example is provided by the pair $M \subset \Gamma$, defined as the images of the matrices

$$\begin{bmatrix} e_0 + e_2 & e_1 \\ 0 & e_2 \end{bmatrix} : \not{k}^2 \to \not{k}^2 \otimes W, \qquad \begin{bmatrix} e_0 & 0 \\ e_1 & 0 \\ e_2 & e_1 \\ 0 & e_2 \end{bmatrix} : \not{k}^4 \to \not{k}^2 \otimes W$$

respectively. Then (M, Γ) is stable but neither M nor Γ.

6.4. By the above lemma the projection $X \xrightarrow[\pi]{} Y$ maps $X^{ss} \to Y^{ss}$ and $X^{ss} \subset \pi^{-1}(Y^{ss})$. Since X^{ss} is open and $\pi^{-1}(Y^{ss})$ is irreducible as a bundle over Y^{ss}, also X^{ss} is irreducible and smooth.

6.5. PROPOSITION. *If* $(z, M, \Gamma) \in X$ *is semi-stable, the pair* $(\langle\sigma(\Gamma)\rangle,$ $\langle\det M^*\rangle) \in \mathbb{P}S^2W_z^\vee \times \mathbb{P}S^2W_z$ *is a Poncelet pair and determines a pair of Poncelet conics* $S(\Gamma) \subset \mathbb{P}W_z$, $C^\vee(M) \subset \mathbb{P}W_z^\vee$.

Proof. For the proof we use the normal forms of the spaces $\Gamma \subset k^2 \otimes W_z$ given in 5.3. Let $e_0, e_1, e_2 \in W_2$ and $z_0, z_1, z_2 \in W_z^\vee$ be dual bases, as in 5.3.

Case 1. $S(\Gamma)$ is regular and its form is $\sigma(\Gamma) = z_0z_2 - z_1^2$. Then with the notations of 3.3 the Poncelet quadric

$$Q_{\sigma(\Gamma)} = \{c_{00}c_{22} - c_{01}c_{12} + c_{02}c_{11} + c_{11}^2 = 0\} \subset \mathbb{P}S^2W_z.$$

We have to verify that $\det M^* = \sum_{i\le j} c_{ij}e_ie_j$ satisfies this condition. As $M \subset \Gamma$ is a 2-dim. subspace, we have

$$M^* = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix} \begin{bmatrix} e_0 & 0 \\ e_1 & e_0 \\ e_2 & e_1 \\ 0 & e_2 \end{bmatrix},$$

where we use the normal form of Γ^*. When $a_{ij} = \alpha_i\beta_j - \alpha_j\beta_i$ we get $c_{00} = a_{12}$, $c_{01} = a_{13}$, $c_{02} = a_{14} - a_{23}$, $c_{11} = a_{23}$, $c_{12} = a_{24}$, $c_{22} = a_{34}$. Inserted into the formula:

$$c_{00}c_{22} - c_{01}c_{12} + c_{02}c_{11} + c_{11}^2 = a_{12}a_{34} - a_{13}a_{24} + (a_{14} - a_{23})a_{23} + a_{23}^2 = 0$$

this expression vanishes because of the Plücker relation of the a_{ij}.

Case 2. $S(\Gamma)$ is a pair of lines, $z_0z_2 = 0$, and the matrix Γ^* has the form (II$'$) say. Now by 3.3

$$Q_{\sigma(\Gamma)} = \{c_{00}c_{22} = 0\} \subset \mathbb{P}S^2W_z$$

and from

$$M^* = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix} \begin{bmatrix} e_0 & \\ e_1 & \\ e_2 & e_1 \\ & e_2 \end{bmatrix}$$

we obtain $c_{00} = 0$. This shows that again $\langle\det M^*\rangle \in Q_{\sigma(\Gamma)}$. If Γ^* has the form (II$''$) we would get $c_{22} = 0$.

Case 3. $S(\Gamma)$ is a double line; this can be treated as case 2.

6.6. By the last proposition the morphism

$$X^{ss} \to \mathbb{P}S^2\mathbb{W}^\vee \times_Z \mathbb{P}S^2\mathbb{W},$$
$$(z, M, \Gamma) \to (\langle\sigma(\Gamma)\rangle, \langle\det M^*\rangle)$$

has its image contained in the Poncelet quadric bundle

$$\begin{array}{cl} Q & \subset \mathbb{P}S^2\mathbb{W}^\vee \times_Z \mathbb{P}S^2\mathbb{W} \\ \downarrow & \\ C(\mathbb{G}) & \end{array}$$

such that for the fibres over a point z we have the Poncelet bundle

$$Q_z \subset \mathbb{P}S^2 W_z^\vee \times \mathbb{P}S^2 W_z$$

of the plane $\mathbb{P}W_z$, see 3.3. Again we have

6.6.1. Proposition. *The induced morphism* φ

$$\begin{array}{ccc} X^{ss}//\mathrm{SL}(2) & \xrightarrow[\approx]{\varphi} & Q \\ \downarrow & & \downarrow \\ Y^{ss}//\mathrm{SL}(2) & \xrightarrow[\approx]{} & C(\mathbb{G}) \end{array}$$

is an isomorphism.

Proof. Clearly the morphism factorizes through the good quotient. The surjectivity can be shown directly by constructing $M \subset \Gamma$ for a pair of conics using the normal forms, or simply by remarking that it must have a dense image and that $X^{ss}//\mathrm{SL}(2)$ is projective, whereas Q is irreducible.

The morphism is also *injective*. Since we know this already for $Y^{ss}//\mathrm{SL}(2) \to C(\mathbb{G})$, we have to show injectivity in the fibres for fixed $\sigma(\Gamma)$. If $M_1, M_2 \subset \Gamma$ and $\langle\det M_1^*\rangle = \langle\det M_2^*\rangle$, we can assume

$$M_\nu^* = A_\nu \circ \Gamma^*$$

with the same Γ^*. When $a_{ij}^\vee$ are the Plücker coordinates of A_ν and c_{ij}^ν the coefficients of $\det M_\nu^*$, the formulas of the proof of 6.5 show that $a_{ij}^2 = \lambda a_{ij}^1$ in the different cases of $S(\Gamma)$. But this proves that the matrices A_1, A_2 span the same subspaces, i.e. $M_1 = M_2$.

Finally as in the previous cases $X^{ss}//\mathrm{SL}(2) \to Q$ must be an isomorphism, since the quotient is integral and Q is normal.

6.7. By the general theory of good quotients there are open sets $Q^s \subset Q$ and $C(\mathbb{G})^s \subset C(\mathbb{G})$ such that their inverse images in the parameter spaces are X^s and Y^s respectively. (However X^s is not mapped necessarily into Y^s, nor Q^s into $C(\mathbb{G})^s$.) But we know from Lemma 6.3.1 that $Q|C(\mathbb{G})^s \subset Q^s$, since the stable conics are the regular ones in $C(\mathbb{G})$. Therefore the semi-stable but non-stable points can only lie over $C(\mathbb{G}) \setminus C(\mathbb{G})^s$, i.e. in the fibres over the degenerate conics in G. It turns out that we even have:

6.7.1. THEOREM. *The non-stable points of Q are exactly the singular points of Q.*

The proof follows easily from the description of the singular points $(S, C^\vee) \in Q$ in 3.12 on the one hand, and from the characterisation of the semi-stable points $(z, M, \Gamma) \in X$ in 6.3.1 on the other hand. However one has to take special care of those points (z, M, Γ) which are stable without M, Γ being stable.

6.7.2. REMARK. The stable points (z, M, Γ) for which neither M nor Γ are stable correspond exactly to those pairs $(S, C^\vee)$ which are smooth points of Q but with both conics S and $C^\vee$ singular. The corresponding sheaves in $\overline{M(0, 2)}$ are stable and will be treated in 10.4, 10.7, see also 10.5.

6.8. *Points in X^{ss} parametrising bundle monads.* The projective variety X has been constructed in such a way that it completes the space of monads (D_2) of bundles in 2 and simultaneously serves as a parameter space of Q. We are going to identify the part of X^{ss} which consists of monads for bundles. Let Q^0 be the complement of the 4 Weil divisors Q_e, Q_α, Q_β, Q_0 in Q as defined in §4. Then Q^0 consists entirely of pairs of regular conics and maps onto $C^0(\mathbb{G})$. Let $X^0 \subset X^s$ resp. $Y^0 \subset Y^s$ be the inverse images of Q^0 resp. $C^0(\mathbb{G})$ in X resp. Y. We then have the diagram of SL(2)-quotients

$$\begin{array}{ccc} X^0 & \longrightarrow & Q^0 \\ \downarrow & & \downarrow \\ Y^0 & \longrightarrow & C^0(\mathbb{G}) \end{array}.$$

6.8.1. Lemma. (i) *The open and dense set* $X^0 \subset X^{\mathrm{ss}}$ *is the set of monads of bundles in* $M(0, 2)$.

(ii) *The open and dense set* $Y^0 \subset Y^{\mathrm{ss}}$ *is the set of right arrows of such monads, and is isomorphic to* $G_2^0(\mathscr{k}^2 \otimes V)$, *see Remark* 1.4.

Proof. If (M, N) denotes a monad (D_2) for a bundle, we have $N \in G_2^0(\mathscr{k}^2 \otimes V)$ and $M \subset \Gamma = \mathrm{Ker}(\mathscr{k}^2 \otimes \bigwedge^2 V \to N^\vee \otimes \bigwedge^2 V)$. The pair of conics (S, C) associated to the bundle and thus to (M, N) had been described in 2.4, see also 1.3, as $S = S(\Gamma) = \mathbb{G} \cap \mathbb{P}W$ and $C = \{\det M^* = 0\} \cap \mathbb{P}W$, where $\mathbb{P}W = \mathbb{P}W_z$ is the plane of S. Then $x = (z, M, \Gamma)$ is a point of X belonging to X^0, since S and the polar dual $C^\vee$ of C form by this description exactly the pair $(S, C^\vee)$ associated to x. If on the other hand $x = (z, M, \Gamma) \in X^0$ is given, then $y = (z, \Gamma) \in Y^0$ defines a regular plane conic section $S(\Gamma) = \mathbb{G} \cap \mathbb{P}W_z$. By the proof of 5.8.1 we find that there is an $N \in G_2^0(\mathscr{k}^0 \otimes V)$ defining Γ as its kernel. (In fact Y^0 is the image of the imbedding ε of 5.7.) Now the pair (M, N) is a bundle monad (D_2) by 2.4.2, since both conics $S(\Gamma)$, $C^\vee(M)$ are regular.

7. The universal kernel sheaf over Y^{ss}. By the construction of Y we have got the following diagram of morphisms

$$
\text{(11)} \qquad
\begin{tikzcd}
& Y^{\mathrm{ss}} \arrow[r, hook] \arrow[dl, "\nu"'] \arrow[d, "\gamma"] & G_4^{\mathrm{ss}}(\mathscr{k}^2 \otimes \mathbb{W}) \arrow[r, hook] & Z \times G_4^{\mathrm{ss}}(\mathscr{k}^2 \otimes \bigwedge^2 V) \arrow[dll, "\mathrm{pr}"] \\
G_2^{\mathrm{ss}}(\mathscr{k}^2 \otimes V) & G_4^{\mathrm{ss}}(\mathscr{k}^2 \otimes \bigwedge^2 V) & &
\end{tikzcd}
$$

Let $\mathbb{N}$ resp. $\mathbb{T}$ denote the tautological subbundles on the Grassmannians respectively. Let furthermore p and q denote the first and second projection of $\mathbb{P}_3 \times T$ for any second space T. We define the sheaves $\mathscr{N}$, $\mathscr{A}$ and $\mathscr{C}$ as kernel, image and cokernel of the composed homomorphism

$$\mathscr{k}^2 \otimes p^*\Omega^1(1) \hookrightarrow \mathscr{k}^2 \otimes V^\vee \otimes \mathscr{O}_{P_3 \times Y^{\mathrm{ss}}} \to q^*\nu^*\mathbb{N}^\vee,$$

which is derived from the imbedding $\Omega^1(1) \subset V^\vee \otimes \mathscr{O}_{\mathbb{P}_3}$ and the canonical epimorphism $\mathscr{k}^2 \otimes V^\vee \otimes \mathscr{O}_{G_2} \to \mathbb{N}^\vee$. Therefore we have the exact sequence

$$\text{(12)} \qquad 0 \to \mathscr{N} \to \mathscr{k}^2 \otimes p^*\Omega^1(1) \xrightarrow{\mathscr{A}} q^*\nu^*\mathbb{N}^\vee \to \mathscr{C} \to 0.$$

We can transform this sequence into the equivalent sequence $(12')$ by the same construction as in 5.2 (9), where $\mathbb{A}$ is the universal quotient on $G_2(\mathscr{k}^2 \otimes V)$,

$$(12') \qquad 0 \to \mathscr{N} \to q^*\nu^*\mathbb{A}^\vee \to \mathscr{k}^2 \otimes p^*\mathscr{O}_{\mathbb{P}_3}(1) \to \mathscr{C} \to 0.$$

If $y \in Y^{ss}$, we denote by

$$\mathscr{N}_y, \quad \mathscr{A}_y, \quad \mathscr{C}_y$$

the sheaves induced on the fibre $\mathbb{P}_3 \times \{y\} \simeq \mathbb{P}_3$ and call them the fibres of the sheaves respectively.

7.1. Proposition. (i) *The sheaves* $\mathscr{A}$ *and* $\mathscr{N}$ *are flat over* Y^{ss}.

(ii) $q_*\mathscr{N}(1) = \gamma^*\mathbb{T} \otimes \bigwedge^4 V^\vee$ *is locally free, and for a point* $y = (z, \Gamma) \in Y^{ss}$ *we have* $\Gamma \otimes \bigwedge^4 V^\vee = \Gamma\mathscr{N}_y(1)$.

(iii) $\operatorname{Supp}\mathscr{C}$ *is finite over the exceptional set* $E \subset Y^s$, *where* $E = \nu^{-1}(E_0)$, $E_0 = G_2^{ss}(\mathscr{k}^2 \otimes V) \setminus G_2^s(\mathscr{k}^2 \otimes V)$.

7.1.1. Remark. One could do the same construction of $\mathscr{N}$ and $\mathscr{A}$ over $G_2^{ss}(\mathscr{k}^2 \otimes V)$. However in this case $\mathscr{N}$ would not be flat. The modification $Y^{ss} \xrightarrow[\nu]{} G_2^{ss}$ is necessary to obtain a flat sheaf, and indeed our $\mathscr{N}$ can be considered the flattening of the corresponding sheaf over G_2^{ss}. On the other hand the modification ν was necessary in order to extend the quadric bundle of Poncelet conics across the boundary of the completion R, see 5.9.

Proof. (a) First we are going to show (iii). If $N = \nu(y)$ we have an exact sequence

$$0 \to \mathscr{K}_N \to \mathscr{k}^2 \otimes \Omega^1(1) \to N^\vee \otimes \mathscr{O}_{\mathbb{P}_3} \to \mathscr{C}_N \to 0$$

with $\mathscr{C}_N = \mathscr{C}_y$ and the

7.1.2. Lemma. (α) N *is stable iff* $\mathscr{C}_N = 0$.

(β) *If* $N \in E_0$ *then* $\mathscr{C}_N$ *is a skyscraper sheaf* $\mathscr{k}_x$ *or* $\mathscr{k}_x \oplus \mathscr{k}_y$ *or an extension* $0 \to \mathscr{k}_x \to \mathscr{C}_N \to \mathscr{k}_x \to 0$ *where* $\mathscr{k}_x = \mathscr{O}/\mathscr{m}(x)$ *on* $\mathbb{P}_3$.

Proof. (α) It follows from the stability criterion 5.1.1 that N is stable iff $N \cap (\xi \otimes V) = 0$ for any $\xi \in \mathscr{k}^2$. On the other hand $\mathscr{C}_N = 0$ iff $N \cap (\mathscr{k}^2 \otimes z) = 0$ for any $z \in V$, see proof of Theorem 2.1 and 0.2. It is immediate to see that these two conditions are equivalent.

(β) If N is only semi-stable it must be represented by a matrix $N^*\colon \mathscr{k}^2 \to \mathscr{k}^2 \otimes V$ of the form

$$\begin{bmatrix} x & 0 \\ y & x' \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x & 0 \\ 0 & x' \end{bmatrix}$$

up to equivalence, with $y \notin \operatorname{Span}(x, y')$. Now it is easy to verify that the cokernel of

$$(13) \qquad k^2 \otimes \Omega^1(1) \xrightarrow[N^{*\vee}]{} k^2 \otimes \mathcal{O} \to \mathcal{C}_N \to 0$$

is k_x or $k_x \oplus k_y$, or an extension of k_x by itself, by using the equivalent presentation

$$(13') \qquad k^6 \otimes \mathcal{O} \xrightarrow[A^{*\vee}]{} k^2 \otimes \mathcal{O}(1) \to \mathcal{C}_N \to 0,$$

where $A^{*\vee}$ can be determined as the kernel of $N^{*\vee}\colon k^2 \otimes V^\vee \to k^2$.

Now (iii) follows directly from this lemma.

(b) To prove (ii) we first remark that $q_* p^* \Omega^1(2) = \bigwedge^2 V^\vee \otimes \mathcal{O}_{Y^{ss}}$ and

$$q_*(p^*\mathcal{O}_{\mathbb{P}_3}(1) \otimes q^*\nu^*\mathbb{N}^\vee) = q_*p^*\mathcal{O}_{\mathbb{P}_3}(1) \otimes \nu^*\mathbb{N}^\vee = V^\vee \otimes \nu^*\mathbb{N}^\vee$$

by the projection formula. Therefore we obtain the exact sequence

$$0 \to q_*\mathcal{N}(1) \to k^2 \otimes \bigwedge^2 V^\vee \otimes \mathcal{O}_{Y^{ss}} \to V^\vee \otimes \nu^*\mathbb{N}^\vee.$$

On the other hand by the definition of ν we have

$$\Gamma \in \operatorname{Ker}\left(k^2 \otimes \bigwedge^2 V \to N^\vee \otimes \bigwedge^3 V\right) \quad \text{if } N = \nu(z, \Gamma),$$

and therefore, using $\bigwedge^i V^\vee \simeq \bigwedge^{4-i} V$,

$$\gamma^*\mathbb{T} \otimes \bigwedge^4 V^\vee \subset q_*\mathcal{N}(1).$$

The quotient sheaf $\mathcal{R} = q_*\mathcal{N}(1)/\gamma^*\mathbb{T} \otimes \bigwedge^4 V^\vee$ is supported on E, since for $N \notin E_0$ the space Γ is the kernel. But $\mathcal{R}$ is also a subsheaf of the quotient bundle $k^2 \otimes \bigwedge^2 V^\vee \otimes \mathcal{O}_{Y^{ss}}/\gamma^*\mathbb{T} \otimes \bigwedge^4 V^\vee$. Since Y^{ss} is irreducible, $\mathcal{R} = 0$.

(Remark. If we knew flatness already then the base change homomorphism $\Gamma \otimes \bigwedge^4 V^\vee \to \Gamma\mathcal{N}_y(1)$ would be an isomorphism already. We are going to prove this directly, which then implies flatness.)

(c) Lemma. *For any point* $y \in Y^{ss}$, $H^1(\mathbb{P}_3 \times U, m(y)\mathcal{N}(1)) = 0$ *for sufficiently small neighborhoods* $U(y) \subset Y^{ss}$ (*for the proof see* (e)).

Using this lemma we find that $\Gamma\otimes\bigwedge^4 V^\vee \to \Gamma\mathcal{N}_y(1)$ is onto by the diagram

$$\begin{tikzcd}
\Gamma(U, q_*\mathcal{N}(1)) \arrow[r] \arrow[d, equal] & \Gamma\mathcal{N}_y(1) \arrow[r] & 0 \\
\Gamma(U, \gamma^*\Gamma\otimes\bigwedge^4 V^\vee) \arrow[r] & \Gamma\otimes\bigwedge^4 V^\vee \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

On the other hand from

$$q^*\gamma^*\Gamma\otimes\bigwedge^4 V^\vee \underset{\approx}{\longrightarrow} q^*q_*\mathcal{N}(1) \to \mathcal{N}(1)$$

we get the diagram

$$\begin{tikzcd}
(q^*\gamma^*\Gamma\otimes\bigwedge^4 V^\vee)_y \arrow[r] \arrow[d, equal] & \mathcal{N}_y(1) \\
\Gamma\otimes\bigwedge^4 V^\vee\otimes\mathcal{O}_{\mathbb{P}_3} \arrow[r] & k^2\otimes\Omega^1(2) \arrow[u]
\end{tikzcd},$$

which induces the diagram on sections

$$\begin{tikzcd}
& \Gamma\mathcal{N}_y(1) \arrow[d] \\
\Gamma\otimes\bigwedge^4 V^\vee \arrow[ur] \arrow[r] \arrow[d, equal] & k^2\otimes\Gamma\Omega^1(2) \arrow[d, equal] \\
\Gamma \arrow[r, hook] & k^2\otimes\bigwedge^2 V.
\end{tikzcd}$$

This proves that $\Gamma\otimes\bigwedge^4 V^\vee \to \Gamma\mathcal{N}_y(1)$ is an isomorphism.

(d) *Proof of the flatness.* We put $\mathcal{T}or_i(\mathcal{F}, y) = \mathcal{T}or_i(\mathcal{F}, \mathcal{O}_y/m(y))$ for any $\mathcal{O}_{\mathbb{P}_3\times Y}$-module. From (12) we get the exact sequences, $N = \nu(y)$, on $\mathbb{P}_3$

$$0 \to \mathcal{T}or_1(\mathcal{C}, y) \to \mathcal{A}_y(1) \to N^\vee\otimes\mathcal{O}(1) \to \mathcal{C}_N \to 0,$$
$$0 \to \mathcal{T}or_1(\mathcal{A}(1), y) \to \mathcal{N}_y(1) \to k^2\otimes\Omega^1(2) \to \mathcal{A}_y(1) \to 0.$$

By the previous diagram $\Gamma\mathcal{N}_y(1) \to k^2\otimes\Gamma\Omega^1(2)$ is *injective*, and hence $\mathcal{T}or_1(\mathcal{A}(1), y)$ has no sections. But since $\mathcal{C}_N$ is a sky-scraper sheaf, also $\mathcal{T}or_1(\mathcal{A}(1), y)$ is a sky-scraper and hence must be zero. This proves that $\mathcal{A}$ is flat over Y^{ss}. Now also $\mathcal{N}$ must be flat over Y^{ss} since $p^*\Omega^1(1)$ is a bundle.

(e) *Proof of Lemma* (c). Since $\operatorname{Supp}\mathscr{C} \to Y^{\mathrm{ss}}$ is finite we have $H^i(\mathbb{P}_3 \times U, \mathscr{C}) = 0$ for $i > 0$, and the same is true for any coherent sheaf $\mathscr{F}$ with $\operatorname{Supp}\mathscr{F} \subset \operatorname{Supp}\mathscr{C}$. If $E \to F$ is a homomorphism of vector bundles there is the Eagon-Northcott complex

$$\cdots \to \bigwedge^{f+3} E \otimes S^2 F^\vee \to \bigwedge^{f+2} E \otimes F^\vee \to \bigwedge^{f+1} E \to E \otimes \bigwedge^{f} F \to F \otimes \bigwedge^{f} F \to 0$$

which is exact wherever $E \to F$ is onto. We consider this complex in the case $q^*\nu^*\mathbb{A}^\vee(-1) \to \mathscr{k}^2 \otimes \mathscr{O}_{\mathbb{P}_3 \times Y^{\mathrm{ss}}}$, see $(12')$, which has $\mathscr{N}(-1)$ as kernel.

Putting $q\mathscr{O}(d) = \mathscr{k}^q \otimes p^*\mathscr{O}_{\mathbb{P}_3}(d)$ for the moment, the Eagon-Northcott complex is locally over Y of the form

$$\xrightarrow{\alpha_4} q_3\mathscr{O}(-4) \xrightarrow{\alpha_3} q_2\mathscr{O}(-3) \xrightarrow{\alpha_2} 6\mathscr{O}(-1) \xrightarrow{\alpha_1} 2\mathscr{O} \to \mathscr{C} \to 0.$$

Let $\mathscr{Z}_i = \operatorname{Ker}\alpha_i$, $\mathscr{B}_i = \operatorname{Im}\alpha_{i+1}$, $\mathscr{C}_i = \mathscr{Z}_i/\mathscr{B}_i$, and let us write $H^j\mathscr{F}$ for $H^j(\mathbb{P}_3 \times U, \mathscr{F})$. Since the complex is exact away from $\operatorname{Supp}\mathscr{C}$, we have $\operatorname{Supp}\mathscr{C}_i \subset \operatorname{Supp}\mathscr{C}$ and $H^j\mathscr{C}_i = 0$ for $j > 0$.

Since $H^4\mathscr{F} = 0$ for any $\mathscr{F}$ and $H^3\mathscr{O}(-3) = 0$, we get the following chain of vanishings.

$$\begin{gathered} H^3\mathscr{Z}_3(2) = H^3\mathscr{B}_3(2) = 0, \quad H^2\mathscr{Z}_2(2) = H^2\mathscr{B}_2(2) = 0, \\ H^1\mathscr{B}_1(2) = 0, \quad H^1\mathscr{N}(1) = H^1\mathscr{Z}_1(2) = 0. \end{gathered}$$

By the same method we even get

$$H^i(\mathbb{P}_3 \times U, \mathscr{N}(d)) = 0 \quad \text{for } d \geq 2 - i, \ i > 0.$$

To get the vanishing of the lemma we consider a local resolution

$$m_2\mathscr{O}_y \xrightarrow{\beta_2} m_1\mathscr{O}_y \xrightarrow{\beta_1} m_0\mathscr{O}_y \xrightarrow{\beta_0} \mathscr{m}(y) \to 0$$

of the maximal ideal on $U \subset Y^{\mathrm{ss}}$ and put $\mathscr{K}_i = \operatorname{Ker}\beta_i$. We obtain the exact sequence

$$\begin{gathered} 0 \to \mathscr{N}_0 \longrightarrow \mathscr{m}(y) \otimes \mathscr{N}(1) \to \mathscr{m}(y)\mathscr{N}(1) \to 0 \\ 0 \to \mathscr{N}_1 \to \mathscr{K}_0 \otimes \mathscr{N}(1) \to m_0\mathscr{N}(1) \to \mathscr{m}(y) \otimes \mathscr{N}(1) \to 0 \\ 0 \to \mathscr{N}_2 \to \mathscr{K}_1 \otimes \mathscr{N}(1) \to m_1\mathscr{N}(1) \to \mathscr{K}_0 \otimes \mathscr{N}(1) \to 0, \end{gathered}$$

where the $\mathscr{N}_i$ are $\mathscr{T}\!or_1(\mathscr{N}(1), \mathscr{K}_{i-1})$. Since $\mathscr{N}$ is locally free outside $\operatorname{Supp}\mathscr{C}$, $\operatorname{Supp}\mathscr{N}_i \subset \operatorname{Supp}\mathscr{C}$ and again $H^j\mathscr{N}_i = 0$. As in the previous part we get from $H^i\mathscr{N}(1) = 0$ the desired vanishing

$$H^1(\mathbb{P}_3 \times U, m(y)\mathscr{N}(1)) = 0.$$

This *completes the proof* of the proposition. By the same calculation we even get

$$H^i(\mathbb{P}_3 \times U, m(y)\mathscr{N}(d)) = 0 \quad \text{for } i > 0,\ e \geq 3 - i.$$

Hence we have the

7.2. COROLLARY. *For any* $y \in Y^{ss}$, $H^i\mathscr{N}_y(d) = 0$ *for* $i > 0$, $d \geq 2 - i$.

Proof. We have

$$H^i(\mathbb{P}_3 \times U, \mathscr{N}(d)) \to H^i\mathscr{N}_y(d) \to H^{i+1}(\mathbb{P}_3 \times U, m(y)\mathscr{N}(d)).$$

7.3. *Evaluation map and the sheaf* $\mathscr{G}$. The isomorphism $\gamma^*\Gamma \otimes \bigwedge^4 V^\vee = q_*\mathscr{N}(1)$ induces the homomorphism (called evaluation map)

$$0 \to q^*\gamma^*\mathbb{T} \otimes \bigwedge^4 V^\vee \to \mathscr{N}(1) \to \mathscr{G}(1) \to 0$$

with cokernel $\mathscr{G}(1)$.

7.3.1. LEMMA. *The evaluation map is injective and* $\mathscr{G}$ *is flat over* Y^{ss}.

Proof. If $y \in Y^0 \simeq G_2^0(k^2 \otimes V) =$ set of regular bundle epimorphism, then $\mathscr{N}_y$ is a bundle and $\mathscr{G}_y$ is a line bundle on a quadric in $\mathbb{P}_3$. This proves that $\operatorname{rank}\mathscr{G}_y = 0$. Since $\operatorname{rank}\Gamma = 4 = \operatorname{rank}\mathscr{N}$, the kernel must have $\operatorname{rank} = 0$ and thus is 0. To show that $\mathscr{G}$ is flat we consider the sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{T}or_1(\mathscr{G}, y) \arrow[r] & \Gamma \otimes \bigwedge^4 V^\vee \otimes \mathscr{O} \arrow[r] \arrow[dr, "\varphi"'] & \mathscr{N}_y(1) \arrow[r] & \mathscr{G}_y(1) \arrow[r] & 0 \\
& & & k^2 \otimes \Omega^1(2), & &
\end{tikzcd}$$

where we have put $\mathscr{T}or_1(\mathscr{G}, y) = \mathscr{T}or_1(\mathscr{G}, \mathscr{O}_Y/m(y))$, as in 7.1.2, (d). Since φ is injective, the sheaf $\mathscr{T}or_1 = 0$. This proves flatness.

7.4. *The display of* $\mathscr{N}_y$. If $y \in Y^{ss}$ we write $\mathscr{T}_y = \mathscr{T}or_1(\mathscr{C}, y)$. Clearly $\operatorname{Supp}\mathscr{T}_y \subset \operatorname{Supp}\mathscr{C}_y$. From the defining sequence (12) we obtain the exact diagram for the fibre sheaves on $\mathbb{P}_3$:

$$
(14)\qquad
\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
 & 0 \arrow[d] & & \mathscr{T}_y \arrow[d] & & \\
0 \arrow[r] & \mathscr{N}_y \arrow[r] \arrow[d] & k^2 \otimes \Omega^1(1) \arrow[r] \arrow[d, equal] & \mathscr{A}_y \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & \mathscr{K}_n \arrow[r] \arrow[d] & k^2 \otimes \Omega^1(1) \arrow[r] & N^\vee \otimes \mathscr{O} \arrow[r] \arrow[d] & \mathscr{C}_n \arrow[r] \arrow[dl, equal] & 0 \\
 & \mathscr{T}_y \arrow[d] & & \mathscr{C}_y \arrow[d] & & \\
 & 0 & & 0 & &
\end{tikzcd}
$$

By the flatness of $\mathscr{A}$, $\chi\mathscr{A}_y(m) = 2\chi\mathscr{O}(m)$ is constant and we obtain $h^0\mathscr{T}_y = h^0\mathscr{C}_y$. In particular the skyscraper $\mathscr{T}_y \neq 0$ iff $\mathscr{C}_y \neq 0$.

7.5. *Zero sets of section of* $\mathscr{N}_y(1)$. We are now able to generalise the results for kernels of bundle monads in Lemma 1.3 to any of the sheaves $\mathscr{N}_y$.

7.5.1. Proposition. (1) *Let* $y \in Y^{ss}$ *and* $s \in \Gamma\mathscr{N}_y(1)$. *If the zero scheme* $Z(s)$ *is neither empty nor a point, it is a line belonging to the conic* $S(\Gamma)$.

(2) *The conic* $S(\Gamma)$ *is exactly the set of all zero lines of sections of* $\mathscr{N}_y(1)$.

Proof. Since supports of $\mathscr{C}_y$, $\mathscr{T}_y$ are 0-dimensional we find $\mathscr{E}xt^1(\mathscr{A}_y, \mathscr{O}) = 0$. It follows that $s \in \Gamma\mathscr{N}_y(1)$ has the same zero scheme as a section of $\mathscr{N}_y(1)$ and as a section of $k^2 \otimes \Omega^1(2)$, since $k^2 \otimes \Omega^1(1)^\vee \to \mathscr{N}_y(1)^\vee$ is onto. Let now $\gamma \in \Gamma \subset k^2 \otimes \bigwedge^2 V$, $\gamma = (\xi, \eta)$, correspond to s. A point $\langle z \rangle \in \mathbb{P}V$ is a zero of s iff $(\xi \wedge z, \eta \wedge z) = 0$. If now $Z(s)$ is not 0-dimensional, then ξ and η must be linearly dependent and by the definition of the conic $S(\Gamma)$ in 4.2 define a point of $S(\Gamma)$. Conversely any point of $S(\Gamma)$ comes from some $y \in \Gamma$ with a line as zero locus.

7.6. *Conjugate conic* S^0 *and* $R^1\mathscr{N}$. If $S \in C(\mathbb{G})$ is a conic we define the "conjugate" conic $S^0 \in C(\mathbb{G})$ as follows. If $S = \mathbb{G} \cap \mathbb{P}W$ we let $S^0 = \mathbb{G} \cap \mathbb{P}W^{\perp}$ where $W^{\perp}$ is the orthogonal of W with respect to the quadratic form of $\mathbb{G}$. Then S is regular iff S^0 is regular. If S is a double line then also S^0 is a double line with the same reduced line but with a different plane. If however $S \subset \mathbb{P}W \subset \mathbb{G}$ we define $S^0 = S$. It can be shown that this map $S \mapsto S^0$ is an involutive morphism of $C(\mathbb{G})$.

REMARK. $\Gamma \to \Gamma^0$ can be defined by continuity.

We are now going to generalise Proposition 1.5, (ii) to any kernel sheaf of our construction:

7.6.1. PROPOSITION. *Let* $y \in Y^{ss}$ *and* $S = S(\Gamma)$. *Then* $S^0 = \operatorname{Supp} R^1\mathscr{N}_y = \operatorname{Supp} R^1\mathscr{G}_y$ *as reduced schemes.*

Proof. Since it seems complicated to show that the family $R_1\mathscr{N}_y$ and their supports form a flat family, we proceed to calculate $R^1\mathscr{G}_y$ in the different cases of $S(\Gamma)$, which also gives a beautiful insight into the structure of those sheaves.

Case 1. $y \in Y^0$ *and defines a regular conic* $S(\Gamma) \in C^0(\mathbb{G})$ was treated in 1.5.

Case 2. y *defines a regular conic* $S(\Gamma) \subset \mathbb{P}W_z\mathbb{G}$ *in a* β*-plane.* Since the entries of Γ are decomposed we can choose a basis $e_0, e_1, e_2 \in W_z$ s.t. Γ is represented by the matrix

$$\Gamma^* = \begin{bmatrix} e_{01} & \\ e_{12} & e_{01} \\ e_{02} & e_{12} \\ & e_{02} \end{bmatrix}, \quad N^{\vee *} = \begin{bmatrix} e_0 & e_1 \\ -e_2 & e_0 \end{bmatrix},$$

where $e_{ij} = e_i \wedge e_j$, and then $N^{\vee}$ is represented by the matrix $N^{\vee *}$ in the above form, see (10) in 5.8. By 0.3 the homomorphism induced is surjective, i.e. $\mathscr{C}_y = 0$ and we have the exact sequence

$$0 \to \mathscr{N}_y \to k^2 \otimes \Omega^1(1) \xrightarrow[N^{\vee *}]{} k^2 \otimes \mathscr{O} \to 0.$$

If we apply $R^{\cdot}$ we get, see 0.4,

$$\mathscr{k}^2 \otimes Q^{\vee} \xrightarrow[N^{\vee *}]{} \mathscr{k}^2 \otimes \mathscr{O}_{\mathbb{G}} \to R^1\mathscr{N}_y \to 0.$$

We can now calculate this homomorphism as follows. Let p_{ij} be dual to e_{ij}, i.e. the homogeneous Plücker-coordinates of $\mathbb{P}\bigwedge^2 V$, and let $\mathbb{G}_{ij} = \{p_{ij} \neq 0\}$. If $x = \sum x_i e_i$ defines the homomorphism $Q^{\vee} \xrightarrow[x]{} \mathscr{O}_{\mathbb{G}}$ and if $Q^{\vee}|G_{ij} \simeq \mathscr{k}^2 \otimes \mathscr{O}_{\mathbb{G}}$ is trivialised, this homomorphism can be expressed by the matrix

$$\begin{bmatrix} x_k - \dfrac{P_{ik}}{P_{ij}}x_j + \dfrac{P_{jk}}{P_{ij}}x_i \\ x_l - \dfrac{P_{il}}{P_{ij}}l_j + \dfrac{P_{jl}}{P_{ij}}x_i \end{bmatrix} : 2\mathscr{O}_{\mathbb{G}} \to \mathscr{O}_{\mathbb{G}}.$$

This follows immediately if we choose a basis of $(V/U)^{\vee}$, the fibre of $Q^{\vee}$ at $U \in \mathbb{G}_{ij}$, where $p_{\mu\nu}$ are the Plücker coordinates of U, and k, l are complementary to i, j. If we choose for example $\mathbb{G}_{01}$, we get the homomorphism

$$\mathscr{k}^2 \otimes \mathscr{k}^2 \otimes \mathscr{O}_{\mathbb{G}} \to \mathscr{k}^2 \otimes \mathscr{O}_{\mathbb{G}} \to R^1\mathscr{N}_y \to 0,$$

$$\left[\begin{array}{c|c} p_{12} & -p_{02} \\ p_{13} & -p_{03} \\ \hline -1 & p_{12} \\ 0 & p_{13} \end{array}\right] \cdot p_{01}^{-1}.$$

The Fitting ideal of $R^1\mathscr{N}_y|\mathbb{G}_{01}$ therefore is generated by p_{13}, p_{03}, $p_{12}^2 - p_{02}$ which is the ideal of the conic $S(\Gamma)$ parametrised by $s^2e_{01} + ste_{12} + t^2e_{02}$. Since $S = S^0$ here, this settles Case 2.

Case 3. y defines a regular conic $S(\Gamma) \subset \mathbb{P}W_z \subset \mathbb{G}$ in an α-plane. Again upon choosing a basis of V we can assume that Γ, $N^{\vee}$ are represented by the matrices

$$\Gamma^* = \begin{bmatrix} e_{01} & \\ e_{03} & e_{01} \\ e_{02} & e_{02} \\ & e_{03} \end{bmatrix}, \quad N^{\vee *} = \begin{bmatrix} e_0 & \\ & e_0 \end{bmatrix}.$$

In this case $\mathscr{K}_N = \mathscr{k}^2 \otimes \mathscr{Z}$ where $\mathscr{Z}$ is the kernel in

(15) $$0 \to \mathscr{Z} \to \Omega^1(1) \xrightarrow[e_0]{} \mathscr{O} \to \mathscr{k}_{e_0} \to 0$$

and $\mathcal{N}_y$ is the kernel in

$$0 \to \mathcal{N}_y \to k^2 \otimes \mathcal{Z} \to k^2 \otimes k_{e_0} \to 0 \tag{16}$$

where we use that here $\mathcal{T}_{e_0} = k^2 \otimes k_{e_0}$.

To proceed further we have to digress into computations for the sheaf $\mathcal{Z}$. If we write x for e_0 and apply the same transformation as in 5.2 (9), we find that $\mathcal{Z}$ is the first syzygy of $m(x)(1)$, i.e. we have exact sequences

$$\begin{gathered} 0 \to \mathcal{Z} \to (V/x)^\vee \otimes \mathcal{O} \to \mathcal{M}(1) \to 0, \\ 0 \to \bigwedge^3 (V/x)^\vee \otimes \mathcal{O}(-2) \to \bigwedge^2 (V/x)^\vee \otimes \mathcal{O}(-1) \to \mathcal{Z} \to 0. \end{gathered} \tag{17}$$

From (17) we get $\Gamma\mathcal{Z}(1) = \bigwedge^2 (V/x)^\vee$ and that

$$\operatorname{Hom}(\mathcal{Z}, k_x) = \bigwedge^2 (V/x).$$

Hence if $\xi \in \bigwedge^2 (V/x)$ induces $\mathcal{Z} \xrightarrow[\xi]{} k_x$, the same element gives the induced homomorphism $\Gamma\mathcal{Z}(1) \xrightarrow[\xi]{} k$. Moreover, if we apply R^0 we get

$$\begin{tikzcd} R^0\mathcal{Z} \arrow[r, "R^0\xi"] \arrow[d, equal] & R^0 k_x \arrow[d, equal] \\ \bigwedge^3 (V/x)^\vee \otimes \mathcal{O}_{\mathbb{G}}(-1) \arrow[r] \arrow[d] & \mathcal{O}_{P_x} \arrow[dl, "\xi"] \\ \bigwedge^3 (V/x)^\vee \otimes \mathcal{O}_{P_x} & \end{tikzcd}$$

where $P_x = \mathbb{P}(V/x)$ is the α-plane of x and ξ is identified with an element of $(V/x)^\vee = \bigwedge^2 (V/x)$.

Now we are able to calculate $R^1\mathcal{N}_y$ in Case 3. First we determine the homomorphism in (16) by the induced sequence

$$\begin{tikzcd} 0 \arrow[r] & \Gamma\mathcal{N}_y(1) \arrow[r] \arrow[d, equal] & k^2 \otimes \Gamma\mathcal{Z}(1) \arrow[r] \arrow[d, equal] & k^2 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & k^4 \otimes \bigwedge^4 V^\vee \arrow[r, "\Gamma^*"'] & k^2 \otimes \bigwedge^2 (V/x)^\vee \arrow[r, "A"'] & k^2 \arrow[r] & 0 \end{tikzcd}.$$

It follows that A is the matrix

$$A = \begin{bmatrix} e_{12} & e_{13} \\ -e_{23} & e_{12} \end{bmatrix}.$$

Passing now to $R^0\mathscr{Z}$ we get the diagram

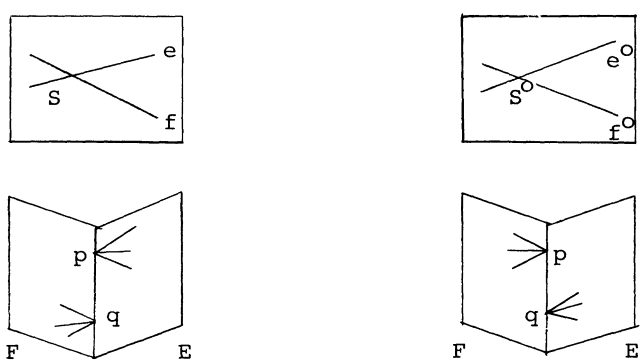
$$\begin{tikzcd}
k^2 \otimes R^0\mathscr{Z} \arrow[r] \arrow[d] & k^2 \otimes R^0 k_{e_0} \arrow[r] \arrow[d, equal] & R^1\mathscr{N}_y \arrow[r] \arrow[d, equal] & 0 \\
k^2 \otimes \mathscr{O}_{P_{e_0}}(-1) \arrow[r, "A"'] & k^2 \otimes \mathscr{O}_{P_{e_0}} \arrow[r] & R^1\mathscr{N}_y \arrow[r] & 0
\end{tikzcd}$$

Now under $\bigwedge^2(V/e_0) \simeq (V/e_0)^\vee$ we have $e_{12} \leftrightarrow e_3^\vee$, $e_{13} \leftrightarrow -e_2^\vee$, $e_{23} \leftrightarrow e_1^\vee$ and hence

$$\det A = e_3^{\vee 2} - e_1^\vee e_2^\vee.$$

But this is exactly the equation of the conic $S(\Gamma)$ in the α-plane $P_{e_0} = \mathbb{P}(V/e_0) = \mathbb{P}\langle e_{01}, e_{03}, e_{02}\rangle$ which is given by $s^2e_{01}+ste_{03}+t^2e_{02}$. Since here also $S = S^0$, this proves Case 3.

Case 4. y defines a pair of lines $S(\Gamma)$. We assume that $S(\Gamma)$ is a plane section as in (D), 0.4, since the other situations are only special cases of this. If S is the union of the two lines e, f which define the pencil of lines in E through p, F through q respectively, then S^0 consists of the lines e_0, f_0 which describe the pencil of lines in E through q, F through p, respectively.



Let $\mathscr{G}_y$ be the quotient of $\mathscr{N}_y$ s.t. $\operatorname{Supp}\mathscr{G}_y = E \cup F$ and $R^1\mathscr{N}_y = R^1\mathscr{G}_y$. It will be shown in 10.2 that $\mathscr{G}_y$ is an extension

$$0 \to m_E(q)(-1) \to \mathscr{G}_y \to m_F(p)(-1) \to 0$$

where $\mathscr{m}_F(p) \subset \mathscr{O}_F$ is the ideal sheaf of p in the plane F. Choosing two generating sections of $\mathscr{m}_F(p)$ we obtain the resolution

$$(18) \qquad 0 \to \mathscr{O}_F(-3) \xrightarrow[(a,b)]{} \mathscr{k}^2 \otimes \mathscr{O}_F(-2) \to \mathscr{m}_F(p)(-1) \to 0.$$

Since $R^1\mathscr{O}(-m-2) = S^2S \otimes \bigwedge^2 S$, where S is the tautological subbundle on $\mathbb{G}$, it is easy to derive that $R^1\mathscr{O}_F(-m-2)$ is the restriction of $R^1\mathscr{O}(-m-2)$ to the β-plane P_F of all lines in F, and that the homomorphism $\mathscr{O}_F(-3) \to \mathscr{O}_F(-2)$ becomes contraction with $a: S \otimes \bigwedge^2 S|P_F \to \bigwedge^2 S|P_F$. Hence from (18) we get the sequence

$$(19) \quad 0 \to S \otimes \mathscr{O}_{P_F}(-1) \xrightarrow[(a,b)]{} \mathscr{k}^2 \otimes \mathscr{O}_{P_F}(-1) \to R^1(\mathscr{m}_F(x)(-1)) \to 0.$$

This shows that the support of $R^1\mathscr{m}_F(p)(-1)$ is the line f^0 and that the homomorphism (a, b) must be injective. Indeed $R^1\mathscr{m}_F(p)(-1) = \mathscr{O}_{f^0}$, since $S|P_F \simeq \Omega^1_{P_F}(1)$ and (19) can be transformed into

$$0 \to \mathscr{O}_{P_F}(-1) \xrightarrow[\alpha]{} \mathscr{O}_{P_F} \to R^1\mathscr{m}_F(p)(-1) \to 0$$

as in 5.2 (9), where α is the equation of $f^0 \subset P_F$.

Similarly we obtain $R^1\mathscr{m}_E(q)(-1) = \mathscr{O}_{e^0}$.

Now we consider the resolution of $\mathscr{G}$ which can be constructed from the resolution of the ends.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{O}_E(-3) & \longrightarrow & \mathscr{O}_E(-3) \oplus \mathscr{O}_F(-3) & \longrightarrow & \mathscr{O}_F(-3) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{k}^2 \otimes \mathscr{O}_E(-2) & \longrightarrow & \mathscr{k}^2 \otimes \mathscr{O}_E(-2) \oplus \mathscr{k}^2 \otimes \mathscr{O}_F(-2) & \longrightarrow & \mathscr{k}^2 \otimes \mathscr{O}_F(-2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{m}_E(q)(-1) & \longrightarrow & \mathscr{G}_y & \longrightarrow & \mathscr{m}_F(p)(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

If we apply R^1 we find that the sequence

$$0 \to R^1\mathscr{m}_E(q)(-1) \to R^1\mathscr{G}_y \to R^1\mathscr{m}_F(p)(-1) \to 0$$

is exact because the left-hand side of (19) is injective. This proves that $R^1\mathscr{G}_y$ is an extension of $\mathscr{O}_{f^0}$ by $\mathscr{O}_{e^0}$, and that $\operatorname{Supp} R^1\mathscr{N}_y = S^0$.

8. The universal sheaf $\mathscr{F}$ over X^{ss}. Recall that in 6.3 we have defined X to be the flag subvariety

$$X \subset Z \times G_2\left(\mathscr{k}^2 \otimes \bigwedge^2 V\right) \times G_4\left(\mathscr{k}^2 \otimes \bigwedge^2 V\right)$$

of all triples (z, M, Γ) satisfying $(z, \Gamma) \in Y$ and $M \subset \Gamma \subset \mathscr{k}^2 \otimes W_z$. Let

$$\begin{tikzcd} & X \arrow[dl, "\mu"'] \arrow[dr, "\gamma"] \arrow[rr, "\pi"] & & Y \arrow[d, "\gamma_Y"] \\ G_2(\mathscr{k}^2 \otimes \bigwedge^2 V) & & & G_4(\mathscr{k}^2 \otimes \bigwedge^2 V) \end{tikzcd}$$

be the projections in $\mathbb{M}$, $\mathbb{T}$ the tautological subbundles respectively. By the definition of X we get the exact sequence

$$0 \to \mu^*\mathbb{M} \to \gamma^*\mathbb{T} \to \mathbb{B} \to 0$$

on X where $\mathbb{B}$ is defined as the quotient bundle. As before we denote by p, resp. q, the first and second projection of $\mathbb{P}_3 \times X^{ss}$, and we denote $(\mathrm{id} \times \pi)^*\mathscr{N}$ again by $\mathscr{N}$, so that we have $\mathscr{N}_x = \mathscr{N}_{\pi(x)}$ by abuse of notation and similarly for $\mathscr{G}$. Since we had the inclusion $q^*\gamma_y^*\mathbb{T} \hookrightarrow \mathscr{N}(1)$ we obtain the exact diagram (up to the factor $\otimes \bigwedge^4 V^\vee$ in the top row)

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & q^*\mu^*\mathbb{M} \arrow[r] \arrow[d, equal] & q^*\gamma^*\mathbb{T} \arrow[r] \arrow[d] & q^*\mathbb{B} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & q^*\mu^*\mathbb{M} \arrow[r] & \mathscr{N}(1) \arrow[r] \arrow[d] & \mathscr{F}(1) \arrow[r] \arrow[d] & 0, \\ & & \mathscr{G}(1) \arrow[r, equal] \arrow[d] & \mathscr{G}(1) \arrow[d] & \\ & & 0 & 0 & \end{tikzcd}$$

in which $\mathscr{F}$ is defined as cokernel. Since $\mathbb{B}$ is a bundle and $\mathscr{G}$ is flat we conclude that also $\mathscr{F}$ is flat over X^{ss}. Thus for any point

$x = (z, M, \Gamma)$ we have the diagram (up to $\otimes \bigwedge^4 V^\vee$)

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \otimes \mathscr{O} \arrow[r] \arrow[d, equal] & \Gamma \otimes \mathscr{O} \arrow[r] \arrow[d] & (\Gamma/M) \otimes \mathscr{O} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \otimes \mathscr{O} \arrow[r] & \mathscr{N}_y(1) \arrow[r] \arrow[d] & \mathscr{F}_x(1) \arrow[r] \arrow[d] & 0 \\
 & & \mathscr{G}_y(1) \arrow[r, equal] \arrow[d] & \mathscr{G}_y(1) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
\tag{21}
$$

on $\mathbb{P}_3$. We also recall that we have a monad display generalising (D_2) of §2:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \otimes \Omega^3(3) \arrow[r] \arrow[d, equal] & \mathscr{N}_x \arrow[r] \arrow[d] & \mathscr{F}_x \arrow[r] \arrow[d] & 0 \\
0 \arrow[r, "A"'] & M \otimes \Omega^3(3) \arrow[r] & \not{k}^2 \otimes \Omega^1(1) \arrow[r] \arrow[d] & \mathscr{M}_x \arrow[r] \arrow[d] & 0 \\
 & & \mathscr{A}_y \arrow[r, equal] \arrow[d] & \mathscr{A}_y \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
\tag{22}
$$

where $\mathscr{M}_x$ is defined as the cokernel and $\mathscr{A}_y$ comes with the definition of $\mathscr{N}_y$, see 7.4.

8.1. Proposition. *For any $x \in X^{ss}$ the sheaf $\mathscr{F}_x$ is semi-stable of rank 2 with Chern classes $c_1 = 0$, $c_2 = 2$, $c_3 = 0$ on $\mathbb{P}_3$.*

Therefore the family $\mathscr{F}$ defines a morphism of X^{ss} into the Maruyama scheme which will be discussed in 8.3.

Proof. We only prove here that $\mathscr{F}_x$ is torsionfree with the Chern classes indicated. Semi-stability will follow from the geometric description of the sheaves in §§9 and 10.

It is enough to prove that $\mathscr{M}_{\langle x\rangle}$ is torsionfree by diagram (22). Since depth $\mathscr{M}_{\langle x\rangle} \geq 2$ everywhere, it is enough to show that $\mathscr{M}$ is locally free outside a curve, see f.e. **[Si-Tr]**. If M^* is a matrix representing M, and the homomorphism of the fibre over $\langle x\rangle \in \mathbb{P}V$,

$$k^2 \otimes \langle x\rangle \xrightarrow{M\overset{*}{\wedge} x} k^2 \otimes \bigwedge^2 V \wedge x$$

is degenerate, i.e. has rank < 2, then for any $y \in V$ also the matrix $M^* \wedge x \wedge y$ has determinant zero and therefore vanishes on the α-plane $P_{\langle x\rangle}$. If now M^* is degenerate on a surface, its $\det M^*$ would be identically zero on the Grassmannian, which contradicts semi-stability of M, see Proposition 6.1.1. This proves that $\mathscr{F}_x$ is torsionfree. The calculation of rank and Chern classes follows immediately from the diagrams.

8.2. *Cohomology of $\mathscr{F}_x$*. The cohomology dimensions $h^i\mathscr{N}_y(d)$, $h^i\mathscr{F}_x(d)$ can be summarised in the following tables.

d	h^0	h^1	h^2	h^3
≥ 2	*			
1	4			
0		2		
-1		2		
-2		t	t	
-3		t	t	
≤ -4		t	t	*

for $\mathscr{N}_y$

d	h^0	h^1	h^2	h^3
≥ 2	*			
1	2			
0		2		
-1		2		
-2		t	t	
-3		t	$t+2$	
≤ -4		t	*	*

for $\mathscr{F}_x$

Here $t = h^0\mathscr{T}_y = h^0\mathscr{C}_y \leq 2$, see 7.1.2.

Proof. We fix x and y and omit the index. It was shown in Corollary 7.2 that $h^i\mathscr{N}(d) = 0$ for $i > 0$ and $d \geq 2 - i$.

Since $H^i\mathcal{N}(d) = H^i\mathcal{F}(d)$ for $i > 0$ and $d \geq -2$, the same is true for $\mathcal{F}$. Next we show that also $0 = h^3\mathcal{N}(d) = h^3\mathcal{F}(d)$ for $d = -2, -3$, which settles the case h^3: From the display (14) of $\mathcal{N}_y = \mathcal{N}$ in 7.4 we obtain easily $0 = H^2\mathcal{A}(d) = H^3\mathcal{N}(d)$ for these d.

Next we state that $h^2\mathcal{N}(d) = t$ for $d \leq -2$, which also follows from the same display by $h^0\mathcal{C} = h^1\mathcal{A}(d) = h^2\mathcal{N}(d)$.

The case $h^2\mathcal{N}(-1) = 0$ is more subtle. To obtain this we note that $h^2\mathcal{N}(-1) = h^2\mathcal{K}(-1)$ and that for $\mathcal{K}$ we can replace the row of $\mathcal{K}$ in (14), 7.4 by the row

$$0 \to \mathcal{K} \to k^6 \otimes \mathcal{O} \xrightarrow[R]{} k^2 \otimes \mathcal{O}(1) \to \mathcal{C} \to 0,$$

where the matrix R is the kernel of

$$0 \to k^6 \to k^2 \otimes V^\vee \xrightarrow[N^{*\vee}]{} k^2 \to 0,$$

see (9) in the case of a plane. Now it is easy to see that in the few cases of degenerate $N^{*\vee}$ the induced homomorphism $k^2 \otimes \Gamma\mathcal{O} \to \Gamma\mathcal{C}(-1)$ is onto which implies $H^2\mathcal{K}(-1) = H^1\,\mathrm{Im}(R)(-1) = 0$. By this the case $h^2\mathcal{N}(d)$ is settled for all d.

For $d \geq -2$ we also have $H^2\mathcal{N}(d) = H^2\mathcal{F}(d)$, and for $d = -3$ the exact sequence

$$0 \to H^2\mathcal{N}(-3) \to H^2\mathcal{F}(-3) \to M \otimes H^3\mathcal{O}(-3) \to 0,$$

and hence $h^2\mathcal{F}(-3) = 2 + t$.

Finally $h^1\mathcal{N}(-1) = h^1\mathcal{F}(-1) = 2$ and $h^1\mathcal{N} = h^1\mathcal{F} = 2$ follows from Riemann-Roch and $h^0\mathcal{N} = h^0\mathcal{F} = 0$. Of course $h^0\mathcal{N}(1) = 4$ and $h^0\mathcal{F}(1) = 2$ by our earlier result.

8.3. *Morphism* $Q \xrightarrow[\varphi]{} \overline{M(0,2)}$. Let $\overline{M}(2; 0, 2, 0)$ be the Maruyama scheme of all semi-stable coherent rank 2 sheaves on $\mathbb{P}_3$ with Chern classes $c_1 = 0$, $c_2 = 2$, $c_3 = 0$ which contains $M(0,2)$ as an open part. Let $\overline{M(0,2)}$ be its closure. The family $\mathcal{F}_x$, $x \in X^{ss}$, provides us with a morphism $X^{ss} \to \overline{M}(2; 0, 2, 0)$. Since by our construction $\mathcal{F}_x \cong \mathcal{F}_x$, if $O(x) = O(x')$, this morphism is SL(2)-equivariant and factors through the good quotient $Q = X^{ss}//\,\mathrm{SL}(2)$. By the description of $M(0,2)$ in 2.4.2 the open set $Q \setminus Q_0 \cup Q_\alpha \cup Q_\beta \cup Q_e$, see 4, maps isomorphically onto $M(0,2)$. Therefore we have a surjective birational morphism $Q \xrightarrow[\varphi]{} \overline{M(0,2)}$. We are going to investigate how far it is from being an isomorphism.

Let $Q \xrightarrow{\pi} C(\mathbb{G})$ be the projection of the quadric bundle and let $\Sigma_0' \subset \Sigma_0 \subset C(\mathbb{G})$ be the subvarieties of all double lines, resp. of all singular conics. We write

$$Q_{\mathrm{exc}} = \pi^{-1}(\Sigma_0') \cap \operatorname{Sing} Q.$$

By 4.2 this is 2-dimensional over Σ_0' and indeed a $\mathbb{P}_2$-bundle.

8.4. Proposition. (1) *$Q \setminus Q_{\mathrm{exc}} \underset{\varphi}{\longrightarrow} \overline{M(0,2)}$ is injective.*

(2) *The fibres of $Q_{\mathrm{exc}} \underset{\varphi}{\longrightarrow} \overline{M(0,2)}$ are in the $\mathbb{P}_1$'s of double structures of the conics in Σ_0'.*

Proof. (a) The injectivity on $Q \setminus Q_{\mathrm{exc}}$ will follow when we prove that the pair $(S, C^\vee)$ of conics given by $x \in X^{ss}$ is determined by the class $[\mathscr{F}_x]$ through $\operatorname{Supp} R^1\mathscr{F}_x$ and $\operatorname{Supp} R^1\mathscr{F}_x(-1)$. Since $R^1\mathscr{N}_y = R^1\mathscr{F}_x$ the reduced conic S is already determined by $S^0 = \operatorname{Supp} R^1\mathscr{F}_x$, see 7.6.

(b) If $x = (z, M, \Gamma) \in X^{ss}$ denote $W = W_z$, $\mathscr{F} = \mathscr{F}_x$, $S = S(\Gamma)$, and $C^\vee = C^\vee(M)$. We consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W^\perp & \longrightarrow & \bigwedge^2 V & \longrightarrow & W' & \longrightarrow & 0 \\ & & \downarrow\wr & & \| & & \|\wr & & \\ 0 & \longrightarrow & \left(\bigwedge^2 V/W\right)^\vee & \longrightarrow & \bigwedge^2 V^\vee & \longrightarrow & W^\vee & \longrightarrow & 0 \end{array}$$

where the vertical arrow in the middle is the quadratic form of the Grassmannian, which identifies the orthogonal $W^\perp$ with $(\bigwedge^2 V/W)^\vee$, and provides an isomorphism of the cokernel W' with $W^\vee$. If we take any splitting of the first sequence we get a projection

$$\mathbb{P}\bigwedge^2 V \setminus \mathbb{P}W^\perp \to \mathbb{P}W^\perp \subset \mathbb{P}\bigwedge^2 V.$$

(c) From diagram (22) we get the exact sequence

$$0 \to R^0\mathscr{M}(-1) \to k^2 \otimes \mathscr{O}_{\mathbb{G}}(-1) \xrightarrow{M^*} k^2 \otimes \mathscr{O}_{\mathbb{G}} \to R^1\mathscr{M}(-1) \to 0,$$

where M^* is a matrix representing M, which also is a homomorphism on the Grassmannian by $k^2 \otimes \bigwedge^4 V^\vee \otimes \bigwedge^2 U \to k^2$ for $U \in G_2V$, and which we also denote by M^*. It follows that $R^0\mathscr{M}(-1) = 0$ and

$$\operatorname{Supp} R^1\mathscr{M}(-1) = \{\det M^* = 0\}.$$

Moreover from the display (14) we obtain that $R^0\mathscr{A}(-1) = R^0\mathscr{T}$, $R^1\mathscr{A}(-1) = R^0\mathscr{C}$, and from (22) the exact sequence

$$0 \to R^0\mathscr{T} \to R^1\mathscr{F}(-1) \to R^1\mathscr{M}(-1) \to R^0\mathscr{C} \to 0.$$

Since $\operatorname{Supp}\mathscr{T} = \operatorname{Supp}\mathscr{C}$, also $\operatorname{Supp} R^0\mathscr{T} = \operatorname{Supp} R^0\mathscr{C}$, and the sequences show that

$$J = \operatorname{Supp} R^1\mathscr{F}(-1) = \operatorname{Supp} R^1\mathscr{M}(-1) = \{\det M^* = 0\}.$$

REMARK. M^* is determined by $R^1\mathscr{M}(-1)$ through $k^2 \to k^2 \otimes \Gamma\mathscr{O}_{\mathbb{G}}(1)$ up to equivalence.

(d) LEMMA. *There is a unique quadric hypersurface $\widetilde{J} \subset \mathbb{P}\bigwedge^2 V$ such that $J = \mathbb{G} \cap \widetilde{J}$ and $\widetilde{J}$ is singular along $S^0 \subset J$.*

Proof. Let f be the equation of any quadric hypersurface $\widetilde{J}$ with $J = \mathbb{G} \cap \widetilde{J}$ and let q be the equation of $\mathbb{G}$. Since S^0 is contained in the singular locus Sing(J) (because $S^0 \subset \mathbb{P}W^\perp \cap \mathbb{G} \subset J$ and M^* vanishes on $\mathbb{P}W^\perp$), for any $p \in S^0$ there is a unique $\lambda(p) \in k$ such that

$$\frac{\partial f}{\partial p_{ij}}(p) = \lambda(p)\frac{\partial q}{\partial p_{ij}}(p)$$

for all derivatives with respect to the Plücker coordinates of $\mathbb{P}\bigwedge^2 V$. Because $\mathbb{G}$ is regular, λ is a regular function on S^0 and hence constant. Then $\widetilde{J} = \{f - \lambda q = 0\}$ is the unique hypersurface of the lemma.

Since however $\{\det M^* = 0\}$ has the properties of $\widetilde{J}$ in the lemma, $\widetilde{J} = \{\det M^* = 0\}$. On the other hand the conic $C^\vee(M)$ in $\mathbb{P}W^\vee$ has exactly the same equation, see 6.5. If $\mathbb{P}W^\vee \subset \mathbb{P}\bigwedge^2 V$ by some splitting in the diagram of (b), we obtain

$$C^\vee = \mathbb{P}W^\vee \cap \widetilde{J}.$$

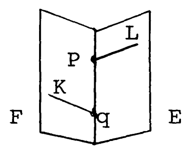
Therefore the injectivity of φ on $Q \setminus Q_{\text{exc}}$ will be proved if the plane $\mathbb{P}W$ or $\mathbb{P}W^\perp$ can be determined by $\mathscr{F}$.

(e) If we consider $Q \setminus \pi^{-1}(\Sigma_0')$ clearly $\mathbb{P}W$ is determined by $S^0 = \operatorname{Supp} R^1\mathscr{F}$, since each S and S^0 is a pair of distinct lines. In this case the injectivity follows if we show that $\operatorname{Supp} R^1\mathscr{F}$ and $\operatorname{Supp} R^1\mathscr{F}(-1)$ are invariants of the class $[\mathscr{F}] \in \overline{M(0,2)}$. If $\mathscr{F}$ is stable, there is nothing to prove. If $\mathscr{F}$ is semi-stable and non-stable then the pair

$(S, C^\vee)$ is semi-stable, see 3.12, 10.5 with S and $C^\vee$ both degenerate. It is shown in 10.5 that in this case $\mathcal{F}$ is an extension of the type

$$0 \to \mathcal{I}_{L\cup q} \to \mathcal{F} \to \mathcal{I}_{K\cup p} \to 0,$$

where $\mathcal{I}_{L\cup q}$, $\mathcal{I}_{K\cup p}$ are ideal sheaves of a line and a point as indicated in the figure, which is determined by $(S, C^\vee)$.



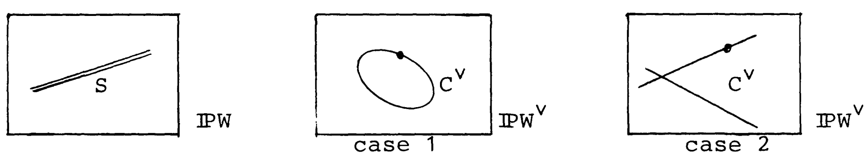
(If $q \in L$ then $\mathcal{I}_{L\cup q}$ is the ideal sheaf of the line L with a multiple structure in q with tangent vector in the plane E, similarly for $K\cup p$.) It follows first from 7.6.1, Case 4, that, if we consider the sheaf $\mathcal{N}$, $\operatorname{Supp} R^1\mathcal{N}$ is independent of the extension class; hence the same is true for $R^1\mathcal{F} = R^1\mathcal{N}$. Second we have $\mathcal{I}_{L\cup q}(-1) \subset \mathcal{I}_L(-1) \subset \mathcal{O}(-1)$ and hence $R^0\mathcal{I}_{L\cup q}(-1) = 0$. Therefore from the extension sequence of $\mathcal{F}$ we also obtain the short exact sequence

$$0 \to R^1\mathcal{I}_{L\cup q}(-1) \to R^1\mathcal{F}(-1) \to R^1\mathcal{I}_{K\cup p}(-1) \to 0,$$

which shows that the support of $R^1\mathcal{F}(-1)$ is independent of the extension. This proves injectivity of $\varphi|Q \setminus \pi^{-1}(\Sigma_0')$.

(f) Let us now consider the regular points over Σ_0', i.e. $\pi^{-1}(\Sigma_0') \setminus Q_{\text{exc}}$. Because these correspond to stable points $x \in X^{ss}$ with $\mathcal{F}_x$ stable, see 1, the supports of $R^1\mathcal{F}_x$, $R^1\mathcal{F}_x(-1)$ are determined by $[\mathcal{F}_x]$. But here we have to show that $\mathcal{F}_x$ determines the plane $\mathbb{P}W$ or $\mathbb{P}W^\perp$.

Now in the case of stable pairs $(S, C^\vee)$ with S a double line we only have two cases of $C^\vee$ as indicated in the picture, see 3.12.



In Case 1 $\mathbb{P}W^\perp = \operatorname{Sing}\tilde{J}$ since $\mathbb{P}W^\perp \subset \operatorname{Sing}\tilde{J}$ and the latter is 2-dimensional. Therefore the plane is determined by $\mathcal{F}$ in this case.

In Case 2, $\tilde{J}$ has an equation $a \cdot b = 0$ such that one factor, a say, is not in $S^0 = S \subset W \subset \bigwedge^2 V$ as a 2-space. This implies

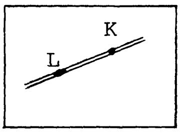
$$S^\perp \not\subset a^\perp \cap b^\perp = \operatorname{Sing}\tilde{J}.$$

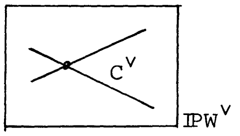
Because here $S^\perp$ and $\operatorname{Sing}\widetilde{J}$ both are 3-dimensional and contain $\mathbb{P}W^\perp$

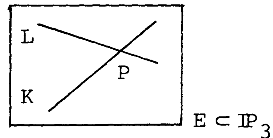
$$\mathbb{P}W^\perp = S^\perp \cap \operatorname{Sing}\widetilde{J}.$$

This again proves that $\mathbb{P}W^\perp$ is determined by $\mathscr{F}$, since $S = S^0$ and $\widetilde{J}$ are determined by $\mathscr{F}$.

(g) Finally we consider the restriction $\varphi|Q_{\mathrm{exc}}$. If $(S, C^\vee) \in Q_{\mathrm{exc}}$ the conics are of the type







with the singular point of $C^\vee$ being the point $S \in \mathbb{P}W^\vee$ and $C^\vee$ determining two points $L, K \in S$. The triple (S, L, K) determines a pair of lines in $\mathbb{P}_3$ with a plane E and a point $p \in E$. The sheaf $\mathscr{F} = \mathscr{F}_x$ coming from a point x defining $(S, C^\vee)$ again is an extension

$$0 \to \mathscr{I}_{L\cup p} \to \mathscr{F} \to \mathscr{I}_{K\cup p} \to 0,$$

see (e), and $[\mathscr{F}] = [\mathscr{I}_{L\cup p} \oplus \mathscr{I}_{K\cup p}]$ in $\overline{M(0,2)}$, where the double structure of p in one of the lines shows in the direction of the plane E. But now the class $[\mathscr{F}]$ cannot remember the plane (whereas the extension class of $\mathscr{F}$ can, as can be shown easily). Thus $(S, C^\vee) \underset{\varphi}{\longrightarrow} [\mathscr{F}]$ forgets the plane $\mathbb{P}W$, but $[\mathscr{F}]$ determines the triple (S, L, K). This shows that $\varphi|Q_{\mathrm{exc}}$ blows down the $\mathbb{P}_1$'s of double structures of the conics $S \in \Sigma_0'$, see 4.2, 3.9.

8.5. REMARK. An "Orbit-Lemma" is true for $Q \setminus Q_{\mathrm{exc}}$: Let $x, x' \in X^{ss}$ with $q(x), q(x') \notin Q_{\mathrm{exc}}$. Then $\mathscr{F}_x \cong \mathscr{F}_{x'}$ iff $O(x) = O(x')$.

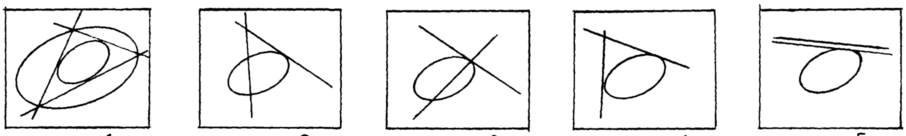
9. Sheaves in the boundary with regular conic S. In this section we start with the detailed geometric description of the sheaves representing boundary points of the moduli space. Since we fix a semi-stable parameter point $x = (z, M, \Gamma)$ in each case with $y = (z, \Gamma)$ and associated space N, we drop the indices and write

$$\mathscr{F} = \mathscr{F}_x, \quad \mathscr{N} = \mathscr{N}_y, \quad \mathscr{G} = \mathscr{G}_y, \quad \mathscr{K} = \mathscr{K}_N, \quad \mathscr{C} = \mathscr{C}_N, \quad \mathscr{T} = \mathscr{T}_y$$

and also

$$S = S(\Gamma), \quad C^\vee = C^\vee(M), \quad W = W_z.$$

In this section we assume that S is a regular conic, which could be a regular plane section, or contained in a β- or α-plane, see 9.1, 9.2, 9.3 respectively. It is convenient in this case to consider the Poncelet conic C in the same plane $\mathbb{P}W$ of S, which is the polar dual of $C^\vee$ w.r.t. S. Since the Poncelet condition for degenerate C just means that one of the lines of C must be tangent to S, we have to consider the following cases:

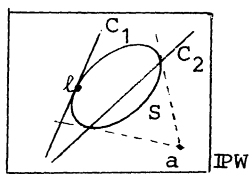


case 1 case 2 case 3 case 4 case 5

9.1. *The sheaves in* $Q_e \setminus Q_\alpha \cup Q_\beta \cup Q_0$. The pairs (S, C) in this set are characterised by S to be a regular plane section and C to be singular. In such a case the homomorphism defined by N is a regular epimorphism as in 1.1, 1.2, and we have $\mathscr{N} = \mathscr{K}$, $\mathscr{C} = \mathscr{T} = 0$, and $\mathscr{G} = \mathscr{O}_Q(-2, 1)$, where Q here denotes the quadric defined by S.

9.1.1. Proposition. *The sheaves* $\mathscr{F}$ *in* Case 2/3 *are exactly those which can be obtained by an "elementary transformation"*

$$0 \to \mathscr{F} \to \mathscr{E}' \xrightarrow[\pi]{} \mathscr{O}_l(1) \to 0,$$



where $\mathscr{E}' \in M(0, 1)$ *is an instanton bundle with* $c_2\mathscr{E}' = 1$ *(i.e. a null-correlation bundle),* l *is a line in* $\mathbb{P}_3$ *and* π *is an epimorphism. The data* $(\mathscr{E}', l, \pi)$ *are in* 1:1 *correspondence with the pairs* $(S, C) \in Q_e \setminus Q_\alpha \cup Q_\beta \cup Q_e$ *as follows*

(i) $\mathscr{E}'$ *is the bundle in* $M(0, 1) = \mathbb{P}\bigwedge^2 V \setminus G_2V$ *determined by the pole* a *of the component* C_2 *of* C *in the plane* $\mathbb{P}W$,

(ii) l *is the tangent point of the component* C_1,

(iii) *the epimorphism* π *corresponds (in a way described in the proof) to the plane* $\mathbb{P}W$ *through* a, l *and intersecting* $\mathbb{G}$ *regularly in* S.

Cases 2 *and* 3 *can be distinguished by* $\mathscr{E}'|l \simeq 2\mathscr{O}_l$ *and* $\mathscr{E}'|l \simeq \mathscr{O}_l(-1) \oplus \mathscr{O}_l(1)$.

Corollary 1. *Each such* $\mathscr{F}$ *is* μ*-stable, since* $\mathscr{E}'$ *is* μ*-stable.*

COROLLARY 2. *Let* $B \to \mathbb{G} \times M(0, 1)$ *be the projective bundle of homomorphisms* $\mathscr{E}' \to \mathscr{O}_l(1)$ *mod scalars for* $(l, \mathscr{E}') \in \mathbb{G} \times M(0, 1)$ *and let* B^0 *be the open part of epimorphisms. Then the elementary transformation gives us an isomorphism*

$$B^0 \hookrightarrow Q_e|C^0(\mathbb{G})$$

onto the open part of the boundary component Q_e *defined by* 9.1, *Cases* 2 *and* 3.

Proof. (a) Let $\mathscr{F}$ be given. Since S is regular we can choose a basis e_i of V such that the space N can be presented by the matrix

$$N^* = \begin{bmatrix} e_0 & e_2 \\ e_1 & e_3 \end{bmatrix},$$

see 1.1, 1.2, and such that $l = e_{01} = e_0 \wedge e_1$. Moreover since $C^\vee(M)$ is a pair of lines, the matrix representing M must have the shape

$$M^* = \begin{bmatrix} l & 0 \\ b & a \end{bmatrix},$$

such that $M^* \wedge N^{*t} = 0$. By our convention $\bigwedge^2 V \simeq \bigwedge^2 V^\vee$, the conic C has the equation $l \circ a = 0$ in $\mathbb{P}W \simeq \mathbb{P}W^\vee$ (duality given by $\mathbb{G}$ or S). By this form of M^* we obtain the exact diagram

$$(23)\quad \begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \Omega^3(3) \arrow[r, "l"] \arrow[d] & \Omega^1(1) \arrow[r] \arrow[d] & \mathscr{F}' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & k^2 \otimes \Omega^3(3) \arrow[r, "\left(\begin{smallmatrix} l & \\ b & a \end{smallmatrix}\right)"'] \arrow[d] & k^2 \otimes \Omega^1(1) \arrow[r] \arrow[d] & \mathscr{M} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Omega^3(3) \arrow[r, "a"'] \arrow[d] & \Omega^1(1) \arrow[r] \arrow[d] & \mathscr{E}' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

with cokernels $\mathscr{F}$, $\mathscr{M}$, $\mathscr{E}'$ respectively. On the other hand the monad

(22) of $\mathscr{F}$ in 8. gives the mid row of the exact diagram

(24)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathscr{F}' \arrow[r, equal] \arrow[d] & \mathscr{F}' \arrow[d] & \\
0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d, equal] & \mathscr{M} \arrow[r] \arrow[d] & k^2 \otimes \mathscr{O} \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{E}' \arrow[r] \arrow[d] & \mathscr{C}oker \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

In this the composite $\mathscr{F}' \to k^2 \otimes \mathscr{O}$ is still injective, which follows from the upper right-hand square of (23) and from

$$\begin{tikzcd}
k^2 \otimes \Omega^1(1) \arrow[r] \arrow[dr, "{\left[\begin{smallmatrix} e_0 & e_1 \\ e_2 & e_3 \end{smallmatrix}\right]}"'] & \mathscr{M} \arrow[d] \\
 & k^2 \otimes \mathscr{O}
\end{tikzcd},$$

since $l = e_{01}$. This shows that (24) is exact. Since a is indecomposable, $\mathscr{E}'$ is a typical bundle of $M(0, 1) = \mathbb{P}\bigwedge^2 V \setminus \mathbb{G}$, see **[Ha2]**. It remains to identify the cokernel. By the definition of $\mathscr{F}' \to k^2 \otimes \mathscr{O}$ we get the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
0 \arrow[r] & \Omega^3(3) \arrow[r, "l"] & \Omega^1(1) \arrow[r] \arrow[d, equal] & \mathscr{F}' \arrow[r] \arrow[d] & 0 & \\
 & & \Omega^1(1) \arrow[r, "{(e_0, e_1)}"'] & k^2 \otimes \mathscr{O} \arrow[r] \arrow[d] & \mathscr{O}_l(1) \arrow[r] & 0, \\
 & & & \mathscr{C}oker \arrow[d] & & \\
 & & & 0 & &
\end{tikzcd}$$

which shows that $\mathscr{C}oker = \mathscr{O}_l(1)$. (In particular we have obtained the two equivalent descriptions of $\mathscr{F}'$ which is a sheaf of the boundary of $M(0, 1)$.)

(b) It is also easy to verify that the subspace $W \subset \bigwedge^2 V$ is isomorphic under $\bigwedge^2 V \simeq \bigwedge^2 V^\vee$ to the kernel of the composed map

$$\bigwedge^2 V^\vee = \Gamma\Omega^1(2) \to \Gamma\mathscr{E}'(1) \to \Gamma\mathscr{O}_l(1),$$

using the above matrices. This shows that the plane $\mathbb{P}W$ is determined by π. Conversely we had just constructed π from l, a and the plane. Thus we have established (i), (ii), (iii) if $\mathscr{F}$ is given.

(c) Let now an elementary transformation be given. We can find a monad for $\mathscr{F}$ by going backwards in the diagrams (23) and (24). First we can determine l, a and the plane $\mathbb{P}W$ by π as in (b). Let $\mathscr{F}'$ be defined as the kernel of $k^2 \otimes \mathscr{O} \to \mathscr{O}_l(1)$, and define $\mathscr{M}$ as the pullback in diagram (24). Since $\mathscr{F}'$ and $\mathscr{E}'$ have the resolutions as in (23), we get the resolution of $\mathscr{M}$ by adding up. Then $0 \to \mathscr{F} \to \mathscr{M} \to k^2 \otimes \mathscr{O}$ and the resolution of $\mathscr{M}$ give us a monad. To see that this defines a pair (S, C) of the above type, we consider the composed homomorphism

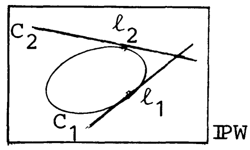
$$k^2 \otimes \Omega^1(1) \xrightarrow[N^{*\vee} = \begin{pmatrix} e_0 & e_1 \\ v & w \end{pmatrix}]{} k^2 \otimes \mathscr{O} \to 0,$$

which must have the entries e_0, e_1 in its first row because of $l = e_{01}$ and $M^* \wedge N^{*\vee} = 0$, and which must be an epimorphism. By 0.2 we must have $\dim\langle e_0, e_1, v, w\rangle = 4$ or $= 3$ in a special configuration. If $\dim = 3$ it would follow that the entries l, a, b of M^* are contained in a β-plane, see 7.6, Case 2, and a would be decomposable. Therefore N^* defines a regular conic S and M^*, by its shape, a degenerate conic C as in Cases 2 or 3.

Case 4. If the conic C consists of two tangents we get a degenerate case of the elementary transformation by replacing $\mathscr{E}'$ by a sheaf of the type $\mathscr{F}'$ considered above. Thus a sheaf $\mathscr{F}$ in 9.1, Case 4, is given as the kernel of an epimorphism π

$$0 \to \mathscr{F} \to 2\mathscr{O} \xrightarrow{\pi} \mathscr{O}_{l_1}(1) \oplus \mathscr{O}_{l_2}(1) \to 0,$$

where π corresponds to the plane $\mathbb{P}W$ through the axis $\overline{l_1, l_2}$. The proof is a special case of the one just made. Note that by this we extend the morphism of elementary transformation to $\overline{M(0, 1)}$. If we compactify this along the direction of the epimorphisms we would leave the set of regular conics S.



Note that also in this case $\mathscr{F}$ is stable, since it is easy to show that any sheaf $\mathscr{F}'$ as above is stable (but not μ-stable any more): If $\mathscr{F}_0' \subset \mathscr{F}'$ is a sub-sheaf of rank 1 with $\mathscr{F}'/\mathscr{F}_0'$ torsionfree, we can assume $c_1\mathscr{F}_0' = 0$ and hence $\mathscr{F}_0' \subset \mathscr{O}$ an ideal sheaf. The diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{F}_0' & \longrightarrow & \mathscr{O} & \longrightarrow & \mathscr{O}_Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{F}' & \longrightarrow & k^2 \otimes \mathscr{O} & \longrightarrow & \mathscr{O}_l(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{F}'/\mathscr{F}_0' & \longrightarrow & \mathscr{O} & \longrightarrow & \mathscr{C}oker & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

shows that $\mathrm{Supp}(\mathscr{C}oker)$ is at most 0-dimensional if $Z \neq \emptyset$, and that then $\chi\mathscr{F}_0'(m) < \frac{1}{2}\chi\mathscr{F}'(m)$ for large m.

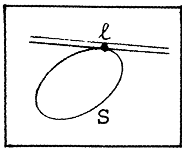
Case 5. Again this is a degeneration of Case 3 or Case 4. We obtain here an exact sequence

$$0 \to \mathscr{F} \to 2\mathscr{O} \xrightarrow{\pi} \mathscr{R} \to 0,$$

where $\mathscr{F}^{\vee\vee} = 2\mathscr{O}$ as in the previous case and where $\mathscr{R}$ is supported on L as an $\mathscr{O}$-module extension

$$0 \to \mathscr{O}_l(1) \to \mathscr{R} \to \mathscr{O}_l(1) \to 0.$$

Here both the extension and the epimorphism depend on the plane $\mathbb{P}W$, but we omit further details. Again $\mathscr{F}$ is stable.



$\mathbb{P}W$

9.2. *The sheaves in* $Q_\beta \setminus Q_0$. These are the sheaves corresponding to a pair $(S, C^\vee)$ where S is a regular conic in a β-plane. Since $\mathbb{P}W$ is a β-plane, we have $W = \bigwedge^2 U$ where $\mathbb{P}U \subset \mathbb{P}V$ is a plane, which can be considered now the dual of $\mathbb{P}W$ by $\bigwedge^2 U^\vee \simeq U$. The dual conic $S^\vee \subset \mathbb{P}U$ can now be considered as the base conic for the Poncelet property and we can also consider $C^\vee(M)$ as a conic in $\mathbb{P}U$ given by the equation

$$\det M^* = 0 \quad \text{in } \mathbb{P}U,$$

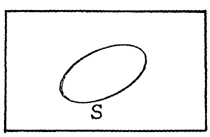
where the entries of M^* are elements of $W = \bigwedge^2 U \simeq U^\vee$. Moreover we choose a basis e_i of V such that

$$U = \langle e_0, e_1, e_2 \rangle, \quad U \oplus \langle e_3 \rangle = V,$$

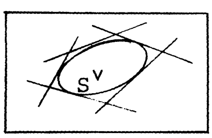
and such that the matrices Γ^*, N^* representing the spaces Γ, N are given by

$$\Gamma^* = \begin{bmatrix} e_{01} & 0 \\ e_{12} & e_{01} \\ e_{02} & e_{12} \\ 0 & e_{02} \end{bmatrix}, \quad N^{*\vee} = \begin{bmatrix} e_0 & e_1 \\ -e_2 & e_0 \end{bmatrix}$$

as in Case 2 of 7.6.



$\mathbb{P}W \subset \mathbb{G}$



$P = \mathbb{P}U \subset \mathbb{P}_3$

9.2.1. PROPOSITION. (1) *The sheaves* $\mathscr{F}$ *in* $Q_\beta \setminus Q_0$ *are elementary transformations of the type*

$$0 \to \mathscr{F} \to k^2 \otimes \mathscr{O} \to \mathscr{R}(1) \to 0$$

where $\mathscr{R}$ *is supported by the conic* $C^\vee \subset P \subset \mathbb{P}_3$ *and is the cokernel of* M^* :

$$0 \to k^2 \otimes \Omega_P^2(2) \xrightarrow[M^*]{} k^2 \otimes \mathscr{O}_P \to \mathscr{R} \to 0,$$

where M^* *is the matrix of* M *with entries in* $W = \bigwedge^2 U$ ($\mathscr{R}$ *is a Cohen-Macaulay module on* $C^\vee$).

(2) *The sheaf* $\mathscr{G}$ *is a stable rank-*2 *bundle on* P *with* $c_1\mathscr{G}(1) = 0$, $c_2\mathscr{G}(1) = 2$ *and is the kernel of* $N^{*\vee}$,

$$0 \to \mathscr{G} \to k^2 \otimes \Omega_P^1(1) \xrightarrow{N^{*\vee}} k^2 \otimes \mathscr{O}_P \to 0$$

(*with entries of* N^* *in* U). *Its jumping lines are the points of* S, *which at the same time are the zero loci of sections of* $\mathscr{N}(1)$.

REMARK. These sheaves are well understood, see **[Ba]**.

(3) The Poncelet relation of $S^\vee$ with $C^\vee$ in P has its expression in the exact sequence

$$0 \to \mathscr{G}(-1) \to (\Gamma/M) \otimes \Omega^2 P(2) \to \mathscr{R} \to 0$$

obtained in (25) of the proof.

(4) The restriction of $\mathscr{F}$ to P splits into

$$\mathscr{F}|P = \mathscr{G} \oplus \mathscr{R}.$$

(5) Each such $\mathscr{F}$ is stable.

Proof. (a) We first remark that the homomorphism $\Omega^3(3) \xrightarrow[a]{} \Omega^1(1)$ defined by $a \in \bigwedge^2 U \subset \bigwedge^2 V$ splits into

$$\Omega_P^2(2) \xrightarrow[(a,0)]{} \mathscr{O}_P \oplus \Omega_P^1(1)$$

when restricted to the plane $P = \mathbb{P}U$, see 0.1.

(b) Let us consider now the diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{k}^4 \otimes \Omega^3(3) \arrow[r] \arrow[d, equal] & \mathscr{N} \arrow[r] \arrow[d] & \mathscr{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{k}^4 \otimes \Omega^3(3) \arrow[r, "\Gamma^*"'] & \mathscr{k}^2 \otimes \Omega^1(1) \arrow[r] \arrow[d, "N^{*\vee}"] & \widetilde{\mathscr{G}} \arrow[r] \arrow[d] & 0, \\
 & & \mathscr{k}^2 \otimes \mathscr{O} \arrow[r, equal] \arrow[d] & \mathscr{k}^2 \otimes \mathscr{O} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$

which is defined by Γ^*. From this it can be proved first that $z_3\mathscr{G} = 0$, i.e. $\mathscr{G}$ is an $\mathscr{O}_P$-module ($z_0, \dots, z_3 \in V^\vee$ are the dual coordinates). If we restrict this diagram to P we obtain the splitting as indicated in the diagram, where we identify $\mathscr{T}or_1(\mathscr{G}, \mathscr{O}_P) = \mathscr{G}(-1) = \mathscr{T}or_1(\widetilde{\mathscr{G}}, \mathscr{O}_P)$,

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & & \mathscr{N}|P \arrow[r] \arrow[d] & \mathscr{G} \arrow[r] \arrow[d] \arrow[dl, dashed] & 0 \\
0 \arrow[r] & \mathscr{G}(-1) \arrow[r] & \mathscr{k}^4 \otimes \Omega_P^2(2) \arrow[r] \arrow[r, bend left, "0"] \arrow[r, bend right, "\Gamma^*"'] \arrow[dr, "0"'] & \mathscr{k}^2 \otimes \mathscr{O}_P \oplus \mathscr{k}^2 \otimes \Omega_P^1(1) \arrow[r] \arrow[r, bend right] \arrow[d, "N^{*\vee}"] & \widetilde{\mathscr{G}}|P \arrow[r] \arrow[d, "\approx"'] & 0 \\
 & & & \mathscr{k}^2 \otimes \mathscr{O}_P \arrow[r, equal] \arrow[d] & \mathscr{k}^2 \otimes \mathscr{O}_P \arrow[d] & \\
 & & & 0 & 0 &
\end{tikzcd}
$$

It follows that

$$\mathscr{N}|_P \simeq \mathscr{k}^2 \otimes \mathscr{O}_P \oplus \mathscr{G},$$

and that we obtain two different presentations of the sheaf $\mathscr{G}$ which are equivalent by using a transformation based on $\Omega_P^1(1) \subset U^\vee \otimes \mathscr{O}_P$ as in (9) of 5.2.

(c) Let now the sheaf $\mathscr{R}$ be defined as the cokernel of M^* as homomorphism on P and as in the proposition. Then we obtain the exact diagram

(25)

$$\begin{tikzcd}
 & & p \arrow[d] & & & \\
 & 0 \arrow[r] \arrow[d] & M \otimes \Omega_P^2(2) \arrow[r] \arrow[d] & k^2 \otimes \mathscr{O}_P \arrow[r] \arrow[d, equal] & \mathscr{R} \arrow[r] & 0 \\
\mathscr{O} \arrow[r] & \mathscr{G}(-1) \arrow[r] \arrow[d] & \Gamma \otimes \Omega_P^2(2) \arrow[r] \arrow[d] & k^2 \otimes \mathscr{O}_P \arrow[r] & 0 & \\
 & (\Gamma/M) \otimes \Omega_P^2(2) \arrow[r, equal] \arrow[d] & (\Gamma/M) \otimes \Omega_P^2(2) \arrow[d] & & & \\
 & \mathscr{R} \arrow[d] & 0 & & & \\
 & 0 & & & &
\end{tikzcd}$$

and in particular the sequence (3) of the proposition.

(d) If we restrict display (22) of $\mathscr{F}$ to P we obtain

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \otimes \Omega_P^2(2) \arrow[r] \arrow[d, equal] & k^2 \otimes \mathscr{O}_P \oplus \mathscr{G} \arrow[r] \arrow[d] & \mathscr{F}|_P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \otimes \Omega_P^2(2) \arrow[r] & k^2 \otimes \mathscr{O}_P \oplus k^2 \otimes \Omega_P^1(1) \arrow[r] \arrow[d, shift right, "0"'] \arrow[d, shift left, "N^{*\vee}"] & \mathscr{M}|_P \arrow[r] \arrow[d] & 0 \\
 & & k^2 \otimes \mathscr{O}_P \arrow[r, equal] \arrow[d] & k^2 \otimes \mathscr{O}_P \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

;

thereby obtaining the splitting

$$\mathscr{F}|_P = \mathscr{R} \oplus \mathscr{G}.$$

(e) Finally we consider the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & \Gamma/M \otimes \Omega^3(2) \arrow[r] \arrow[d, "z_3"] & \mathscr{F}(-1) \arrow[r] \arrow[d, "z_3"] \arrow[dl, dashed, "\alpha"'] & \mathscr{G}(-1) \arrow[r] \arrow[d, dashed, "0"] & 0 \\
 & 0 \arrow[r] & \Gamma/M \otimes \Omega^3(3) \arrow[r] \arrow[d] \arrow[dr, "\beta"'] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{G} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{G}(-1) \arrow[r] & \Gamma/M \otimes \Omega^2_P(2) \arrow[r] \arrow[d] & \mathscr{R} \oplus \mathscr{G} \arrow[r] \arrow[d] & \mathscr{G} \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd}$$

Since $z_3\mathscr{G}(-1) = 0$ the multiplication by z_3 lifts to α, and by the splitting of the bottom row we obtain the exact diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Gamma/M \otimes \Omega^3(2) \arrow[r] \arrow[d, equal] & \mathscr{F}(-1) \arrow[r] \arrow[d, "\alpha"] & \mathscr{G}(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma/M \otimes \Omega^3(2) \arrow[r] & \Gamma/M \otimes \Omega^3(3) \arrow[r] \arrow[d, "\beta"] & \Gamma/M \otimes \Omega^2_P(2) \arrow[r] \arrow[d] & 0 \\
 & & \mathscr{R} \arrow[r, equal] \arrow[d] & \mathscr{R} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Altogether this proves (1), ... , (4) of the proposition. For the proof of stability we can take a subsheaf $\mathscr{F}' \subset \mathscr{F}$ with $\mathscr{F}/\mathscr{F}'$ torsionfree, with $c_1\mathscr{F}' = 0$, $\operatorname{rank}\mathscr{F}' = 1$, since $2\mathscr{O}$ is μ-semi-stable. Now a diagram analogous to the one of 9.1, Case 4, shows that the $\mathscr{C}oker$ is 0-dimensional and thus $\chi\mathscr{F}'(m) < \frac{1}{2}\chi\mathscr{F}(m)$ for large m.

REMARK 1. The sequence (3) of the proposition describes the Poncelet situation in terms of bundles in the plane P, see **[Ba]** and also **[Tr2]**. Since $\mathscr{E}xt^1(\mathscr{G}, \mathscr{O}) = \mathscr{G}^\vee(1)$, $\mathscr{G}^\vee = \mathscr{G}(2)$, and $\mathscr{E}xt^2_{\mathscr{O}}(\mathscr{R}, \mathscr{O}) = \mathscr{R}^\vee(2)$ (dual of $\mathscr{R}$ on its support), we also obtain the exact sequence

$$0 \to (\Gamma/M)^\vee \otimes \mathscr{O}_P \to \mathscr{G}(2) \to \mathscr{R}^\vee(1) \to 0,$$

where $\Gamma^\vee = \Gamma\mathscr{G}(2)$. This shows that we get all Poncelet curves if we vary the 2-dimensional subspaces of $\Gamma\mathscr{G}(2)$.

REMARK 2. We can also investigate the epimorphisms $k^2 \otimes \mathscr{O} \xrightarrow[\pi]{} \mathscr{R}(1) \to 0$ for a given conic $C^\vee$ in P as in 9.1. The result is that the pencils $\subset \mathbb{P}\Gamma\mathscr{R}(1) = \mathbb{P}_3$ describe the 4-dimensional family of regular conics $S^\vee \subset P$ to which $C^\vee$ is Poncelet related. Thus also in Case 9.2 the epimorphisms π correspond to the regular conics. Moreover the elementary transformations investigated here also extend the ones of 9.1 to the case of β-planes.

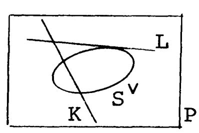
Now we can describe the different situations of the conic $C^\vee$.

Case 1: in which $C^\vee$ is regular. Then $\mathscr{R} = \mathscr{O}_{C^\vee}(1)$ is the line bundle of degree 1 on $C^\vee$.

Cases 2, 3: in which $C^\vee$ is a pair of lines. In this case the matrix M^* cannot be split and defines $\mathscr{R}$ as a nontrivial extension

$$0 \to \mathscr{O}_L \to \mathscr{R} \to O_K \to 0,$$

where L, K are the two lines of $C^\vee$ in the plane P.

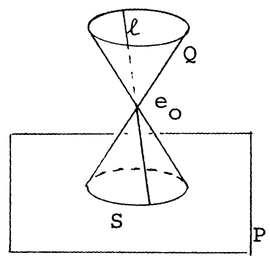


Case 4: in which $C^\vee$ consists of a pair of tangents. Here $\mathscr{R}$ is the direct sum $\mathscr{O}_L \oplus \mathscr{O}_K$.

Case 5: in which $C^\vee$ is a double tangent. Now $\mathscr{R}$ can be a nontrivial extension again depending on M^*, $0 \to \mathscr{O}_L \to \mathscr{R} \to \mathscr{O}_L \to 0$.

9.3. *The sheaves in* $Q_\alpha \backslash Q_0$. These are the sheaves corresponding to a pair $(S, C^\vee)$ where S is regular in an α-plane and thus determines a cone $Q \subset \mathbb{P}_3$. Any plane $P = \mathbb{P}U$ in $\mathbb{P}_3$ not passing through the vertex e_0 serves as a base of the cone which is isomorphic to the α-plane $\mathbb{P}W$, and we assume

$$W = e_0 \wedge U.$$



The conic $Q \cap P$ can be identified with the given conic S. Now we can choose a basis $e_0, \dots, e_3$ such that

$$U = \langle e_1, e_2, e_3 \rangle \quad \text{and} \quad \Gamma^* = \begin{bmatrix} e_{01} & 0 \\ e_{03} & e_{01} \\ e_{02} & e_{03} \\ 0 & e_{02} \end{bmatrix},$$

see 7.6, Case 3. Then the equation of S in P is $z_2^2 - z_1 z_3 = 0$, where the z_j denote the dual coordinates, and the matrix N^* is necessarily a direct product

$$N^* = \begin{bmatrix} e_0 & \\ & e_0 \end{bmatrix}.$$

As shown in 7.6, Case 3, we have in this case $\mathscr{C} = \mathscr{T} = k^2 \otimes k_{e_0} =$ 2 times the skyscraper sheaf $k_{e_0} = \mathscr{O}/m(e_0)$, and

$$\mathscr{K} = k^2 \otimes \mathscr{Z},$$

where $\mathscr{Z}$ is the first syzygy of the ideal sheaf $m(e_0)(1)$, or equivalently

$$0 \to \mathscr{Z} \to \Omega^1(1) \xrightarrow[e_0]{} \mathscr{O} \to k_{e_0} \to 0.$$

We first investigate the sheaf $\mathscr{G}$, which of course by §7 is determined by the cone Q alone. Let $\tilde{\mathscr{G}}$ be the cokernel in:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Gamma \otimes \Omega^3(3) \arrow[r] \arrow[d, equal] & \mathscr{N} \arrow[r] \arrow[d] & \mathscr{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma \otimes \Omega^3(3) \arrow[r] & \mathscr{K} \arrow[r] \arrow[d] & \tilde{\mathscr{G}} \arrow[r] \arrow[d] & 0. \\
 & & \mathscr{T} \arrow[r, equal] \arrow[d] & \mathscr{T} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

9.3.0. PROPOSITION. *$\tilde{\mathscr{G}}$ is the ideal sheaf $\mathscr{J}_{l,Q} \subset \mathscr{O}_Q$ of any line of the cone and $\mathscr{G} = m(e_0)\mathscr{J}_{l,Q}$ ($\tilde{\mathscr{G}}$ is a reflexive Cohen-Macaulay module of the singularity e_0).*

Note that in this case, we have only $\mathscr{G}(2) \subset \tilde{\mathscr{G}}(2) \simeq \mathscr{E}xt^1_{\mathscr{O}}(\mathscr{G}, \mathscr{O})$.

Proof. We have $\Gamma\mathscr{K}(1) = k^2 \otimes W \subset k^2 \otimes \bigwedge^2 V$ and thus the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \Gamma \otimes \mathscr{O} \arrow[r, equal] \arrow[d] & \Gamma \otimes \mathscr{O} \arrow[d] & \\
0 \arrow[r] & k^2 \otimes \mathscr{O}(-1) \arrow[r] \arrow[d, equal] & k^2 \otimes W \otimes \mathscr{O} \arrow[r] \arrow[d] & \mathscr{K}(1) \arrow[r] \arrow[d] & 0, \\
0 \arrow[r] & k^2 \otimes \mathscr{O}(-1) \arrow[r, "\varphi"'] & k^2 \otimes \mathscr{O} \arrow[r] \arrow[d] & \tilde{\mathscr{G}}(1) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

in which φ becomes the matrix $\left(\begin{smallmatrix} z_3 & -z_2 \\ -z_1 & z_3 \end{smallmatrix}\right)$ by the entries of Γ^* and the canonical resolution of $\mathscr{K}$ as $\mathscr{k}^2 \otimes Z$. But such a φ is exactly the resolution of an ideal $\mathscr{I}_{l,W}$ with $l = \{z_2 = z_3 = 0\}$, say. It follows that $\widetilde{\mathscr{G}}/m(e_0)\widetilde{\mathscr{G}}$ has dimension 2 and hence is isomorphic to $\mathscr{T}$. Therefore $\mathscr{G} \simeq m(e_0)\widetilde{\mathscr{G}}$.

The different situations of C or $C^\vee$ (we also identify C with a conic in the plane $P \simeq \mathbb{P}W$) can now be interpreted by the structure of the bidual sheaf $\mathscr{F}^{\vee\vee}$. The situation is similar to that in 9.1 except that all sheaves are singular in the vertex and for elementary transformations we have to consider lines on the cone.

9.3.1. PROPOSITION. *Let $\mathscr{F}$ correspond to a Poncelet pair $(S, C^\vee)$ with S a regular α-conic. Then for $0 \to \mathscr{F} \to \mathscr{F}^{\vee\vee} \to \mathscr{R} \to 0$, we have,*

(i) *$\mathscr{F}^{\vee\vee}$ is locally free outside the vertex e_0 and $c_1\mathscr{F}^{\vee\vee} = 0$.*

(ii) *$\mathscr{F}^{\vee\vee} = i_*\pi^*(\mathscr{F}^{\vee\vee}|P)$, where $\mathbb{P}_3 \xleftarrow{i} \mathbb{P}_3 \setminus \{e_0\} \xrightarrow{\pi} P$, and thus $\mathscr{F}^{\vee\vee}$ is determined by the 2-bundle $\mathscr{F}^{\vee\vee}|P$ with Chern classes $c_1 = 0$, $0 \le c_2 \le 2$.*

(iii)

$$c_2\mathscr{F}^{\vee\vee} = \begin{cases} 2 & \textit{if } C \textit{ is regular}, \\ 1 & \textit{if } C \textit{ is a tangent and a secant}, \\ 0 & \textit{if } C \textit{ is a pair of tangents.} \end{cases}$$

(iv) *$\mathscr{F}$ is stable in each of the different cases of C, which will be described below.*

Proof. (a) The sheaf $\mathscr{Z}$ is also the kernel of $(V/e_0)^\vee \otimes \mathscr{O} \to m(e_0)(1)$ and $\mathscr{Z}|P = \Omega^1_P(1)$. It follows immediately that

$$i_*\pi^*(\mathscr{Z}|P) = \mathscr{Z}.$$

Moreover, the homomorphism

$$\mathscr{Z}^\vee \to \mathscr{O}(1) = i_*\pi^*(\Omega^1_P(2) \xrightarrow{a} \mathscr{O}_P(1))$$

can be described by

$$\begin{tikzcd} (V/e_0) \otimes \mathscr{O} \arrow[r, "e_0\wedge a"] \arrow[d] & e_0 \wedge \bigwedge^2 V \otimes \mathscr{O} \subset \bigwedge^3 V \otimes \mathscr{O} \arrow[d] \\ \mathscr{Z} \arrow[r] & \mathscr{O}(1) \end{tikzcd},$$

and the dual of any homomorphism $\Omega^3(3) \xrightarrow[e_0\wedge a]{} \mathscr{Z} \subset \Omega^1(1)$ is of this form.

(b) Let now $\widetilde{\mathscr{F}}$ denote the cokernel of

$$\begin{array}{l} 0 \to \mathscr{k}^2 \otimes \Omega^3(3) \xrightarrow[M^*]{} \mathscr{k}^2 \otimes \mathscr{Z} \to \widetilde{\mathscr{F}} \to 0 \\ \qquad\qquad\qquad\qquad\quad \cap \\ \qquad\qquad\qquad\quad \mathscr{k}^2 \otimes \Omega^1(1), \end{array}$$

where as before M^* represents M. Then from $0 \to \mathscr{F} \to \widetilde{\mathscr{F}} \to \mathscr{T} \to 0$ we get $\widetilde{\mathscr{F}}^\vee = \mathscr{F}^\vee$. In our case moreover $M^* = e_0 \wedge A$ for some $\mathscr{k}^2 \xrightarrow[A]{} \mathscr{k}^2 \otimes U$ and thus we have the diagram

$$(26)\qquad \begin{array}{ccccccc} & & & \mathscr{k}^2 \otimes (V/e_0) \otimes \mathscr{O} & \xrightarrow{e_0\wedge A} & \mathscr{k}^2 \otimes \bigwedge^3 V \otimes \mathscr{O} & \\ & & & \downarrow & & \downarrow & , \\ 0 \longrightarrow & \mathscr{F}^\vee & \longrightarrow & \mathscr{k}^2 \otimes \mathscr{Z}^\vee & \longrightarrow & \mathscr{k}^2 \otimes \mathscr{O}(1) & \end{array}$$

which, after restriction to P, gives the exact sequence

$$(26_{\mathrm{P}})\qquad 0 \to \mathscr{F}^\vee|P \to \mathscr{k}^2 \otimes \Omega^1_P(2) \xrightarrow{A'} \mathscr{k}^2 \otimes \mathscr{O}_P(1) \to \Phi \to 0.$$

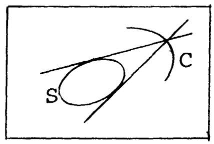
(c) This proves the proposition: Since $\mathscr{F}^\vee|P$ is reflexive on P, it is locally free. We must have

$$i_*\pi^*(\mathscr{F}^\vee|P) = \mathscr{F}^\vee$$

by (a) and diagram (26). Hence $\mathscr{F}^\vee$ is locally free on $\mathbb{P}_3 \setminus \{e_0\}$. Taking the dual of this identity yields (i), (ii). From (26_{P}) we get $c_1(\mathscr{F}^\vee|P) = 0$, $c_2(\mathscr{F}^\vee|P) = 2 - h^0\Phi$, where we note that Φ must have 0-dimensional support. This proves (iii), since it is shown below that $h^0\Phi = 0, 1, 2$ in the different cases of C.

We are going now to describe $\mathscr{F}^{\vee\vee} \to \mathscr{R}$ in the different cases of C. Note first that the conic $C^\vee \subset \mathbb{P}U^\vee \simeq PW^\vee$ has the equation $\det A = 0$, where as above $M^* = e_0 \wedge A$ with entries in U. This follows from our convention $\bigwedge^2 V \simeq \bigwedge^2 V^\vee$ and $\bigwedge^2 W \simeq W^\vee$; $W = e_0 \wedge U$.

Case 1: in which $C^\vee$ is regular. In this case the entries of M^* or A span the space W or U and the sheaf $\mathscr{M}$ in display (22) must be locally free except at e_0. Since $\operatorname{Supp}\mathscr{E} = \{e_0\}$, it follows from the same display that also $\mathscr{F}$ is locally free on $\mathbb{P}_3\setminus\{e_0\}$. Moreover, if we consider (26_P) in this case, we see by the form of the matrix A that $\Phi = 0$ and hence $\mathscr{F}^\vee|P$ and $\mathscr{F}|P$ are bundles with Chern-classes $c_1 = 0$, $c_2 = 2$. Its jumping lines are exactly the points of $C^\vee$ as can be calculated from its representing matrix A. Since $C^\vee$ is the polar dual of C w.r.t. S, the jumping lines are the polars of points of C w.r.t. S.



There is a unique subspace $L^\vee \subset \Gamma(\mathscr{F}|P)(1)$ s.t.

$$0 \to L^\vee \otimes \mathscr{O}_P(1) \to \mathscr{F}|P \to \mathscr{O}_S(-1) \to 0, \tag{27}$$

the cokernel of the evaluating homomorphism is $\mathscr{O}_S(-1)$, and this sequence is nothing but the restriction of the sequence

$$0 \to (\Gamma/M)\otimes\mathscr{O}(-1) \to \mathscr{F} \to \mathscr{G} \to 0.$$

If we start with (27) and apply $i_*\pi^*$ we obtain the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L^\vee \otimes \mathscr{O}(-1) \arrow[r] \arrow[d, equal] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L^\vee \otimes \mathscr{O}(-1) \arrow[r] & \mathscr{F}^{\vee\vee} \arrow[r] \arrow[d] & \widetilde{\mathscr{G}} \arrow[r] \arrow[d] & 0, \\
 & & \mathscr{T} \arrow[r, equal] \arrow[d] & \mathscr{T} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

in which $\mathscr{F}$ is the pullback of $\mathscr{G} \subset \widetilde{\mathscr{G}}$. Therefore $\mathscr{R} = \mathscr{T}$ in this case.

REMARK. If we pursue the point of view of elementary transformations $\mathscr{F}^{\vee\vee} \xrightarrow[h]{} \mathscr{T}$, the sheaves $\mathscr{F}^{\vee\vee}$ and $\mathscr{T}$ can be defined by $C \subset P$ and e_0, and then the epimorphism h corresponds to a conic S which is regular and in Poncelet relation with C or $C^\vee$.

Stability. In Case 1 the sheaf $\mathscr{F}^{\vee\vee}$ has no non-zero section and thus μ-stable, hence also $\mathscr{F}$ is μ-stable.

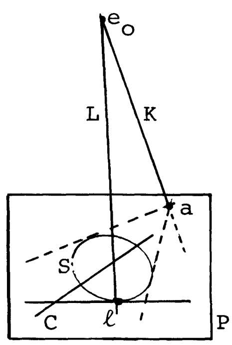
Cases 2/3: in which C is a pair of lines, a tangent and a secant of S. Let L be the line joining the tangent point with the vertex and K be the line joining the pole of the secant with the vertex. In this case the sheaf $\mathscr{F}$ can be described as follows:

(a) $\mathscr{R}$ is the structure sheaf

$$\mathscr{R} = \mathscr{O}_Q/\mathscr{G} = \mathscr{O}_Q/m(e_0)\mathscr{I}_{L,Q}$$

of the line L with a multiple point in e_0 and we have the exact sequence

$$0 \to \mathscr{T} \to \mathscr{R} \to \mathscr{O}_L \to 0.$$



(b) The restriction $\mathscr{F}^{\vee\vee}|P$ is the unique 2-bundle on P with $c_2(\mathscr{F}^{\vee\vee}|P) = 1$ such that its jumping lines are the lines in P through the pole a. (Such bundles are never stable, since $h^0(\mathscr{F}^{\vee\vee}|P) = 1$, see **[Ba]**.)

(c) There is a unique subspace $L^\vee \subset \Gamma(\mathscr{F}^{\vee\vee}(1)|P)$ s.t. the evaluation map yields the sequence

$$0 \to L^\vee \otimes \mathscr{O}_P(-1) \to \mathscr{F}^{\vee\vee}|P \to \mathscr{O}_S \to 0.$$

Pulling this up via $i_*\pi^*$ we get the pullback diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L^\vee \otimes \mathscr{O}(-1) \arrow[r] \arrow[d, equal] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{G} \arrow[r] \arrow[d] & \mathscr{O} \\
0 \arrow[r] & L^\vee \otimes \mathscr{O}(-1) \arrow[r] & \mathscr{F}^{\vee\vee} \arrow[r] \arrow[d] & \mathscr{O}_Q \arrow[r] \arrow[d] & 0. \\
 & & \mathscr{R} \arrow[r, equal] \arrow[d] & \mathscr{R} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

(d) $\mathscr{F}$ is *stable* (although $\mathscr{F}^{\vee\vee}$ is not semi-stable).

Proof. We first investigate the sheaf $\widetilde{\mathscr{F}}$ which was introduced as the cokernel of $M \otimes \Omega^3(3) \to \ell^2 \otimes \mathscr{Z}$. Since now $M^* = e_0 \wedge A$ with (up to equivalence)

$$A = \begin{bmatrix} l & 0 \\ b & a \end{bmatrix},$$

and since we get the exact sequences

$$0 \to \Omega^3(3) \xrightarrow[e_0\wedge a]{} \mathscr{Z} \to \mathscr{J}_K \to 0,$$
$$0 \to \Omega^3(3) \xrightarrow[e_0\wedge l]{} \mathscr{Z} \to \mathscr{J}_L \to 0,$$

the sheaf $\widetilde{\mathscr{F}}$ must be an extension of the kind

$$0 \to \mathscr{J}_L \to \widetilde{\mathscr{F}} \to \mathscr{J}_K \to 0.$$

Taking the bidual gives us a diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & I_L \arrow[r] \arrow[d] & \widetilde{\mathscr{F}} \arrow[r] \arrow[d] & \mathscr{I}_K \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{O} \arrow[r] \arrow[d] & \mathscr{F}^{\vee\vee} \arrow[r] \arrow[d] & \mathscr{I}_K \arrow[r] & 0, \\
 & \mathscr{O}_L \arrow[r, equal] \arrow[d] & \mathscr{O}_L \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \tag{28}
$$

as can be easily checked. Restricting the evaluating sequence $(\Gamma/M) \otimes \mathscr{O}(1) \to \mathscr{F} \to \mathscr{G}$ to P we obtain the diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (\Gamma/M) \otimes \mathscr{O}_P(-1) \arrow[r] \arrow[d, equal] & \mathscr{F}|P \arrow[r] \arrow[d] & \mathscr{G}|P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\Gamma/M) \otimes \mathscr{O}_P(-1) \arrow[r] & \mathscr{F}^{\vee\vee}|P \arrow[r] \arrow[d] & \mathscr{L} \arrow[r] \arrow[d] & 0, \\
 & & \mathscr{O}_L \otimes \mathscr{O}_P \arrow[r, equal] \arrow[d] & \mathscr{O}_L \otimes \mathscr{O}_P \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{29}
$$

where $\mathscr{L}$ is defined as cokernel. Now $\mathscr{L}$ is supported on $S = Q \cap P$ and an $\mathscr{O}_S$-module, since $\mathscr{G}|P = \mathscr{O}_S(-1)$ and $\mathscr{O}_L \otimes \mathscr{O}_P = k_l$. Since by the middle row its depth is $= 1$, it is a line bundle on S. But $h^0\mathscr{L} = 1$, indeed $h^0(\mathscr{F}^{\vee\vee}|P) = 1$. Therefore $\mathscr{L} = \mathscr{O}_S$. Now the proposition can be derived:

It is clear that $c_1(\mathscr{F}^{\vee\vee}|P) = 0$, $c_2(\mathscr{F}^{\vee\vee}|P) = 1$ by (29) and that $h^0(\mathscr{F}^{\vee\vee}|P) = 0$. The jumping lines are exactly those through a, which follows from $0 \to \mathscr{O} \to \mathscr{F}^{\vee\vee} \to \mathscr{J}_K \to 0$ by restricting to P and by investigating the result of $\otimes \mathscr{O}_{L'}$ for a line $L' \subset P$. Finally (c) follows by pulling back the middle row of (29), which also gives the definition of $\mathscr{R}$. Since $\mathscr{O}_L = \mathscr{O}_Q / \widetilde{\mathscr{G}}$ we get (a).

To prove the stability of $\mathscr{F}$ we remark that for any nonzero section of $\mathscr{F}^{\vee\vee}$ the composed homomorphism must be onto $\mathscr{R}$:

$$\begin{tikzcd} \mathscr{O} \arrow[r] \arrow[d] \arrow[dr] & \mathscr{O}_Q \arrow[d] \\ \mathscr{F}^{\vee\vee} \arrow[r] & \mathscr{R} \end{tikzcd}.$$

If now $\mathscr{F}' \subset \mathscr{F}$ is a rank-1 subsheaf with $\mathscr{F}/\mathscr{F}' = \mathscr{F}''$ torsionfree, we can assume that $c_1\mathscr{F}' = 0$. Then we get a diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathscr{F}' \arrow[r] \arrow[d] & \mathscr{O} \arrow[r] \arrow[d] \arrow[dr, "\psi"] & \mathscr{O}_{Z'} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}^{\vee\vee} \arrow[r] \arrow[d] & \mathscr{R} \arrow[r] \arrow[d] & 0, \\ 0 \arrow[r] & \mathscr{F}'' \arrow[r, "\varphi''"'] \arrow[d] & \mathscr{J}_K \arrow[r] \arrow[d] & \mathscr{C}'' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

which is exact, because φ'' is nonzero and injective, and because $\mathscr{F}''$ is torsionfree. Since ψ must be surjective, we conclude that $\mathscr{C}'' = 0$ and $\mathscr{F}'' = \mathscr{J}_K$.

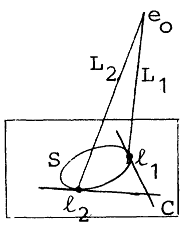
Now

$$\chi\mathscr{F}'(m) = \chi\mathscr{O}(m) - \chi\mathscr{R}(m) = \chi\mathscr{O}(m) - \chi\mathscr{O}_L(m) - 2,$$
$$\chi\mathscr{F}''(m) = \chi\mathscr{O}(m) - \chi\mathscr{O}_K(m).$$

This shows that $\chi\mathscr{F}'(m) < \frac{1}{2}\chi\mathscr{F}(m)$.

Case 4: in which C is a pair of tangents. In this case the bidual $\mathscr{F}^{\vee\vee} = 2\mathscr{O}$, too, and we have the diagram

(30)



$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 2\mathscr{O}(-1) \arrow[r] \arrow[d, equal] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 2\mathscr{O}(-1) \arrow[r] & 2\mathscr{O} \arrow[r] \arrow[d] & \widetilde{\mathscr{G}}(1) \arrow[r] \arrow[d] & 0 \\
 & & \mathscr{R} \arrow[r, equal] \arrow[d] & \mathscr{R} \arrow[d] & \\
 & & {} & {} &
\end{tikzcd}$$

in which $\mathscr{R}$ is defined by the right-hand column, and thus determined by the cone. It fits into the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{G} \arrow[r] \arrow[d, equal] & \widetilde{\mathscr{G}} \arrow[r] \arrow[d] & \mathscr{T} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{T} \arrow[r] & \widetilde{\mathscr{G}}(1) \arrow[r] \arrow[d] & \mathscr{R} \arrow[r] \arrow[d] & 0, \\
 & & \mathscr{O}_{L_1} \oplus \mathscr{O}_{L_2} \arrow[r, equal] \arrow[d] & \mathscr{O}_{L_1} \oplus \mathscr{O}_{L_2} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{31}$$

which is derived in the proof. Also in this case $\mathscr{F}$ is *stable.*

Proof. Because C consists of two tangents we can choose the basis of U so that $l_1 = e_1$, $l_2 = e_2$. By the shape of the matrix Γ^* above we find that the only possibility of $M^* = e_0 \wedge A$ is the direct sum

$$A = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}.$$

It follows from the previous proof that $\widetilde{\mathscr{F}} = \mathscr{I}_{L_1} \oplus \mathscr{I}_{L_2}$ and therefore $\mathscr{F}^{\vee\vee} = \widetilde{\mathscr{F}}^{\vee\vee} = 2\mathscr{O}$. Furthermore the diagram (29) now becomes

$$\tag{29'}\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 2\mathscr{O}_P(-1) \arrow[r] \arrow[d, equal] & \mathscr{F}|P \arrow[r] \arrow[d] & \mathscr{G}|P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 2\mathscr{O}_P(-1) \arrow[r] & 2\mathscr{O}_P \arrow[r] \arrow[d] & \mathscr{L} \arrow[r] \arrow[d] & 0, \\
 & & k_{l_1} \oplus k_{l_2} \arrow[r, equal] \arrow[d] & k_{l_1} \oplus k_{l_2} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

and we conclude that $\mathscr{L} = \mathscr{O}_S(1)$ as in Cases 2/3. Since $\mathscr{G}|P = \mathscr{O}_S(-1)$, the right-hand column becomes the top row of the next diagram

$$\tag{32}\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathscr{O}_S(-1) \arrow[r, "z_3|S"'] \arrow[u] & \mathscr{O}_S(1) \arrow[r] \arrow[u] & k_{e_1} \oplus k_{e_2} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & 2\mathscr{O}_P(-1) \arrow[r, "z_3"'] \arrow[u] & 2\mathscr{O}_P \arrow[r] \arrow[u] & 2\mathscr{O}_{\mathscr{g}} \arrow[r] \arrow[u] & 0, \\
0 \arrow[r] & 2\mathscr{O}_P(-2) \arrow[r, "z_3"'] \arrow[u, "\varphi"'] & 2\mathscr{O}_P(-1) \arrow[r] \arrow[u, "\varphi"'] & 2\mathscr{O}_{\mathscr{g}}(-1) \arrow[r] \arrow[u, "\varphi|\mathscr{g}"'] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where z_3 is the equation of the line $\mathscr{g} \subset P$ through e_1, e_2. The homomorphism φ can be given (up to equivalence) by

$$\varphi = \begin{bmatrix} z_3 & -z_2 \\ -z_1 & z_3 \end{bmatrix},$$

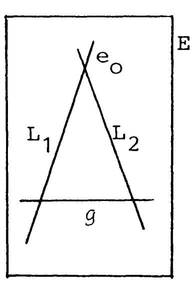
s.t. $\det\varphi = z_3^2 - z_1 z_2$ is the equation of the conic S (see definition of $\widetilde{\mathscr{G}}$). From this we see that

$$\varphi|_{\mathscr{g}} = \begin{bmatrix} 0 & -z_2 \\ -z_1 & 0 \end{bmatrix}.$$

If we apply $i_*\pi^*$ to the last diagram (32) we obtain

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \widetilde{\mathscr{G}} \arrow[r] \arrow[u] & \widetilde{\mathscr{G}}(1) \arrow[r] \arrow[u] & \mathscr{O}_{L_1} \oplus \mathscr{O}_{L_2} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & 2\mathscr{O}(-1) \arrow[r, "z_3"'] \arrow[u] & 2\mathscr{O} \arrow[r] \arrow[u] & 2\mathscr{O}_E \arrow[r] \arrow[u] & 0, \\
0 \arrow[r] & 2\mathscr{O}(-2) \arrow[r, "z_3"'] \arrow[u, "\varphi"'] & 2\mathscr{O}(-1) \arrow[r] \arrow[u, "\varphi"'] & 2\mathscr{O}_E(-1) \arrow[r] \arrow[u, "\varphi|_E"'] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{33}$$

where now E is the plane $z_3 = 0$, spanned by the two lines $L_1 \cup L_2 = Q \cap E$. The top row of (33) gives us the diagram (31) with the definition of $\mathscr{R}$. Diagram (30) follows from diagram (29′) by pulling back via $i_*\pi^*$ again, which first gives the corresponding diagram with $\widetilde{\mathscr{F}} = i_*\pi^*(\mathscr{F}|P)$ and $\widetilde{\mathscr{G}}$, and then imbedding the sequence $0 \to 2\mathscr{O}(-1) \to \mathscr{F} \to \mathscr{G} \to 0$ into its first row.



The proof of stability of $\mathscr{F}$ is reduced to that of Cases 2 and 3 as follows: We have the two diagrams

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{\mathscr{G}} & \longrightarrow & \mathscr{O}_Q & \longrightarrow & \mathscr{O}_{L_1} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{\mathscr{G}} & \longrightarrow & \tilde{\mathscr{G}}(1) & \longrightarrow & \mathscr{O}_{L_1} \oplus \mathscr{O}_{L_2} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathscr{O}_{L_2} & = & \mathscr{O}_{L_2} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{O}_Q & \longrightarrow & \mathscr{R}_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{G} & \longrightarrow & \tilde{\mathscr{G}}(1) & \longrightarrow & \mathscr{R} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathscr{O}_{L_2} & = & \mathscr{O}_{L_2} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

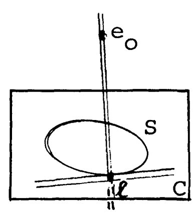
where we use in the first one, that $\mathscr{G} = \mathscr{I}_{L_1, Q}$, and where $\mathscr{R}_1$ in the second is the sheaf $\mathscr{R}$ of the Case 2/3 with $L = L_1$. From the right-hand column of the second we obtain the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d, equal] & \mathscr{F}_1 \arrow[r] \arrow[d] & \mathscr{R}_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F} \arrow[r] & 2\mathscr{O} \arrow[r] \arrow[d] & \mathscr{R} \arrow[r] \arrow[d] & 0 \\
 & & \mathscr{O}_{L_2} \arrow[r, equal] \arrow[d] & \mathscr{O}_{L_2} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

in which $\mathscr{F}_1$ is the pullback and must be isomorphic to $\mathscr{O} \oplus \mathscr{J}_{L_2}$. Now we can use the top row to proceed as in Cases 2/3 to prove stability of $\mathscr{F}$, because any non-zero section of $\mathscr{F}_1$ factorises through $\mathscr{O}_Q$:

$$\begin{tikzcd}
\mathscr{O} \arrow[r] \arrow[d] \arrow[dr] & \mathscr{O}_Q \arrow[d] \\
\mathscr{F}_1 \arrow[r] & \mathscr{R}_1
\end{tikzcd} .$$

Case 5: in which C is a double tangent. This is a special case of



Case 4 and we obtain by the same method that $\mathscr{F}^{\vee\vee} = 2\mathscr{O}$ and the diagrams

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{G} \arrow[r] \arrow[d, equal] & \tilde{\mathscr{G}} \arrow[r] \arrow[d] & \mathscr{T} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{G} \arrow[r] & \tilde{\mathscr{G}}(1) \arrow[r] \arrow[d] & \mathscr{R} \arrow[r] \arrow[d] & 0 \\
 & & \mathscr{O}_{\tilde{\mathscr{L}}} \arrow[r, equal] \arrow[d] & \mathscr{O}_{\tilde{\mathscr{L}}} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 2\mathscr{O}(-1) \arrow[r] \arrow[d, equal] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 2\mathscr{O}(-1) \arrow[r] & 2\mathscr{O} \arrow[r] \arrow[d] & \tilde{\mathscr{G}}(1) \arrow[r] \arrow[d] & 0, \\
 & & \mathscr{R} \arrow[r, equal] \arrow[d] & \mathscr{R} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

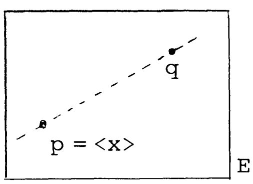
where now $\mathscr{O}_{\tilde{L}}$ denotes the double structure of L in Q. Again by the same method the stability of $\mathscr{F}$ can be proved.

10. Sheaves in the boundary with singular S. If the conic S is degenerate the sheaves $\mathscr{N}$ and $\mathscr{F}$ are of completely different nature. The sheaf $\mathscr{N}$ is always semi-stable and S only determines the stable gradation of $\mathscr{N}$. We are going to describe this gradation first. As in

§9 we drop the indices of the sheaves and conics for a given point $x = (z, M, \Gamma) \in X^{ss}$, $y = (z, \Gamma)$.

If $\mathscr{F}$ is any coherent sheaf on $\mathbb{P}V$ we write as usual $\mathscr{P}(\mathscr{F})(d) = \chi\mathscr{F}(d)/\operatorname{rk}\mathscr{F}$. If $\mathscr{F}$ is semi-stable there is a filtration $0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_n = \mathscr{F}$ by coherent subsheaves, such that $\mathscr{F}_i/\mathscr{F}_{i-1}$ is stable for $1 \le i \le n$ and $\mathscr{P}(\mathscr{F}_i) = \mathscr{P}(\mathscr{F})$, **[Ma2]**. The direct sum $\operatorname{Gr}(\mathscr{F}) = \bigoplus \mathscr{F}_i/\mathscr{F}_{i-1}$ is unique up to isomorphisms and called the stable gradation. In order to describe the stable gradation of the $\mathscr{N}$'s for singular conic S, we consider the following rank-2 sheaves associated to planes in $\mathbb{P}_3$ together with an ordered pair of points in the plane.

10.1. Let $E \subset \mathbb{P}_3$ be a plane and $p, q \in E$. The sheaf $\mathscr{M}(p, E, q) = \mathscr{M}$ is defined by the exact diagram as follows:



$$\begin{tikzcd} & & 0 \arrow[d] & & \\ 0 \arrow[r] & \mathscr{M} \arrow[r] & \mathscr{Z} \arrow[r, "\pi"'] \arrow[d] & k_q \arrow[r] & 0 \\ & & (V/x)^\vee \otimes \mathscr{O} \arrow[d] & & \\ & & m(p)(1) \arrow[d] & & \\ & & 0 & & \end{tikzcd}$$

In this diagram $m(p)(1)$ is the ideal sheaf of p in twist 1, $\mathscr{Z}$ its first syzygy and π the epimorphism defined by the plane E by the

10.1.1. LEMMA. *There is a $1 : 1$ correspondence between $\mathbb{P}\operatorname{Hom}(\mathscr{Z}, k_q)$ and the set of planes E through p, q (if $p = q$ the line $\overline{p, q}$ is replaced by a tangent direction in p).*

Proof. The Koszul resolution of $\mathscr{Z}$ and an epimorphism give rise to a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{O}(-2) \arrow[r, "\alpha"] \arrow[d] \arrow[dr, "e"] & 3\mathscr{O}(-1) \arrow[r] \arrow[d] & \mathscr{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & m(q)(-1) \arrow[r] & \mathscr{O}(-1) \arrow[r] & k_q \arrow[r] & 0 \end{tikzcd}$$

with $\alpha = (z_1, z_2, z_3)$ consisting of a basis of $(V/x)^\vee$, and with the linear form e vanishing in $p = \langle x \rangle$ and q. Conversely any such e factorises through α and $\mathscr{m}(q)(-1)$, thereby defining a non-zero homomorphism $\mathscr{Z} \to \mathscr{k}_q$.

10.1.2. Proposition. *The sheaf $\mathscr{M}(p, E, q)$ has the following properties:*

(i) $\operatorname{rk}\mathscr{M} = 2$ *and* $\mathscr{P}\mathscr{M}(d) = \frac{1}{12}(d+2)(d+3)(2d-1)$.

(ii) $\mathscr{M}$ *has Chern polynomial* $1 - h + h^2 - h^3$.

(iii) $\mathscr{M}$ *is μ-stable.*

(iv) *The sections of $\mathscr{M}(1)$ are in one to one correspondence with the lines in E through p, so that a line is the zero locus of the section.*

(v) $h^0\mathscr{M}(1) = 2$ *and* $\mathscr{M}$ *has the evaluation sequence*

$$0 \to 2\mathscr{O} \to \mathscr{M}(1) \to \mathscr{m}_E(q)(1) \to 0,$$

where $\mathscr{m}_E(q) \subset \mathscr{O}_E$ denotes the ideal sheaf of q in E.

(vi) $h^1\mathscr{M}(d) = 0$ *for* $d \geq 1$.

Proof. (i) and (ii) follow directly from the defining diagram. Since $\mathscr{Z}$ is reflexive with $c_1\mathscr{Z} = -1$ and $h^0\mathscr{Z} = 0$, this sheaf is μ-stable and then also $\mathscr{M}$. That $h^0\mathscr{M}(1) = 2$ and $h^1\mathscr{M}(d) = 0$ for $d \geq 1$ also follow easily from the definition and properties of $\mathscr{Z}$. To prove (v) we consider the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] \arrow[d] & 2\mathscr{O}(-1) \arrow[r] \arrow[d] & \mathscr{M} \arrow[r] \arrow[d] & \mathscr{C} \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{O}(-2) \arrow[r] \arrow[d, "e'"'] \arrow[dr, "e"] & 3\mathscr{O}(-1) \arrow[r] \arrow[d] & \mathscr{Z} \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & \mathscr{m}(q)(-1) \arrow[r] \arrow[d] & \mathscr{O}(-1) \arrow[r] \arrow[d] & \mathscr{k}_q \arrow[r] \arrow[d] & 0 & \\
 & \mathscr{m}_E(q)(-1) \arrow[d] & 0 & 0 & & \\
 & 0 & & & &
\end{tikzcd}$$

used already in the proof of the lemma. The cokernel of e' is $m_E(q)(-1)$ and this is isomorphic to $\mathscr{C}$. From this we also see that any section of $\mathscr{M}(1)$ must have its zero locus in E. Finally to prove (iv) we note that any section of $\mathscr{Z}(1)$ vanishes exactly on a line $L \ni p$ and gives rise to a diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathscr{O}(-1) \arrow[r, equal] \arrow[d] & \mathscr{O}(-1) \arrow[d] & \\
0 \arrow[r] & \mathscr{O}(-2) \arrow[r, "\alpha"'] \arrow[d, equal] & 3\mathscr{O}(-1) \arrow[r] \arrow[d] & \mathscr{Z} \arrow[r] \arrow[d] & 0, \\
0 \arrow[r] & \mathscr{O}(-2) \arrow[r, "\beta"'] & 2\mathscr{O}(-1) \arrow[r] \arrow[d] & \mathscr{I}_L \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where β consists of two independent linear forms with cokernel the ideal of the line they define. Therefore a section of $\mathscr{M}(1)$ must vanish on a line L in E through p, and gives rise to the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathscr{O}(-1) \arrow[r, equal] \arrow[d] & \mathscr{O}(-1) \arrow[d] & & \\
0 \arrow[r] & \mathscr{M} \arrow[r] \arrow[d] & \mathscr{Z} \arrow[r] \arrow[d] & k_q \arrow[r] \arrow[d, equal] & 0, \\
0 \arrow[r] & \mathscr{I}_{L\cup q} \arrow[r] \arrow[d] & \mathscr{I}_L \arrow[r] \arrow[d] & k_q \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

in which either $q \notin L$ or $\mathscr{I}_{L\cup q}$ is the ideal sheaf of L with a multiple structure in q. Conversely given any such line, we can define the section by the last two diagrams.

10.1.3. Corollary. *For any line L with $p \in L \subset E$ there is a diagram*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & \mathscr{O}(-1) \arrow[r] \arrow[d] & 2\mathscr{O}(-1) \arrow[r] \arrow[d] & \mathscr{O}(-1) \arrow[r] & 0 \\
 & & \mathscr{M} \arrow[r, equal] \arrow[d] & \mathscr{M} \arrow[d] & & \\
0 \arrow[r] & \mathscr{O}(-1) \arrow[r] & \mathscr{I}_{L\cup q} \arrow[r] \arrow[d] & \mathscr{m}_E(q)(-1) \arrow[r] \arrow[d] & 0 & \\
 & & 0 & 0 & &
\end{tikzcd}$$

For later use we also need the

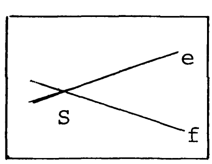
10.1.4. Lemma. *Any non-trivial extension of $\mathscr{m}_E(q)(-1)$ by $\mathscr{O}(-1)$ is of the above form $\mathscr{I}_{L\cup q}$ for some line L as above.*

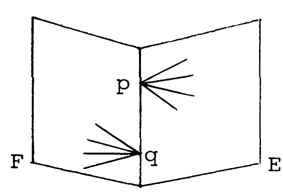
Its proof can be derived from the equalities

$$\operatorname{Ext}^1_{\mathscr{O}}(\mathscr{m}_E(q), \mathscr{O}) = \operatorname{Ext}^1_{\mathscr{O}}(\mathscr{O}_E, \mathscr{O}) = \Gamma\mathscr{O}_E(1)$$

and will be left to the reader.

10.2. *The sheaf $\mathscr{N}$ for singular conic S.* We consider first the generic case in which S is the intersection of G with a plane and consists of two different lines e, f. Then S defines a regulus in $\mathbb{P}_3$ supported by two planes $E \cup F$.





By the construction in 10.1 this configuration defines the sheaves

$$\mathcal{N}^e = \mathcal{M}(p, E, q) \quad \text{and} \quad \mathcal{N}^f = \mathcal{M}(q, F, p).$$

10.2.1. PROPOSITION. *Let $\mathcal{N} = \mathcal{N}_y$ be defined by $y = (z, \Gamma)$ with $S = S(\Gamma)$ as above. Then*

(i) *$\mathcal{N}^e \oplus \mathcal{N}^f$ is the stable gradation of $\mathcal{N}$.*

(ii) *Let Γ be presented by one of the normal forms*

$$\text{(a)} \begin{pmatrix} \xi & 0 \\ \omega & 0 \\ \eta & \omega \\ 0 & \eta \end{pmatrix} \quad \text{(b)} \begin{pmatrix} \xi & \\ \omega & \\ & \omega \\ & \eta \end{pmatrix} \quad \text{(c)} \begin{pmatrix} \xi & \\ \omega & \xi \\ & \omega \\ & \eta \end{pmatrix} \quad \textit{see } 5.3.$$

Then in these different cases $\mathcal{N}$ is an extension:

(a) $0 \to \mathcal{N}^e \to \mathcal{N} \to \mathcal{N}^f \to 0$ *(non-trivial),*

(b) $\mathcal{N} = \mathcal{N}^e \oplus \mathcal{N}_f$,

(c) $0 \to \mathcal{N}^f \to \mathcal{N} \to \mathcal{N}^e \to 0$ *(non-trivial).*

(iii) *The different cases of $\mathcal{N}$ are distinguished by the singularities on $\mathcal{N}$:*

(a) $\operatorname{Sing}\mathcal{N} = \{p\}$, (b) $\operatorname{Sing}\mathcal{N} = \{p, q\}$, (c) $\operatorname{Sing}\mathcal{N} = \{q\}$.

10.2.2. REMARK. At the first glimpse it is a surprise that each of the sheaves $\mathcal{N}^e$, $\mathcal{N}^f$ has two singular points whereas $\mathcal{N}$ has only one in cases (a) and (c), but this is in accordance with the depths of the sheaves in these points.

Proof. By 5.3, Γ has only three normal forms in (ii). Let N be the space associated to Γ by 5.8. Then $N^\vee$ is presented by a matrix of the type

$$\begin{pmatrix} x & 0 \\ y & y' \end{pmatrix}, \quad \begin{pmatrix} x & 0 \\ 0 & y' \end{pmatrix}, \quad \begin{pmatrix} x & x' \\ 0 & y' \end{pmatrix}$$

in the three different cases respectively, where $p = \langle x \rangle$, $q = \langle y' \rangle$. In the direct sum case (b), we then have

$$\mathcal{C} = k_p \oplus k_q = \mathcal{T},$$

and the display diagram (14) gives us $\mathcal{K} = \mathcal{Z}^p \oplus \mathcal{Z}^q$, where $\mathcal{Z}^p$

denotes the syzygy of $m(p)(1)$. In the situation (a) we get the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & \\
0 \arrow[r] & \mathcal{N}^e \arrow[r] \arrow[d] & \mathcal{N} \arrow[r] \arrow[d] & \mathcal{Z}^q \arrow[r] \arrow[d, equal] & k_p \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{Z}^p \arrow[r] \arrow[d] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{Z}^q \arrow[r] & k k_p \arrow[r] & 0 \\
 & k_q \arrow[r, equal] \arrow[d] & \mathcal{T} \arrow[d] & & & \\
 & 0 & 0 & & &
\end{tikzcd}$$

in which the middle row follows from the shape of $\left(\begin{smallmatrix} x & 0 \\ x & y' \end{smallmatrix}\right)$ and a corresponding extension diagram. Here $\mathcal{C} = k_q$, too. In case (c) we get the analogous diagram with p, q interchanged, and in the direct sum case (b) the diagram specialises to

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{N}^e \arrow[r] \arrow[d] & \mathcal{N} \arrow[r] \arrow[d] & \mathcal{N}^f \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{Z}^p \arrow[r] \arrow[d] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{Z}^q \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & k_q \arrow[r] \arrow[d] & \mathcal{T} \arrow[r] \arrow[d] & k_p \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

From these diagrams we easily derive (i), (ii), and also (iii) by looking at the local cohomology groups $H_{\{p\}}\mathcal{N}$, $H_{\{q\}}\mathcal{N}$ etc.

From the extensions in (ii) we can determine also the sheaf $\mathscr{G}$ by the diagram (in case (a) for example)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma' \oplus \Omega^3(3) & \longrightarrow & \Gamma \oplus \Omega^3(3) & \longrightarrow & \Gamma'' \oplus \Omega^3(3) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{N}^e & \longrightarrow & \mathscr{N} & \longrightarrow & \mathscr{N}^f & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & m_E(q)(-1) & \longrightarrow & \mathscr{G} & \longrightarrow & m_F(p)(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where Γ' and Γ'' are defined by the shape of Γ, by which we have an exact sequence

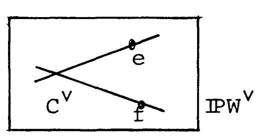
$$\begin{array}{ccccccccc}
0 & \to & \Gamma' & \to & \Gamma & \longrightarrow & \Gamma'' & \to 0. \\
 & & \cap & & \cap & & \cap & \\
0 & \to & W_z & \to & k^2 \oplus W_z & \to & W_z & \to 0.
\end{array}$$

10.3. Lemma. *Let* $x = (z, M, \Gamma) \in X^{ss}$ *be a semi-stable point with* Γ *as in* (a) *or* (b). *With the previous notation the following are equivalent*:

(i) x *is stable,*

(ii) $M \cap \Gamma' = 0$,

(iii) *the pair* $(S, C^\vee(M))$ *is not singular, i.e.* $C^\vee(M)$ *is not a pair of lines passing through the two points* e, $f \in \mathbb{P}W^\vee$ *of* S, *see* 3.12, (i).



Proof. For example in case (a) we have: $M \cap \Gamma' \neq 0$ if and only if M is presented by a matrix of the form

$$\begin{pmatrix} a\xi + b\omega & 0 \\ * & c\omega + d\eta \end{pmatrix},$$

and such M are exactly those which define singular conics $C^\vee$ passing through both points e, f. The cases (b), (c) are proved similarly.

10.4. PROPOSITION. *Let* $x = (z, M, \Gamma)$ *be a* stable *point with* Γ *of type* (a) *or* (b). *Then the sheaf* $\mathscr{F} = \mathscr{F}_x$ *is* stable *and fits into a diagram*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & M \otimes \Omega^3(3) \arrow[r, phantom, "\cong"] \arrow[d] & 2\Omega^3(3) \arrow[d] & \\
0 \arrow[r] & \mathscr{N}^e \arrow[r] \arrow[d, equal] & \mathscr{N} \arrow[r] \arrow[d] & \mathscr{N}^f \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{N}^e \arrow[r] & \mathscr{F} \arrow[r] \arrow[d] & m_F(p)(-1) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

A similar statement holds in case (c) *with* e, f *interchanged.*

Proof. The diagram follows immediately from 10.3 because $M \cap \Gamma\mathscr{N}^e(1) = 0$, and from (v) of the proposition in 10.1. To prove stability, let $\mathscr{F}' \subset \mathscr{F}$ be any rank 1 subsheaf with torsionfree quotient $\mathscr{F}'' = \mathscr{F}/\mathscr{F}'$. We can assume $c_1\mathscr{F}' = 0$, otherwise $P(\mathscr{F}')(d) < \frac{1}{2}P(\mathscr{F})(d)$ would be trivially satisfied. If $\mathscr{N}' = \mathscr{N}^e \cap \mathscr{F}'$, we obtain an exact diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{N}' & \longrightarrow & \mathscr{F}' & \longrightarrow & \mathscr{C}' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{N}^e & \longrightarrow & \mathscr{F} & \longrightarrow & m_F(p)(-1) & \longrightarrow & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{N}'' & \longrightarrow & \mathscr{F}'' & \longrightarrow & \mathscr{C}'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

By definition we have $\chi m_F(p)(d-1) = \binom{d+1}{2} - 1$ and $\chi\mathscr{C}'(d) = \binom{d+1}{2} - 1 - l(d)$ with $l(d) = \chi\mathscr{C}''(d)$. We have to consider

$$\Delta(d) = \frac{1}{2}\chi\mathscr{F}(d) - \chi\mathscr{F}'(d) = \binom{d+3}{3} - (d+2) - \chi\mathscr{N}'(d) - \chi\mathscr{C}'(d)$$
$$= \binom{d+2}{3} + l(d) - \chi\mathscr{N}'(d).$$

Since $\mathrm{rk}\mathscr{N}' = 1$ we have $\mathscr{N}'^{\vee\vee} = \mathscr{O}(c)$ and by the definition of $\mathscr{N}^e$ there is a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{N}' & \longrightarrow & \mathscr{O}(c) & \longrightarrow & \mathscr{L}_q & \longrightarrow & 0 \\ & & \cap & & \cap & & \cap & & , \\ 0 & \longrightarrow & \mathscr{N}^e & \longrightarrow & \mathscr{Z} & \longrightarrow & k_q & \longrightarrow & 0 \end{array}$$

s.t. $\chi\mathscr{N}'(d) = \chi\mathscr{O}(d+c) - \varepsilon$ with $\varepsilon = 0, 1$.

Now $\mathscr{N}^e$ is μ-stable with $c_1 = -1$ and therefore $c = c_1\mathscr{N}' \le -1$. If $c \le -2$, we get immediately

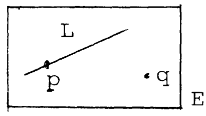
$$\Delta(d) = \binom{d+2}{3} - \binom{d+c+3}{3} + \varepsilon + l(d) > l(d) \ge 0 \quad \text{for } d \gg 0.$$

If however $c = -1$, we only have

$$\Delta(d) = \varepsilon + l(d) \ge 0 \quad \text{for } d \gg 0.$$

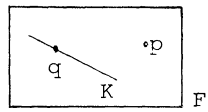
This proves that $\mathscr{F}$ is semi-stable, and that it is even stable if the case $\varepsilon = 0$, $\mathscr{E}'' = 0$ does not occur. But in this case $\mathscr{N}' = \mathscr{O}(-1)$, and by Corollary 10.1.3

$$\mathscr{F}'' = \mathscr{N}'' = \mathscr{I}_{L \cup q}$$



for some line L in E through p and finally

$$\mathscr{F}' = \mathscr{I}_{K \cup p}$$



by Lemma 10.1.4 for some line K in F through q. Therefore we have proved that if $\mathscr{F}$ is not stable under the assumptions of the proposition, it must be an extension of the form

(34) $$0 \to \mathscr{I}_{K \cup p} \to \mathscr{F} \to \mathscr{I}_{L \cup q} \to 0.$$

Now the proof will follow from Lemma 10.3 and from

10.4.1. LEMMA. *If $\mathscr{F}$ is an extension as in* (34) *then $C^\vee \subset \mathbb{P}W^\vee$ is a singular conic through e and f.*

Proof. We use the incidence transform to show that $\operatorname{Supp} R^1\mathscr{F}(-1) = \mathbb{G} \cap \widetilde{J}$ is given by a union $\widetilde{J} = \widetilde{H}_k \cup \widetilde{H}_L$ of two hyperplanes in $\mathbb{P}\bigwedge^2 V$ and s.t. $\widetilde{J} \cap \mathbb{P}W' = C^\vee$ passes through e and f (for notation see 8.4, (b)). This contradicts the assumption of the proposition by 10.3.

First we note that $R^0\mathscr{I}_{L\cup q}(-1) = 0$ since $R^0\mathscr{O}(-1) = 0$, and therefore we obtain the exact sequence

$$0 \to R^1\mathscr{I}_{K\cup p}(-1) \to R^1\mathscr{F}(-1) \to R^1\mathscr{I}_{L\cup q}(-1) \to 0.$$

Therefore it is enough to determine the supports of the ends of this sequence. Since we have the sequence

$$0 \to \mathscr{I}_{L\cup q} \to \mathscr{I}_L \to k_q \to 0,$$

we obtain the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^0 k_q & \longrightarrow & R^1\mathscr{I}_{L\cup q}(-1) & \longrightarrow & R^1\mathscr{I}_L(-1) & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & \mathscr{O}_{P_q} & & & & R^0\mathscr{O}_L(-1) & & \end{array},$$

where P_q is the α-plane of q in $\mathbb{G}$. Further from $0 \to \mathscr{O}_L(-1) \to \mathscr{O}_L \to k_a \to 0$ with some $a \in L$ we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^0\mathscr{O}_L(-1) & \longrightarrow & R^0\mathscr{O}_L & \longrightarrow & R^0 k_a & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathscr{I}_{P_a, H_L} & \longrightarrow & \mathscr{O}_{H_L} & \longrightarrow & \mathscr{O}_{P_a} & \longrightarrow & 0 \end{array}$$

where $H_L \subset \mathbb{G}$ is the cone of all lines in $\mathbb{P}_3$ meeting L. It is a hyperplane section

$$H_L = \mathbb{G} \cap \widetilde{H}_L.$$

Therefore $\operatorname{Supp} R^1\mathscr{I}_L(-1) = H_L$ and hence

$$\operatorname{Supp} R^1\mathscr{I}_{L\cup q}(-1) = H_L \cup P_q.$$

Similarly

$$\operatorname{Supp} R^1\mathscr{I}_{K\cup p} = H_K \cup P_p.$$

But since $p \in L$ and $q \in K$ we have $P_q \subset H_K$ and $P_p \subset H_L$ and thus

$$\operatorname{Supp} R^1\mathscr{F}(-1) = H_L \cup H_K.$$

Finally it is easy to show that the unique hyperplanes $\widetilde{H}_L$, $\widetilde{H}_K$ intersecting $\mathbb{G}$ in H_L, H_K pass through the points e resp. f if we choose an embedding of $\mathbb{P}W^\vee$ as in 8.4 (b). This proves the lemma.

Now we can prove the

10.5. THEOREM. *Let $x = (z, M, \Gamma) \in X^{ss}$ with $S = S(\Gamma)$ singular, and let $\mathscr{F} = \mathscr{F}_x$ be the corresponding sheaf. Then $\mathscr{F}$ is always semi-stable and the following conditions are equivalent:*

(i) *$(S, C^\vee)$ is a singular point of Q.*
(ii) *x is not stable.*
(iii) *$\mathscr{F}$ is not stable.*
(iv) *$\mathscr{F}$ is an extension of the type*

$$0 \to \mathscr{I}_{K\cup p} \to \mathscr{F} \to \mathscr{I}_{L\cup q} \to 0,$$

which also defines the stable filtration of $\mathscr{F}$.

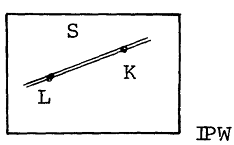
Proof. We restrict ourselves to the generic situation of a singular S, the proof in the other cases is the same. By the previous proof we obtain (iii) $\Rightarrow$ (ii). Since (iv) $\Rightarrow$ (iii) is obvious and (i) $\Leftrightarrow$ (ii) by 10.3, we only have to show that (ii) $\Rightarrow$ (iv). It is sufficient to consider only the case (a). By 10.3 we have $M \cap \Gamma\mathscr{N}^e(1) \neq 0$ if x is not stable, and the dimension of this intersection cannot be 2. Therefore we obtain a diagram

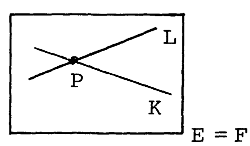
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M' \otimes \Omega^3(3) \arrow[r] \arrow[d] & M \otimes \Omega^3(3) \arrow[r] \arrow[d] & M'' \otimes \Omega^3(3) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{N}^e \arrow[r] \arrow[d] & \mathscr{N} \arrow[r] \arrow[d] & \mathscr{N}^f \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{I}_{L\cup q} \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{I}_{K\cup p} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $M' = M \cap \Gamma\mathscr{N}^e(1)$, M'' is the image of M under $\Gamma\mathscr{N}(1) \to \Gamma\mathscr{N}^f(1)$, and where the cokernels must be of the form $\mathscr{I}_{L\cup q}$ by 10.1.

The other cases of $(S, C^\vee)$ with singular S are treated similarly; one only has to interpret $L\cup q$ in case $q \in L$ as a line L with a double structure in the point q, the tangent plane of which determines the plane E, in which L is contained.

10.6. REMARK. If S is a double line the non-stable pairs $(S, C^\vee)$ are given by two points $L, K \in S$ or two lines in a plane in $\mathbb{P}_3$.

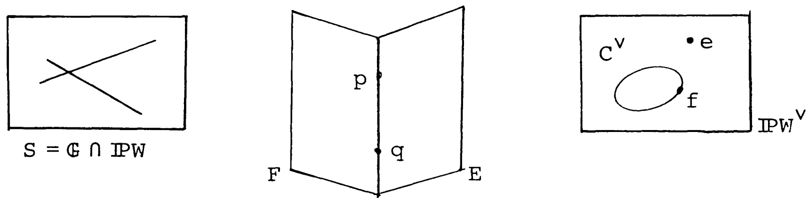




The corresponding sheaf is now an extension

$$0 \to \mathscr{I}_{L\cup p} \to \mathscr{F} \to \mathscr{I}_{K\cup p} \to 0.$$

The class $[\mathscr{F}] = [\mathscr{I}_{L\cup p} \oplus \mathscr{I}_{K\cup p}]$ cannot remember the plane $\mathbb{P}W$ belonging to the $\mathbb{P}_1$-fibration of Q_{exc}, which is blown down. If $L = K$ we obtain the most degenerate element $[\mathscr{I}_{L\cup p} \oplus \mathscr{I}_{L\cup p}]$ in $\overline{M(0, 2)}$.



10.7. We close with some remarks on the embedding $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ for a generic stable pair $(S, C^\vee) \in Q_0$.

We have shown in 10.4 that in such a case the sheaf $\mathcal{F}$ is stable and an extension of $m_F(p)(-1)$ by $\mathcal{N}^e$. Now we can show that for smooth $C^\vee$ through one of the points, say f, there is a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{N}^e & \longrightarrow & \mathcal{F} & \longrightarrow & m_F(p)(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{Z}^p & \longrightarrow & \mathcal{F}^{\vee\vee} & \longrightarrow & m_F(p)(-1) & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & & & \\
 & & k_q & = & k_q & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{35}$$

and that moreover $\mathcal{F}$ is locally free outside q. This implies that $c_3\mathcal{F}^{\vee\vee} = 2$, whereas $c_3\mathcal{F}^{\vee\vee} = 4$ in 9.2 for sheaves in Q_α.

Proof. The first column is the definition of $\mathcal{N}^e$. Next we prove that $\mathcal{F}$ is locally free outside of q in this case and that the cokernel of $\mathcal{F} \subset \mathcal{F}^{\vee\vee}$ is k_q. For this we consider the display diagram (22). One can show that $\mathcal{M}$ is locally free outside q if $C^\vee$ is regular as in the picture above, and that $\mathcal{M}$ is reflexive. Moreover $\mathcal{E} = \mathcal{T} = k_q$ in that case, so that also $\mathcal{A}$ is locally free outside q and therefore $\mathcal{F}$, too. Finally we have the diagram

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & k_q \arrow[d] & \\
0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{M} \arrow[r] \arrow[d, equal] & \mathscr{A} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}^{\vee\vee} \arrow[r] \arrow[d] & \mathscr{M} \arrow[r] & 2\mathscr{O} & \\
 & \mathscr{R} \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}
\tag{36}
$$

from which it follows that $\mathscr{R} = k_q$. Going back to (35) the induced homomorphism $k_q \to k_q$ must be nonzero and hence an isomorphism, for otherwise k_q would inject into $m_F(p)(-1)$, which is not possible. This proves the diagram (35).

If in the previous example however $C^\vee$ is a smooth conic through both of the points e and f, we see by the analogue of diagram (36) that now the kernel and cokernel of $\mathscr{A} \to 2\mathscr{O}$ is $k_p \oplus k_q$ and equals $\mathscr{R}$. Thus in this case $\mathscr{M}$ and $\mathscr{F}^{\vee\vee}$ are singular at p and q and $c_3\mathscr{F}^{\vee\vee} = 4$.

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