PACIFIC JOURNAL OF MATHEMATICS
Vol. 146, No. 1, 1990

OPERATOR-VALUED FEYNMAN INTEGRALS VIA CONDITIONAL FEYNMAN INTEGRALS

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In this paper we use the concept of the conditional Feynman integral to obtain the analytic operator-valued Feynman integral of various functions.

1. Introduction. In **[1]** Cameron and Storvick introduced a very general analytic operator-valued function space "Feynman integral", $J_q^{\mathrm{an}}(F)$, which mapped an $L_2(\mathbb{R}^\nu)$ function ψ into an $L_2(\mathbb{R}^\nu)$ function $(J_q^{\mathrm{an}}(F)\psi)(\vec{\xi})$. Further work involving the $L_2 \to L_2$ theory includes **[2, 3, 16–18]**. In **[4, 19]** the existence of the Feynman integral as an operator from $L_1(\mathbb{R})$ to $L_\infty(\mathbb{R})$ was studied. Finally in **[20]**, an $L_p \to L_{p'}$ theory, $1/p + 1/p' = 1$, was developed for $1 < p \leq 2$. Related stability results were established in **[10, 25]**.

In **[15]**, Chung and Skoug introduced the concept of a conditional Feynman integral. In this paper we further develop this concept and proceed to express operator-valued Feynman integrals in terms of conditional Feynman integrals. In particular we show that various operator-valued Feynman integrals can be obtained using the formula

$$(1.1)\qquad (J_q^{\mathrm{an}}(F)\psi)(\vec{\xi}) = \int_{\mathbb{R}^\nu} E^{\mathrm{anf}_q}(F|X)(\vec{\xi})(\vec{\eta}) \left[\frac{q}{2\pi i T}\right]^{\nu/2} \cdot \exp\left\{\frac{qi}{2T}\|\vec{\eta} - \vec{\xi}\|^2\right\} \psi(\vec{\eta})\, d\vec{\eta}$$

where $E^{\mathrm{anf}_q}(F|X)$ is the conditional analytic Feynman integral of F given X. Thus $J_q^{\mathrm{an}}(F)$ can be interpreted as an integral operator with kernel

$$\left[\frac{q}{2\pi i T}\right]^{\nu/2} \exp\left\{\frac{qi}{2T}\|\vec{\eta} - \vec{\xi}\|^2\right\} E^{\mathrm{anf}_q}(F|X)(\vec{\xi})(\vec{\eta}).$$

In **[5]**, Cameron and Storvick introduced a Banach algebra $S(\nu)$ of functions on Wiener space which are a kind of stochastic Fourier transform of Borel measures on $L_2^\nu[0, T]$. In §3 of this paper we show that for all F in $S(\nu)$, $J_q^{\mathrm{an}}(F)$ is given by (1.1) and can be interpreted as a bounded linear operator from $L_1(\mathbb{R}^\nu)$ to $L_\infty(\mathbb{R}^\nu)$. In this setting we also obtain some stability results.

A very important class of functions in Quantum Mechanics are functions on Wiener space $C_0^\nu[0, T]$ of the form

$$F(\vec{x}) = \exp\left\{\int_0^T \theta(s, \vec{x}(s))\, ds\right\} \tag{1.2}$$

where $\theta: [0, T] \times \mathbb{R}^\nu \to \mathbb{C}$. In §§4 and 5, using a useful series expansion formula, we show that for appropriate θ, $J_q^{\mathrm{an}}(F)$ exists as an operator from L_1 to L_∞ and is given by (1.1).

2. Definitions and preliminaries. Let ν be a positive integer. Let $C^\nu[0, T]$ denote the space of $\mathbb{R}^\nu$-valued continuous functions on $[0, T]$ and let $C_0^\nu[0, T]$ denote ν-dimensional Wiener space; that is the set of all functions $\vec{x}(t)$ in $C^\nu[0, T]$ such that $\vec{x}(0) = \vec{0}$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0^\nu[0, T]$ and let m denote ν-dimensional Wiener measure. $(C_0^\nu[0, T], \mathcal{M}, m)$ is a complete measure space and we denote the Wiener integral of a Wiener measurable function F by

$$\int_{C_0^\nu} F(\vec{x}) m(d\vec{x})$$

whenever the integral exists.

A set $E \in \mathcal{M}$ is said to be scale-invariant measurable **[11, 21]** provided $\rho E \in \mathcal{M}$ for each $\rho > 0$ and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho N) = 0$ for each $\rho > 0$. A property which holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.).

Next we give Yeh's definition of the conditional Wiener integral **[29]**.

DEFINITION 1. Let X be an $\mathbb{R}^\nu$-valued Wiener measurable function on $C_0^\nu[0, T]$ and let F be a complex-valued Wiener integral on $C_0^\nu[0, T]$. Let P_X be the probability distribution of X, i.e., for all $B \in \mathcal{B}^\nu$, the Borel sets in $\mathbb{R}^\nu$, $P_X(B) = m(X^{-1}(B))$. The conditional Wiener integral of F given X is by definition the equivalence class of Borel measurable and P_X-integrable functions ϕ on $\mathbb{R}^\nu$, modulo null functions on $(\mathbb{R}^\nu, \mathcal{B}^\nu, P_X)$, such that for all $B \in \mathcal{B}^\nu$,

$$\int_{X^{-1}(B)} F(\vec{x}) m(d\vec{x}) = \int_B \phi(\vec{\eta}) P_X(d\vec{\phi}).$$

By the Radon-Nikodym Theorem such a function ϕ exists and is determined up to a null function on $(\mathbb{R}^\nu, \mathscr{B}^\nu, P_X)$. We let $E(F|X)$ denote a representative of the equivalence class and so for all $B \in \mathscr{B}^\nu$,

$$\text{(2.1)} \qquad \int_{X^{-1}(B)} F(\vec{x})m(d\vec{x}) = \int_B E(F|X)(\vec{\eta})P_X(d\vec{\eta}).$$

REMARK. In [27], Park and Skoug showed that if F is Borel measurable and Wiener integrable and if $X(\vec{x}) = \vec{x}(T)$, then the conditional Wiener integral $E(F|X)$ can be expressed in terms of an ordinary Wiener integral by the formula

$$\text{(2.2)} \qquad E(F|X)(\vec{\eta}) = \int_{C_0^\nu} F\left(\vec{x}(\cdot) - \frac{\cdot}{T}\vec{x}(T) + \frac{\cdot}{T}\vec{\eta}\right) m(d\vec{x}).$$

We are now ready to define the conditional analytic Feynman integral of a function F given X.

DEFINITION 2. Let $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_+^\sim$ denote respectively the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part. Let F: $C^\nu[0, T] \to \mathbb{C}$ be such that for each $\lambda > 0$,

$$\int_{C_0^\nu} |F(\lambda^{-1/2}\vec{x} + \vec{\xi})|m(d\vec{x}) < \infty$$

for a.e. $\vec{\xi} \in \mathbb{R}^\nu$. Let X: $C^\nu[0, T] \to \mathbb{R}^\nu$ be such that for each $\lambda > 0$ and a.e. $\vec{\xi} \in \mathbb{R}^\nu$, $X(\lambda^{-1/2}\vec{x} + \vec{\xi})$ is a Wiener measurable function of $\vec{x}$ on $C_0^\nu[0, T]$; i.e., for a.e. $\vec{\xi}$ in $\mathbb{R}^\nu$, $Y(\vec{x}) \equiv X(\lambda^{-1/2}\vec{x} + \vec{\xi})$ is scale-invariant measurable on $C_0^\nu[0, T]$. For $\lambda > 0$ and $\vec{\xi} \in \mathbb{R}^\nu$, let

$$J_\lambda(\vec{\xi}, \vec{\eta}) \equiv E(F(\lambda^{-1/2}\vec{x} + \vec{\xi})|X(\lambda^{-1/2}\vec{x} + \vec{\xi}))(\vec{\eta})$$

denote the conditional Wiener integral of $F(\lambda^{-1/2}\vec{x} + \vec{\xi})$ given $X(\lambda^{-1/2}\vec{x} + \vec{\xi})$. If for a.e. $\vec{\eta} \in \mathbb{R}^\nu$, there exists a function $J_\lambda^*(\vec{\xi}, \vec{\eta})$, analytic in λ on $\mathbb{C}_+$ such that $J_\lambda^*(\vec{\xi}, \vec{\eta}) = J_\lambda(\vec{\xi}, \vec{\eta})$ for all $\lambda > 0$, then $J_\lambda^*(\vec{\xi}, \cdot)$ is defined to be the conditional Wiener integral of F given X with parameter λ and we write

$$E^{\text{anw}_\lambda}(F|X)(\vec{\xi})(\vec{\eta}) = J_\lambda^*(\vec{\xi}, \vec{\eta}).$$

If for fixed real $q \neq 0$, the limit

$$\lim_{\lambda \to -iq} E^{\text{anw}_\lambda}(F|X)(\vec{\xi})(\vec{\eta})$$

exists for a.e. $\vec{\eta} \in \mathbb{R}^\nu$ where $\lambda \to -iq$ through $\mathbb{C}_+$, we will denote the value of this limit by $E^{\mathrm{anf}_q}(F|X)(\vec{\xi})(\cdot)$ and call it the conditional analytic Feynman integral of F given X with parameter q.

We finish this section by stating the definition of the analytic operator-valued Feynman integral as an element of $\mathscr{L}(L_1(\mathbb{R}^\nu), L_\infty(\mathbb{R}^\nu))$.

DEFINITION 3. Let $F: C^\nu[0, T] \to \mathbb{C}$. Given $\lambda > 0$, ψ in $L_1(\mathbb{R}^\nu)$ and $\vec{\xi}$ in $\mathbb{R}^\nu$, let

$$(I_\lambda(F)\psi)(\vec{\xi}) \equiv \int_{C_0^\nu} F(\lambda^{-1/2}\vec{x} + \vec{\xi})\psi(\lambda^{-1/2}\vec{x}(T) + \vec{\xi})m(d\vec{x}).$$

If $I_\lambda(F)\psi$ is in $L_1(\mathbb{R}^\nu)$ as a function of $\vec{\xi}$ and if the correspondence $\psi \to I_\lambda(F)\psi$ gives an element of $\mathscr{L}(L_1(\mathbb{R}^\nu), L_\infty(\mathbb{R}^\nu))$, the space of continuous linear operators from $L_1(\mathbb{R}^\nu)$ to $L_\infty(\mathbb{R}^\nu)$, we say that the operator-valued function space integral $I_\lambda(F)$ exists. Next suppose there exists an $\mathscr{L}$-valued function which is analytic in $\mathbb{C}_+$ and agrees with $I_\lambda(F)$ on $(0, \infty)$; then this $\mathscr{L}$-valued function is denoted by $I_\lambda^{\mathrm{an}}(F)$ and is called the analytic operator-valued Wiener integral of F associated with λ. Finally, for $\lambda = -iq \in \mathbb{C}_+$, suppose there exists an operator $J_q^{\mathrm{an}}(F)$ in $\mathscr{L}(L_1(\mathbb{C}^\nu), L_\infty(\mathbb{R}^\nu))$ such that for every ψ in $L_1(\mathbb{R}^\nu)$,

$$\|J_q^{\mathrm{an}}(F)\psi - I_\lambda^{\mathrm{an}}(F)\psi\|_\infty \to 0$$

as $\lambda \to -iq$ through $\mathbb{C}_+$; then $J_q^{\mathrm{an}}(F)$ is called the analytic operator-valued Feynman integral of F with parameter q.

Finally we state the following well-known integration formula

$$\text{(2.3)} \qquad \int_{\mathbb{R}^\nu} \exp\left\{-\frac{b}{2}\|\vec{\eta}\|^2 + i\langle\vec{\eta}, \vec{\xi}\rangle\right\} d\vec{\eta} = \left[\frac{2\pi}{b}\right]^{\nu/2} \exp\left\{-\frac{1}{2b}\|\vec{\xi}\|^2\right\}, \quad \operatorname{Re} b > 0$$

which we use several times in this paper.

3. The $S(\nu)$ theory. In [**5**] Cameron and Storvick introduced a Banach algebra $S(\nu)$ of functions on ν-dimensional Wiener space each of which is a type of a stochastic Fourier transform of bounded $\mathbb{C}$-valued Borel measures. They showed that the analytic (but scalar-valued) Feynman integral exists for all elements of $S(\nu)$. Further work on $S(\nu)$ includes [**7**, **8**, **13**, **22**, **23**, **24**].

The Banach algebra $S(\nu)$ consists of functions on $C_0^\nu[0, T]$ expressible in the form

$$(3.1) \qquad F(\vec{x}) = \int_{L_2^\nu[0,T]} \exp\left\{ i \sum_{j=1}^{\nu} \int_0^T v_j(s)\, \tilde{d}x_j(s) \right\} d\sigma(\vec{v})$$

for s-a.e. $\vec{x} = (x_1, \dots, x_\nu)$ in $C_0^\nu[0, T]$ where σ is an element of $M(L_2^\nu[0, T])$, the space of $\mathbb{C}$-valued, countably additive Borel measures on $L_2^\nu[0, T]$ and the integrals $\int_0^T v_j(s)\, \tilde{d}x_j(s)$ are Paley-Wiener-Zygmund (P.W.Z.) stochastic integrals [**23**, p. 280].

REMARK. If F is in $S(\nu)$ then F is scale-invariant measurable and s-a.e. defined on $C_0^\nu[0, T]$. Furthermore there is a natural way of regarding F as defined on $C^\nu[0, T]$: If $\vec{x}$ in $C_0^\nu[0, T]$ is such that $F(\vec{x})$ is defined, then by (3.1), $F(\vec{x} + \vec{\xi}) = F(\vec{x})$ for all $\vec{\xi} \in \mathbb{R}^\nu$.

First, for F in $S(\nu)$ and $X(\vec{y}) = \vec{y}(T)$, we obtain a formula for $E^{\mathrm{anf}_q}(F|X)(\vec{\xi})(\vec{\eta})$.

THEOREM 3.1. *Let $F \in S(\nu)$ be given by* (3.1) *and let $X: C^\nu[0, T] \to \mathbb{R}^\nu$ be given by $X(\vec{y}) = \vec{y}(T)$. Then for all $(\vec{\xi}, \vec{\eta}) \in \mathbb{R}^\nu \times \mathbb{R}^\nu$*

$$(3.2) \qquad E^{\mathrm{anw}_\lambda}(F|X)(\vec{\xi})(\vec{\eta}) = \int_{L_2^\nu[0,T]} \exp\left\{ -\frac{1}{2\lambda T} \sum_{j=1}^{\nu} [T\|v_j\|^2 - b_j^2] + \frac{i}{T}\langle \vec{\eta} - \vec{\xi}, \vec{B}\rangle \right\} d\sigma(\vec{v})$$

for all $\lambda \in \mathbb{C}_+$ and

$$(3.3) \qquad E^{\mathrm{anf}_q}(F|X)(\vec{\xi})(\vec{\eta}) = \int_{L_2^\nu[0,t]} \exp\left\{ -\frac{i}{2qT} \sum_{j=1}^{\nu} [T\|v_j\|^2 - b_j^2] + \frac{i}{T}\langle \vec{\eta} - \vec{\xi}, \vec{B}\rangle \right\} d\sigma(\vec{v})$$

for all real $q \neq 0$ where

$$\vec{B} = (b_1, \dots, b_\nu) = \left(\int_0^T v_1(s)\, ds, \dots, \int_0^T v_\nu(s)\, ds \right).$$

Proof. Using (3.1), (2.2), the Fubini Theorem, (3.4) and a fundamental Wiener integration formula involving P.W.Z. integrals, for all $\lambda > 0$ and all $(\vec{\xi}, \vec{\eta}) \in \mathbb{R}^\nu \times \mathbb{R}^\nu$ we obtain the formula

$$
\begin{aligned}
(3.5)\quad & E(F(\lambda^{-1/2}\vec{x} + \vec{\xi}) | X(\lambda^{-1/2}\vec{x} + \vec{\xi}))(\vec{\eta}) \\
&= \int_{C_0^\nu} \left[\int_{L_2^\nu[0,T]} \exp\left\{ i \sum_{j=1}^{\nu} \int_0^T v_j(s) \tilde{d}[\lambda^{-1/2} x_j(s) - \lambda^{-1/2} \frac{s}{T} x_j(T) \right.\right. \\
&\qquad \left.\left. + \frac{s}{T}(\eta_j - \xi_j)] \right\} {}_{C_0^\nu} d\sigma(\vec{v}) \right] m(d\vec{x}) \\
&= \int_{L_2^\nu[0,T]} \left[\int_{C_0^\nu} \exp\left\{ \frac{i}{\sqrt{\lambda}} \sum_{j=1}^{\nu} \left[\int_0^T v_j(s)\, \tilde{d}x_j(s) \right.\right.\right. \\
&\qquad \left. - \frac{x_j(T)}{T} \int_0^T v_j(s)\, ds \right] \\
&\qquad \left.\left. + \frac{i}{T} \sum_{j=1}^{\nu} (\eta_j - \xi_j) \int_0^T v_j(s)\, ds \right\} m(d\vec{x}) \right] d\sigma(\vec{v}) \\
&= \int_{L_2^\nu[0,T]} \exp\left\{ \frac{i}{T} \langle \vec{\eta} - \vec{\xi}, \vec{B} \rangle \right\} \\
&\qquad \cdot \int_{C_0^\nu} \exp\left\{ \frac{i}{\sqrt{\lambda}} \sum_{j=1}^{\nu} \int_0^T \left[v_j(s) - \frac{b_j}{T} \right] \tilde{d}x_j(s) \right\} m(d\vec{x})\, d\sigma(\vec{v}) \\
&= \int_{L_2^\nu[0,T]} \exp\left\{ \frac{i}{T} \langle \vec{\eta} - \vec{\xi}, \vec{B} \rangle \right\} \\
&\qquad \cdot \exp\left\{ -\frac{1}{2\lambda} \sum_{j=1}^{\nu} \int_0^T \left[v_j(s) - \frac{b_j}{T} \right]^2 ds \right\} d\sigma(\vec{v}) \\
&= \int_{L_2^\nu[0,T]} \exp\left\{ -\frac{1}{2\lambda T} \sum_{j=1}^{\nu} [T\|v_j\|^2 - b_j^2] + \frac{i}{T} \langle \vec{\eta} - \vec{\xi}, \vec{B} \rangle \right\} d\sigma(\vec{v}).
\end{aligned}
$$

Using the Cauchy-Schwarz inequality we see that

$$
b_j^2 = \left[\int_0^T v_j(s)\, ds \right]^2 \le \int_0^T 1^2\, ds \int_0^T v_j^2(s)\, ds = T\|v_j\|^2.
$$

Thus, since $\sigma \in M(L_2^\nu[0, T])$, the last expression on the right-hand side of (3.5) is an analytic function of λ throughout $\mathbb{C}_+$ and is a continuous function of λ for $\lambda \in \mathbb{C}_+^\sim$. Thus (see Definition 2 in §2 above) equations (3.2) and (3.3) are established.

THEOREM 3.2. *Let F and X be as in Theorem 3.1. Then for all real $q \neq 0$, the analytic operator-valued Feynman integral $J_q^{\text{an}}(F)$ exists as an element of $\mathscr{L}(L_1(\mathbb{R}^\nu), L_\infty(\mathbb{R}^\nu))$ and for each $\psi \in L_1(\mathbb{R}^\nu)$ we have*

$$(3.6)\qquad \begin{aligned}&(J_q^{\text{an}}(F)\psi)(\vec{\xi})\\ &= \int_{\mathbb{R}^\nu} E^{\text{anf}_q}(F|X)(\vec{\xi})(\vec{\eta}) \left[\frac{q}{2\pi i T}\right]^{\nu/2}\\ &\quad\cdot \exp\left\{\frac{iq\|\vec{\eta}-\vec{\xi}\|^2}{2T}\right\} \psi(\vec{\eta})\, d\vec{\eta}\end{aligned}$$

for all $\vec{\xi} \in \mathbb{R}^\nu$.

Proof. Let $\psi \in L_1(\mathbb{R}^\nu)$ be given. We can assume that ψ is Borel measurable since if ψ is only Lebesgue measurable then there exists a Borel measurable function ψ_1 such that $\psi_1 = \psi$ a.e. on $\mathbb{R}^\nu$. Moreover ψ_1 is unique up to Borel null sets. But F is also Borel measurable and so using equation (2.2) it is quite easy to see that

$$\begin{aligned}&E(F(\lambda^{-1/2}\vec{x}+\vec{\xi})\psi(\lambda^{-1/2}\vec{x}(T)+\vec{\xi})|X(\lambda^{-1/2}\vec{x}+\vec{\xi}))(\vec{\eta})\\ &\quad= \psi(\vec{\eta})E(F(\lambda^{-1/2}\vec{x}+\vec{\xi})|X(\lambda^{-1/2}\vec{x}+\vec{\xi}))(\vec{\eta}).\end{aligned}$$

Then by the definition of $I_\lambda(F)\psi$ and equation (2.1) it follows that

$$\begin{aligned}(I_\lambda(F)\psi)(\vec{\xi}) &= \int_{C_0^\nu} F(\lambda^{-1/2}\vec{x}+\vec{\xi})\psi(\lambda^{-1/2}\vec{x}(T)+\vec{\xi})m(d\vec{x})\\ &= \int_{\mathbb{R}^\nu} E(F(\lambda^{-1/2}\vec{x}+\vec{\xi})\psi(\lambda^{-1/2}\vec{x}(T)+\vec{\xi})|X^{-1/2}\vec{x}+\vec{\xi}))(\vec{\eta})\\ &\quad\cdot\left[\frac{\lambda}{2\pi T}\right]^{\nu/2}\exp\left\{-\frac{\lambda}{2T}\|\vec{\eta}-\vec{\xi}\|^2\right\}d\vec{\eta}\\ &= \int_{\mathbb{R}^\nu} E(F(\lambda^{-1/2}\vec{x}+\vec{\xi})|X(\lambda^{-1/2}\vec{x}+\vec{\xi}))(\vec{\eta})\left[\frac{\lambda}{2\pi T}\right]^{\nu/2}\\ &\quad\cdot\exp\left\{-\frac{\lambda}{2T}\|\vec{\eta}-\vec{\xi}\|^2\right\}\psi(\vec{\eta})\,d\vec{\eta}\end{aligned}$$

for all $\lambda > 0$. Then, using Theorem 3.1 and Morera's Theorem, we obtain that

$$(3.7)\qquad \begin{aligned}&(I_\lambda^{\text{an}}(F)\psi)(\vec{\xi})\\ &= \int_{\mathbb{R}^\nu} E^{\text{anw}_\lambda}(F|X)(\vec{\xi})(\vec{\eta}) \left[\frac{\lambda}{2\pi T}\right]^{\nu/2}\\ &\qquad \cdot \exp\left\{-\frac{\lambda\|\vec{\eta}-\vec{\xi}\|^2}{2T}\right\} \psi(\vec{\eta})\, d\vec{\eta}\end{aligned}$$

for all $\lambda \in \mathbb{C}_+$ and all $\vec{\xi} \in \mathbb{R}^\nu$.

But since $E^{\text{anw}_\lambda}(F|X)(\vec{\xi})(\vec{\eta})$ is bounded and $\psi \in l_1(\mathbb{R}^\nu)$, we see that the right-hand side of (3.7) is continuous in λ on $\mathbb{C}_+^\sim$. Thus

$$\begin{aligned}\lim_{\lambda\to -iq} (I_\lambda^{\text{an}}(F)\psi)(\vec{\xi}) = \int_{\mathbb{R}^\nu} & E^{\text{anf}_q}(F|X)(\vec{\xi}, \vec{\eta}) \left[\frac{q}{2\pi i T}\right]^{\nu/2}\\ & \cdot \exp\left\{\frac{iq\|\vec{\eta}-\vec{\xi}\|^2}{2T}\right\} \psi(\vec{\eta})\, d\vec{\eta}\end{aligned}$$

for each $\vec{\xi} \in \mathbb{R}^\nu$. Thus $J_q^{\text{an}}(F)$ exists as an element of

$$\mathscr{L}(L_1(\mathbb{R}^\nu), L_\infty(\mathbb{R}^\nu))$$

and (3.6) is established.

The following stability results follow quite readily using equations (3.3) and (3.6).

Theorem 3.3. *Let $\{\sigma_n\}$ be a sequence of elements from $M(L_2^\nu[0,T])$ that converge weakly to $\sigma \in M(L_2^\nu[0,T])$, let F be given by* (3.1) *and for $n = 1, 2, \dots$, let*

$$F_n(\vec{x}) = \int_{L_2^\nu[0,T]} \exp\left\{ i \sum_{j=1}^{\nu} \int_0^T v_j(s)\tilde{d}x(s)\right\} d\sigma_n(\vec{v})$$

for s-a.e. $\vec{x} \in C_0^\nu[0,T]$. Let $\{q_n\}$ be a sequence of real numbers converging to $q \neq 0$ and let $\{\psi_n\}$ be a sequence from $L_1(\mathbb{R}^\nu)$ converging in L_1-norm to $\psi \in L_1(\mathbb{R}^\nu)$. Then as $n \to \infty$:

$$(3.8)\qquad E^{\text{anf}_q}(F_n|X)(\vec{\xi})(\vec{\eta}) \to E^{\text{anf}_q}(F|X)(\vec{\xi})(\vec{\eta}) \quad \textit{for all } (\vec{\xi}, \vec{\eta}) \in \mathbb{R}^\nu \times \mathbb{R}^\nu,$$

(3.9) $E^{\mathrm{anf}_{q_n}}(F|X)(\vec{\xi})(\vec{\eta}) \to E^{\mathrm{anf}_q}(F|X)(\vec{\xi})(\vec{\eta})$ *for all* $(\vec{\xi}, \vec{\eta}) \in \mathbb{R}^\nu \times \mathbf{R}^\nu$,

(3.10) $J_q^{\mathrm{an}}(F_n)\psi \to J_q^{\mathrm{an}}(F)\psi$ *in L_∞-norm on* $\mathbb{R}^\nu$,

(3.11) $J_{q_n}^{\mathrm{an}}(F)\psi \to J_q^{\mathrm{an}}(F)\psi$ *in L_∞-norm on* $\mathbb{R}^\nu$, *and*

(3.12) $J_q^{\mathrm{an}}(F)\psi_n \to J_q^{\mathrm{an}}(F)\psi$ *in L_∞-norm on* $\mathbb{R}^\nu$.

4. A useful series expansion. In this section for F given by (1.2) with minimal conditions on θ and $X(\vec{y}) = \vec{y}(T)$ we obtain a useful series expansion for $E(F(\lambda^{-1/2}\vec{x} + \vec{\xi})|X(\lambda^{-1/2}\vec{x} + \vec{\xi}))(\vec{\eta})$.

THEOREM 4.1. *Let $F(\vec{x})$ be given by* (1.2) *where θ is Borel measurable and where for each $\lambda > 0$*

$$\int_{C_0^\nu} |F(\lambda^{-1/2}\vec{x} + \vec{\xi})| m(d\vec{x}) < \infty$$

for a.e. $\vec{\xi} \in \mathbb{R}^\nu$. Then for each $\lambda > 0$

$$\begin{aligned}(4.1)\quad & E(F(\lambda^{-1/2}\vec{x} + \vec{\xi})|X(\lambda^{-1/2}\vec{x} + \vec{\xi}))(\vec{\eta}) \left[\frac{\lambda}{2\pi T}\right]^{\nu/2} \\ & \quad \cdot \exp\left\{-\frac{\lambda}{2T}\|\vec{\eta} - \vec{\xi}\|^2\right\} \\ & = \sum_{n=0}^{\infty} \int_{\Delta_n(T)} \left[\frac{\lambda^{n+1}}{(2\pi)^{n+1} s_1(s_2 - s_1)\cdots(s_n - s_{n-1})(T - s_n)}\right]^{\nu/2} \\ & \quad \cdot \int_{\mathbb{R}^{n\nu}} \left[\prod_{j=1}^{n} \theta(s_j, \vec{w}_j)\right] \\ & \quad \cdot \exp\left\{-\sum_{j=1}^{n} \frac{\lambda}{2(s_j - s_{j-1})}\|\vec{w}_j - \vec{w}_{j-1}\|^2 \right. \\ & \qquad \left. - \frac{\lambda}{2(T - s_n)}\|\vec{w}_n - \vec{\eta}\|^2\right\} d\vec{w}_1 \ldots d\vec{w}_n \, d\vec{s}\end{aligned}$$

where $\Delta_n(T) = \{\vec{s} = (s_1, \ldots, s_n) : 0 < s_1 < s_2 < \cdots < s_n < T\}$, $s_0 = 0$ *and* $\vec{w}_0 = \vec{\xi}$.

Proof. For notational purposes let $G_\lambda(\vec{\xi}, \vec{\eta})$ denote

$$E(F(\lambda^{-1/2}\vec{x} + \vec{\xi}) | X(\lambda^{-1/2}\vec{x} + \vec{\xi}))(\vec{\eta}).$$

Then

$$\begin{aligned}
G_\lambda(\vec{\xi}, \vec{\eta}) &= E\left[\sum_{n=0}^{\infty} \frac{1}{n!}\left[\int_0^T \theta(s, \lambda^{-1/2}\vec{x}(s) + \vec{\xi})\, ds\right]^n \;\Big|\; \vec{x}(T) = \sqrt{\lambda}(\vec{\eta} - \vec{\xi})\right] \\
&= E\left[\sum_{n=0}^{\infty} \int_{\Delta_n(T)} \prod_{j=1}^{n} \theta(s_j, \lambda^{-1/2}\vec{x}(s_j) + \vec{\xi})\, d\vec{s} \,|\, \vec{x}(T) = \sqrt{\lambda}(\vec{\eta} - \vec{\xi})\right] \\
&= \int_{C_0^\nu} \sum_{n=0}^{\infty} \int_{\Delta_n(T)} \prod_{j=1}^{n} \theta\left(s_j, \lambda^{-1/2}\vec{x}(s_j) + \vec{\xi} - \frac{s_j}{T}(\lambda^{-1/2}\vec{x}(T) + \vec{\xi}) + \frac{s_j}{T}\vec{\eta}\right) d\vec{s}\, m(d\vec{x}) \\
&= \sum_{n=0}^{\infty} \int_{\Delta_n(T)} \int_{C_0^\nu} \prod_{j=1}^{n} \theta\left(s_j, \lambda^{-1/2}\vec{x}(s_j) + \vec{\xi} - \frac{s_j}{T}(\lambda^{-1/2}\vec{x}(T) + \vec{\xi}) + \frac{s_j}{T}\vec{\eta}\right) m(d\vec{x})\, d\vec{s} \\
&= \sum_{n=0}^{\infty} \int_{\Delta_n(T)} [(2\pi)^{n+1} s_1(s_2 - s_1) \cdots (T - s_n)]^{-\nu/2} \\
&\quad \cdot \int_{\mathbb{R}^{\nu(n+1)}} \exp\left\{-\sum_{j=1}^{n+1} \frac{\|\vec{u}_j - \vec{u}_{j-1}\|^2}{2(s_j - s_{j-1})}\right\} \\
&\quad \cdot \prod_{j=1}^{n} \theta\left(s_j, \lambda^{-1/2}\vec{u}_j + \vec{\xi} - \frac{s_j}{T}(\lambda^{-1/2}\vec{u}_{n+1} + \vec{\xi}) + \frac{s_j}{T}\vec{\eta}\right) \\
&\quad \cdot d\vec{u}_1 \cdots d\vec{u}_{n+1}\, d\vec{s}
\end{aligned}$$

with $s_0 = 0$ and $\vec{u}_0 = \vec{0}$. Next let $\vec{w}_0 = \vec{\xi}$,

$$\vec{w}_j = \lambda^{-1/2}\vec{u}_j + \vec{\xi} - \frac{s_j}{T}(\lambda^{-1/2}\vec{u}_{n+1} + \vec{\xi} - \vec{\eta}) \quad \text{for } j = 1, \ldots, n$$

and let $\vec{w}_{n+1} = \lambda^{-1/2}\vec{u}_{n+1} + \vec{\xi}$. Then

$$\begin{aligned} G_\lambda(\vec{\xi}, \vec{\eta}) = \sum_{n=0}^{\infty} \int_{\Delta_n(T)} & \left[\frac{\lambda^{n+1}}{(2\pi)^{n+1} s_1(s_2 - s_1)\cdots(T - s_n)}\right]^{\nu/2} \\ & \cdot \int_{\mathbb{R}^{n\nu}} \left[\prod_{j=1}^{n} \theta(s_j, \vec{w}_j)\right] \\ & \cdot \exp\left\{-\sum_{j=1}^{n} \frac{\lambda \|\vec{w}_j - \vec{w}_{j-1}\|^2}{2(s_j - s_{j-1})}\right\} \\ & \cdot \left[\int_{\mathbb{R}^\nu} \exp\left\{-\frac{\lambda}{T}\langle \vec{w}_{n+1} - \vec{\eta}, \vec{w}_n - \vec{\xi}\rangle - \frac{\lambda s_n}{2T^2}\|\vec{w}_{n+1} - \vec{\eta}\|^2 \right.\right. \\ & \quad \left.\left. - \frac{\lambda}{2(T - s_n)}\left\|\vec{w}_{n+1} - \vec{w}_n - \frac{s_n}{T}(\vec{w}_{n+1} - \vec{\eta})\right\|^2\right\} d\vec{w}_{n+1}\right] \\ & \cdot d\vec{w}_1 \cdots d\vec{w}_n \, d\vec{s}. \end{aligned}$$

Next carrying out the integration with respect to $\vec{w}_{n+1}$ in the above expression, simplifying, and multiplying both sides of the resulting expression by

$$\left[\frac{\lambda}{2\pi T}\right]^{\nu/2} \exp\left\{-\frac{\lambda}{2T}\|\vec{\eta} - \vec{\xi}\|^2\right\}$$

we obtain equation (4.1) which concludes the proof of Theorem 4.1.

Recall that in equation (3.3), for $F \in S(\nu)$, we expressed the conditional Feynman integral $E^{\text{anf}_q}(F|X)(\vec{\xi})(\vec{\eta})$ in terms of an integral over the infinite dimensional space $L_2^\nu[0, T]$. In our next theorem, as an application of Theorem 4.1, we obtain a series expansion of $E^{\text{anf}_q}(F|X)$ in terms of integrals over finite dimensional spaces.

THEOREM 4.2. *Let* $F(\vec{x}) = \exp\{\int_0^T \theta(s, \vec{x}(s))\, ds\}$ *with*

$$\theta(s, \vec{w}) = \int_{\mathbb{R}^\nu} \exp\{i\langle \vec{w}, \vec{v}\rangle\}\, d\mu_s(\vec{v}) \tag{4.2}$$

where $\{\mu_s : 0 \le s \le T\}$ *is a family from* $M(\mathbb{R}^\nu)$ *such that* $\|\mu_s\| \in L_1[0, T]$ *and for each Borel set* B *from* $\mathbb{R}^\nu$, $\mu_S(B)$ *is Borel measurable in* s. *Then for all real* $q \neq 0$,

$$
\begin{aligned}
(4.3)\quad & E^{\mathrm{anf}_q}(F|X)(\vec{\xi})(\vec{\eta}) \\
&= \sum_{n=0}^{\infty} \int_{\Delta_n(T)} \int_{\mathbb{R}^{n\nu}} \exp\Bigg\{ -\frac{i}{2q} \sum_{l=1}^{n} \sum_{j=1}^{l} (2 - \delta_{jl}) s_j \langle \vec{v}_j, \vec{v}_l \rangle \\
&\qquad + i \sum_{j=1}^{n} \langle \vec{\xi}, \vec{v}_j \rangle - \frac{i}{T} \sum_{j=1}^{n} \langle \vec{\xi} - \vec{\eta}, s_j \vec{v}_j \rangle \\
&\qquad + \frac{i}{2qT} \left\| \sum_{j=1}^{n} s_j \vec{v}_j \right\|^2 \Bigg\} \\
&\qquad \cdot d\mu_{S_1}(\vec{v}_1) \cdots d\mu_{S_n}(\vec{v}_n)\, d\vec{s}
\end{aligned}
$$

where δ_{jl} *is the Kronecker delta.*

Proof. We first note that $F(\vec{x})$ is Borel measurable [**24**, Corollary 3.2] and belongs to $S(\nu)$ [**24**, Remark 3.3]. Next using (4.1) and (4.2) we see that for

$$
\begin{aligned}
\lambda > 0\; & E(F(\lambda^{-1/2}\vec{x} + \vec{\xi})|X(\lambda^{-1/2}\vec{x} + \vec{\xi}))(\vec{\eta}) \\
&= \left[\frac{2\pi T}{\lambda}\right]^{\nu/2} \exp\left\{\frac{\lambda}{2T}\|\vec{\eta} - \vec{\xi}\|^2\right\} \\
&\cdot \sum_{n=0}^{\infty} \int_{\Delta_n(T)} \left[\frac{\lambda^{n+1}}{(2\pi)^{n+1} s_1(s_2 - s_1)\cdots(T - s_n)}\right]^{\nu/2} \\
&\cdot \int_{\mathbb{R}^{n\nu}} \left[\int_{\mathbb{R}^{n\nu}} \exp\left\{ i \sum_{j=1}^{n} \langle \vec{w}_j, \vec{v}_j \rangle \right\} d\mu_{S_1}(\vec{v}_1)\cdots d\mu_{S_n}(\vec{v}_n)\right] \\
&\cdot \exp\Bigg\{ -\sum_{j=1}^{n} \frac{\lambda}{2(s_j - s_{j-1})} \|\vec{w}_j - \vec{w}_{j-1}\|^2 \\
&\qquad - \frac{\lambda}{2(T - s_n)} \|\vec{w}_n - \vec{\eta}\|^2 \Bigg\} d\vec{w}_1 \cdots d\vec{w}_n d\vec{s}.
\end{aligned}
$$

Then using the Fubini Theorem and the formula (see equation (2.3))

$$\exp\left\{-\frac{\lambda}{2(T-s_n)}\|\vec{w}_n-\vec{\eta}\|^2\right\} = \left[\frac{T-s_n}{2\pi\lambda}\right]^{\nu/2}\int_{\mathbb{R}^\nu}\exp\left\{i\langle\vec{u}\,,\vec{w}_n-\vec{\eta}\rangle-\frac{T-s_n}{2\lambda}\|\vec{u}\|^2\right\}d\vec{u}$$

we obtain

$$\begin{aligned}
&E(F(\lambda^{-1/2}\vec{x}+\vec{\xi})|X(\lambda^{-1/2}\vec{x}+\vec{\xi}))(\vec{\eta}) \\
&= \left[\frac{T}{2\pi\lambda}\right]^{\nu/2}\exp\left\{\frac{\lambda}{2T}\|\vec{\eta}-\vec{\xi}\|^2\right\} \\
&\cdot\sum_{n=0}^{\infty}\int_{\Delta_n(T)}\left[\frac{\lambda^n}{(2\pi)^n s_1(s_2-s_1)\cdots(s_n-s_{n-1})}\right]^{\nu/2} \\
&\cdot\int_{\mathbb{R}^{(n+1)\nu}}\int_{\mathbb{R}^{n\nu}}\exp\Bigg\{i\sum_{j=1}^{n}\langle\vec{w}_j\,,\vec{v}_j\rangle \\
&\qquad -\sum_{j=1}^{n}\frac{\lambda}{2(s_j-s_{j-1})}\|\vec{w}_j-\vec{w}_{j-1}\|^2+i\langle\vec{u}\,,\vec{w}_n-\vec{\eta}\rangle \\
&\qquad -\frac{T-s_n}{2\lambda}\|\vec{u}\|^2\Bigg\} \\
&\qquad \cdot d\vec{w}_n\cdots d\vec{w}_1 d\vec{u}\,d\mu_{S_1}(\vec{v}_1)\cdots d\mu_{S_n}(\vec{v}_n)\,d\vec{s}.
\end{aligned}$$

Next we carry out the integration with respect to $\vec{w}_n, \vec{w}_{n-1}, \dots, \vec{w}_1$ using the formula

$$\begin{aligned}
&\left[\frac{\lambda}{2\pi(s_j-s_{j-1})}\right]^{\nu/2}\int_{\mathbb{R}^\nu}\exp\left\{-\frac{\lambda}{2(s_j-s_{j-1})}\|\vec{w}_j-\vec{w}_{j-1}\|^2\right. \\
&\qquad \left.+i\langle\vec{w}_j\,,\vec{u}+\vec{v}_n+\cdots+\vec{v}_j\rangle\right\}d\vec{w}_j \\
&= \exp\left\{i\langle\vec{w}_{j-1}\,,\vec{u}+\vec{v}_n+\cdot+\vec{v}_j\rangle-\frac{s_j-s_{j-1}}{2\lambda}\|\vec{u}+\vec{v}_n+\cdots+\vec{v}_j\|^2\right\}
\end{aligned}$$

successively for $j = n, n-1, \cdots, 1$ to obtain

$$\begin{aligned}
&E(F(\lambda^{-1/2}\vec{x} + \vec{\xi})|X(\lambda^{-1/2}\vec{x} + \vec{\xi}))(\vec{\eta}) \\
&= \left[\frac{T}{2\pi\lambda}\right]^{\nu/2} \exp\left\{\frac{\lambda}{2T}\|\vec{\eta} - \vec{\xi}\|^2\right\} \\
&\quad \cdot \sum_{n=0}^{\infty} \int_{\Delta_n(T)} \int_{\mathbb{R}^{\nu} n} \left[\int_{\mathbb{R}^{\nu}} \exp\left\{-i\langle \vec{u}, \vec{\eta}\rangle \right.\right. \\
&\qquad\qquad + i\left\langle \vec{\xi}, \vec{u} + \sum_{j=1}^{n} \vec{v}_j \right\rangle - \frac{T - s_n}{2\lambda}\|\vec{u}\|^2 \\
&\qquad\qquad \left.\left. - \sum_{j=1}^{n} \frac{(s_j - s_{j-1})}{2\lambda}\|\vec{u} + \vec{v}_n + \cdots + \vec{v}_j\|^2 \right\} d\vec{u}\right] \\
&\qquad\qquad \cdot d\mu_{s_1}(\vec{v}_1) \cdots d\mu_{s_n}(\vec{v}_n) d\vec{s} \\
&= \left[\frac{T}{2\pi\lambda}\right]^{\nu/2} \exp\left\{\frac{\lambda}{2T}\|\vec{\eta} - \vec{\xi}\|^2\right\} \\
&\quad \cdot \sum_{n=0}^{\infty} \int_{\Delta_n(T)} \int_{\mathbb{R}^{\nu n}} \exp\left\{ i\left\langle \vec{\xi}, \sum_{j=1}^{n} \vec{v}_j \right\rangle \right. \\
&\qquad\qquad \left. - \sum_{j=1}^{n} \frac{(s_j - s_{j-1})}{2\lambda}\|\vec{v}_n + \cdots + \vec{v}_j\|^2 \right\} \\
&\quad \cdot \left[\int_{\mathbb{R}^{\nu}} \exp\left\{-\frac{T}{2\lambda}\|\vec{u}\|^2 i\langle \vec{u}, \vec{\eta} - \vec{\xi}\rangle - \frac{1}{\lambda}\left\langle \vec{u}, \sum_{j=1}^{n} s_j \vec{v}_j \right\rangle \right\} d\vec{u}\right] \\
&\qquad\qquad \cdot d\mu_{s_1}(\vec{v}_1) \cdots d\mu_{s_n}(\vec{v}_n)\, d\vec{s}.
\end{aligned}$$

But

$$\begin{aligned}
&\int_{\mathbb{R}^{\nu}} \exp\left\{-\frac{T}{2\lambda}\|\vec{u}\|^2 - i\langle \vec{u}, \vec{\eta} - \vec{\xi}\rangle - \frac{1}{\lambda}\left\langle \vec{u}, \sum_{j=1}^{n} s_j \vec{v}_j \right\rangle \right\} d\vec{u} \\
&= \left[\frac{2\pi\lambda}{T}\right]^{\nu/2} \exp\left\{ -\frac{\lambda}{2T}\|\vec{\eta} - \vec{\xi}\|^2 \right. \\
&\qquad \left. + \frac{1}{2\lambda T}\left\|\sum_{j=1}^{n} s_j \vec{v}_j\right\|^2 - \frac{i}{T}\left\langle \vec{\xi} - \vec{\eta}, \sum_{j=1}^{n} s_j \vec{v}_j \right\rangle \right\}.
\end{aligned}$$

Hence

$$(4.4)\quad E(F(\lambda^{-1/2}\vec{x}+\vec{\xi})|X(\lambda^{-1/2}\vec{x}+\vec{\xi}))(\vec{\eta})$$

$$= \sum_{n=0}^{\infty}\int_{\Delta_n(T)}\int_{\mathbb{R}^{n\nu}} \exp\left\{-\sum_{j=1}^{n}\frac{(s_j-s_{j-1})}{2\lambda}\|\vec{\xi}v_n+\cdots+\vec{v}_j\|^2 + i\left\langle\vec{\xi},\sum_{j=1}^{n}\vec{v}_j\right\rangle + \frac{1}{2\lambda T}\left\|\sum_{j=1}^{n}s_j\vec{v}_j\right\|^2 - \frac{i}{T}\left\langle\vec{\xi}-\vec{\eta},\sum_{j=1}^{n}s_j\vec{v}_j\right\rangle\right\} \cdot d\mu_{s_1}(\vec{v}_1)\cdots d\mu_{s_n}(\vec{v}_n)\,d\vec{s}$$

$$= \sum_{n=0}^{\infty}\int_{\Delta_n(T)}\int_{\mathbb{R}^{n\nu}} \exp\left\{-\frac{1}{2\lambda}\sum_{l=1}^{n}\sum_{j=1}^{l}(2-\delta_{jl})s_j\left\langle\vec{v}_j,\vec{v}_l\right\rangle + i\left\langle\vec{\xi},\sum_{j=1}^{n}\vec{v}_j\right\rangle - \frac{i}{T}\left\langle\vec{\xi}-\vec{\eta},\sum_{j=1}^{n}s_j\vec{v}_j\right\rangle + \frac{1}{2\lambda T}\left\|\sum_{j=1}^{n}s_j\vec{v}_j\right\|^2\right\} \cdot d\mu_{s_1}(\vec{v}_1)\cdots d\mu_{s_n}(\vec{v}_n)\,d\vec{s}.$$

Since $F\in S(\nu)$, we know by Theorem 3.1 that the left-hand side of (4.4) has an analytic extension to $\mathbf{C}_+$ and is continuous on $\mathbf{C}_+^{\sim}$. We will show that the same is true for the right-hand side of (4.4). We first show that the series converges absolutely for all $\vec{\xi}$, $\vec{\eta}$ in $\mathbb{R}^{\nu}$ and all $\lambda\in\mathbb{C}_+^{\sim}$. This follows from the fact that

$$\sum_{l=1}^{n}\sum_{j=1}^{l}(2-\delta_{jl})\langle\vec{v}_j,\vec{v}_l\rangle s_j - \frac{1}{T}\left\|\sum_{j=1}^{n}s_j\vec{v}_j\right\|^2 \geq 0$$

since

$$\sum_{n=0}^{\infty}\int_{\Delta_n(T)}\int_{\mathbb{R}^{n\nu}}\left|\exp\left\{-\frac{1}{2\lambda}\sum_{l=1}^{n}\sum_{j=1}^{l}(2-\delta_{jl})s_j\langle\vec{v}_j,\vec{v}_l\rangle + i\left\langle\vec{\xi},\sum_{j=1}^{n}\vec{v}_j\right\rangle - \frac{i}{T}\left\langle\vec{\xi}-\vec{\eta},\sum_{j=1}^{n}s_j\vec{v}_j\right\rangle + \frac{1}{2\lambda T}\left\|\sum_{j=1}^{n}s_j\vec{v}_j\right\|^2\right\}\right| \cdot d\mu_{s_1}(\vec{v}_1)\cdots d\mu_{s_n}(\vec{v}_n)\,d\vec{s}$$

(continues)

(continued)

$$\leq \sum_{n=0}^{\infty} \int_{\Delta_n(T)} \int_{\mathbf{R}^{n\nu}} \exp\left\{ -\frac{1}{2}\operatorname{Re}\left[\frac{1}{\lambda}\right] \sum_{l=1}^{n}\sum_{j=1}^{l}(2-\delta_{jl})s_j\langle \vec{v}_j, \vec{v}_l\rangle + \frac{1}{2T}\operatorname{Re}\left[\frac{1}{\lambda}\right]\left\|\sum_{j=1}^{n} s_j\vec{v}_j\right\|^2\right\} \cdot d\mu_{s_1}(\vec{v}_1)\cdots d\mu_{s_n}(\vec{v}_n)\,d\vec{s}$$

$$\leq \sum_{n=0}^{\infty} \int_{\Delta_n(T)} \int_{\mathbb{R}^{n\nu}} d\mu_{s_1}(\vec{v}_1)\cdots d\mu_{s_n}(\vec{v}_n)\,d\vec{s}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{[0,T]^n} \left[\prod_{j=1}^{n} \|\mu_{s_j}\|\right] d\vec{s} = \sum_{n=0}^{\infty}\frac{1}{n!}\left[\int_0^T \|\mu_s\|\,ds\right]^n$$

$$= \exp\left\{\int_0^T \|\mu_s\| ds\right\} < \infty.$$

Thus using Morea's Theorem and the Dominated Convergence Theorem we obtain that the right-hand side of (4.4) is an analytic function of λ throughout $\mathbb{C}_+$ and is continuous in λ on $\widetilde{\mathbb{C}}_+$. Thus (4.3) is established which completes the proof of Theorem 4.2.

The following corollary is immediate using Theorem 4.2 in conjunction with Theorem 3.2.

COROLLARY 4.1. *Let F be as in Theorem 4.2. Then the conclusions of Theorem 3.2 hold and for $\psi \in L_1(\mathbb{R}^\nu)$, $J_q^{\text{an}}(F)\psi$ is given by (3.6) (and (1.1)) with $E^{\text{anf}_q}(F|X)$ given by (4.3).*

5. The $L_1 \to L_\infty$ theory. In this section, as in **[4, 10, 19]** we restrict our attention to the case $\nu = 1$ since **[20**, section 6] Johnson and Skoug gave counterexamples showing that the $L_1(\mathbb{R}^\nu) \to L_\infty(\mathbb{R}^\nu)$ theory doesn't hold for $\nu > 1$. In **[4, 19]** an $\mathscr{L}(L_1(\mathbb{R}), L_\infty(\mathbb{R}))$ theory of the operator-valued Feynman integral $J_q^{\text{an}}(F)$ was developed for functions of the form

$$F(x) = \exp\left\{\int_0^T \theta(s, x(s))\,ds\right\} \tag{5.1}$$

with appropriate assumptions on θ; the most general being as follows: Let $r \in (2, \infty]$ and let $\theta\colon [0, T] \times \mathbb{R} \to \mathbf{C}$ be a Borel measurable function such that for a.e. s in $[0, T]$, $\theta(s, \cdot)$ is in $L_1(\mathbb{R})$ with L_1-norm $\|\theta(s, \cdot)\|_1$ in $L_r[0, T]$. In this section we will show that for such F, $J_q^{\text{an}}(F)$ is given by the formula

$$(5.2)\quad (J_q^{\text{an}}(F)\psi)(\xi) = \int_{-\infty}^{\infty} E^{\text{anf}_q}(F|X)(\xi)(\eta)\left[\frac{q}{2\pi i T}\right]^{1/2} \exp\left\{\frac{qi}{2T}(\eta-\xi)^2\right\}\psi(\eta)\,d\eta$$

for $\psi \in L_1(\mathbb{R})$.

REMARK. Note that F of the form (5.1) may be unbounded and thus not in $S(1)$ and hence Theorem 3.2 and Corollary 4.1 do not apply to F given by (5.1) with θ as above.

THEOREM 5.1. *Let F be given by (5.1) with θ as above and let $X(y) = y(T)$ for $y \in C[0, T]$. Then for all real $q \neq 0$,*

$$(5.3)\quad E^{\text{anf}_q}(F|X)(\xi)(\eta)\left[\frac{q}{2\pi i T}\right]^{1/2}\exp\left\{\frac{iq}{2T}(\eta-\xi)^2\right\} = \sum_{n=0}^{\infty}\left[\frac{-iq}{2T}\right]^{(n+1)/2} \cdot \int_{\Delta_n(T)} [s_1(s_2-s_1)\cdots(s_n-s_{n-1})(T-s_n)]^{-1/2} \cdot \int_{\mathbb{R}^n}\left[\prod_{j=1}^{n}\theta(s_j, w_j)\right] \cdot \exp\left\{\sum_{j=1}^{n}\frac{iq}{2(s_j-s_{j-1})}(w_j-w_{j-1})^2 + \frac{iq}{2(T-s_n)}(w_n-\eta)^2\right\} dw_1\cdots dw_n\, d\vec{s}$$

where $\Delta_n(T) = \{\vec{s} = (s_1, \cdots, s_n): 0 < s_1 < s_2 < \cdots < s_n < T\}$, $s_0 = 0$ and $w_0 = \xi$. Furthermore $E^{\text{anf}_q}(F|X)(\cdot)(\cdot)$ is in $L_\infty(\mathbb{R}^2)$ and

$$(5.4)\quad \|E^{\text{anf}_q}(F|X)(\cdot)(\cdot)\|_\infty \le \sum_{n=0}^{\infty}\left|\frac{q}{2\pi}\right|^{n/2}\frac{T^{n(2-p)/2p}[\Gamma(1-p/2)]^{(n+1)/p}\left[\int_0^T \|\theta(s, \cdot)\|_1^r\, ds\right]^{n/r}}{(n!)^{1/r}\{\Gamma[(n+1)(1-p/2)]\}^{1/p}}$$

where Γ denotes the gamma function and p is such that $1/p + 1/r = 1$.

Proof. Using equation (4.1) with $\nu = 1$ we see that for each $\lambda > 0$,

$$(5.5)\quad \begin{aligned} &E(F(\lambda^{-1/2}x+\xi)|X(\lambda^{-1/2}x+\zeta))(\eta)\left[\frac{\lambda}{2\pi T}\right]^{1/2} \\ &\cdot \exp\left\{-\frac{\lambda}{2T}(\eta-\zeta)^2\right\} \\ &= \sum_{n=0}^{\infty}\left[\frac{\lambda}{2\pi}\right]^{(n+1)/2} \\ &\cdot \int_{\Delta_n(T)} [s_1(s_2-s_1)\cdots(s_n-s_{n-1})(T-s_n)]^{-1/2} \\ &\cdot \int_{\mathbf{R}^n}\left[\prod_{j=1}^{n}\theta(s_j, w_j)\right] \\ &\cdot \exp\left\{-\sum_{j=1}^{n}\frac{\lambda}{2(s_j-s_{j-1})}(w_j-w_{j-1})^2 \right. \\ &\qquad \left. -\frac{\lambda}{2(T-s_n)}(w_n-\eta)^2\right\} dw_1\cdots dw_n\, d\vec{s}. \end{aligned}$$

For notational purposes let $H_\lambda(\xi, \eta)$ denote the right-hand side of (5.5). Then for all $(\lambda, \xi, \eta) \in \mathbf{C}_+^\sim \times \mathbb{R} \times \mathbb{R}$ we see that

$$\begin{aligned} |H_\lambda(\xi, \eta)| &\le \sum_{n=0}^{\infty}\left[\frac{|\lambda|}{2\pi}\right]^{(n+1)/2} \\ &\cdot \int_{\Delta_n(T)} [s_1(s_2-s_1)\cdots(s_n-s_{n-1})(T-s_n)]^{-1/2} \\ &\cdot \int_{\mathbb{R}^n}\prod_{j=1}^{n}|\theta(s_j, w_j)|dw_1\cdots dw_n\, d\vec{s} \\ &\le \sum_{n=0}^{\infty}\left[\frac{|\lambda|}{2\pi}^{(n+1)/2}\right]\cdot\int_{\Delta_n(T)} [s_1(s_2-s_1)\cdots(s_n-s_{n-1})(T-s_n)]^{-1/2} \\ &\cdot \left[\prod_{j=1}^{n}\|\theta(s_j, \cdot)\|_1\right] d\vec{s} \end{aligned}$$

$$\leq \sum_{n=0}^{\infty} \left[\frac{|\lambda|}{2\pi}^{(n+1)/2} \right]$$

$$\cdot \left\{ \int_{\Delta_n(T)} [s_1(s_2 - s_1)\cdots(s_n - s_{n-1})(T - s_n)]^{-p/2}\, d\vec{s} \right\}^{1/p}$$

$$\cdot \left\{ \int_{\Delta_n(T)} \prod_{j=1}^{n} \|\theta(s_j, \cdot)\|_1^r\, d\vec{s} \right\}^{1/r}.$$

But

$$\int_{\Delta_n(T)} \prod_{j=1}^{n} \|\theta(s_j, \cdot)\|_1^r\, d\vec{s} = \frac{1}{n!} \int_0^T \cdots \int_0^T \prod_{j=1}^{n} \|\theta(s_j, \cdot)\|_1^r\, ds_1 \cdots ds_n$$
$$= \frac{1}{n!} \left[\int_0^T \|\theta(s_j, \cdot)\|_1^r\, ds \right]^n$$

and as was shown in [**19**, p. 652],

$$\int_{\Delta_n(T)} [s_1(s_2 - s_1)\cdots(s_n - s_{n-1})(T - s_n)]^{-p/2}\, d\vec{s}$$
$$= \frac{T^{-p/2}T^{n(2-p)/2}[\Gamma(1-p/2)]^{n+1}}{\Gamma[(n+1)(1-p/2)]}.$$

Thus for all $(\lambda, \xi, \eta) \in \mathbb{C}_+^{\sim} \times \mathbb{R} \times \mathbb{R}$,

(5.6) $$|H_\lambda(\xi, \eta)|$$
$$\leq \sum_{n=0}^{\infty} \left[\frac{|\lambda|}{2\pi} \right]^{(n+1)/2}$$
$$\cdot \frac{T^{n(2-p)/2p}[\Gamma(1-p/2)]^{(n+1)/p} \left[\int_0^T \|\theta(s, \cdot)\|_1^r\, ds) \right]^{n/r}}{(n!)^{1/r}\{\Gamma[(n+1)(1-p/2)]\}^{1/p} T^{1/2}}.$$

But since for large positive w,

$$\frac{1}{\Gamma(w)} < \frac{2e^w\sqrt{w}}{\sqrt{2\pi}w^w},$$

it is not hard to see that the series on the right-hand side of (5.6) converges for each $\lambda \in \mathbb{C}_+^{\sim}$; in fact uniformly on compact subsets of

$\mathbb{C}_+$. Thus the right-hand side of (5.5) is an analytic function of λ on $\mathbb{C}_+$ and continuous on $\mathbb{C}_+^\sim$ which establishes (5.3). The inequality (5.4) follows easily from (5.6) and (5.3).

THEOREM 5.2. *Let F and X be as in Theorem 5.1. Then for all real $q \neq 0$, the analytic operator-valued Feynman integral $J_a^{\mathrm{an}}(F)$ exists as an element of $\mathscr{L}(L_1(\mathbb{R}), L_\infty(\mathbb{R}))$ and for each $\psi \in L_1(\mathbb{R})$ is given by* (5.2).

Proof. By **[19]** we know that $J_q^{\mathrm{an}}(F)$ exists as an element of $\mathscr{L}(L_1(\mathbb{R}), L_\infty(\mathbb{R}))$ (actually as an element of $\mathscr{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$. We need to establish equation (5.2) with $E^{\mathrm{anf}_q}(F|X)(\xi)(\eta)$ given by (5.3). But, proceeding as in the beginning of the proof of Theorem 3.2, we see that for all $\lambda > 0$

$$(I_\lambda(F)\psi)(\xi) = \int_{-\infty}^{\infty} E(F(\lambda^{-1/2}x+\xi)|X(\lambda^{-1/2}x+\xi))(\eta)\psi(\eta) \cdot \left[\frac{\lambda}{2\pi T}\right]^{1/2} \exp\left\{-\frac{\lambda}{2T}(\eta-\xi)^2\right\} d\eta$$

where $E(F(\lambda^{-1/2}x+\xi)|X(\lambda^{-1/2}x+\xi))(\eta)$ is given by (4.1) with $\nu = 1$. But, as was shown in Theorem 5.1, $E(F(\lambda^{-1/2}x+\xi)|X(\lambda^{-1/2}x+\xi))(\eta)$ is an analytic function of λ throughout $\mathbb{C}_+$ and so

$$(I_\lambda^{\mathrm{an}}(F)\psi)(\xi) = \int_{-\infty}^{\infty} E^{\mathrm{anw}_\lambda}(F|X)(\xi)(\eta)\left[\frac{\lambda}{2\pi T}\right]^{1/2} \cdot \exp\left\{-\frac{\lambda}{2T}(\eta-\xi)^2\right\}\psi(\eta)\,d\eta \tag{5.7}$$

for all $\lambda \in \mathbb{C}_+$. Taking the limit of both sides of (5.7) as $\lambda \to -iq$, $\lambda \in \mathbb{C}_+$, establishes (5.2).

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Received November 16, 1988. Research of the first author was partially supported by the BSRIP, and the Ministry of Education, Korea (1988).

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