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A COMPARISON ALGEBRA ON A CYLINDER WITH SEMI-PERIODIC MULTIPLICATIONS

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A necessary and sufficient Fredholm criterion is found for a C^*-algebra of bounded operators on a cylinder, which contains operators of the form $L\Lambda^M$, where $\Lambda = (1-\Delta)^{-1/2}$ and L is an Mth order differential operator whose coefficients are periodic at infinity.

0. Introduction. Let Ω denote the cylinder $\mathbb{R}\times\mathbb{B}$, where $\mathbb{B}$ is a compact Riemannian manifold, Δ_Ω its Laplacian and $\mathscr{H}$ the Hilbert space $L^2(\Omega)$. Cordes [3] found a necessary and sufficient Fredholm criterion for operators in the C^*-subalgebra of $\mathscr{L}(\mathscr{H})$ generated by: (i) multiplications by functions that extend continuously to $[-\infty, +\infty]\times\mathbb{B}$, (ii) $\Lambda = (1-\Delta_\Omega)^{-1/2}$ and (iii) operators of the form $D\Lambda$, where D is either $\partial/\partial t$, $t\in\mathbb{R}$, or a first order differential operator on $\mathbb{B}$ with smooth coefficients. Here we extend this algebra by adjoining the multiplications by 2π-periodic continuous functions to the generators, and a similar Fredholm criterion is obtained.

The commutator ideal $\mathscr{E}_\mathscr{P}$ of the extended algebra $\mathscr{C}_\mathscr{P}$ is proven to be $*$-isomorphic to $\mathscr{SL}\,\overline{\otimes}\,\mathscr{K}_\mathbb{Z}\,\overline{\otimes}\,\mathscr{K}_\mathbb{B}$, where $\mathscr{SL}$ denotes the algebra of singular integral operators on the circle and $\mathscr{K}_\mathbb{Z}$ and $\mathscr{K}_\mathbb{B}$ denote the algebras of compact operators on $L^2(\mathbb{Z})$ and $L^2(\mathbb{B})$, respectively. This allows us to define on $\mathscr{C}_\mathscr{P}$ an operator-valued symbol, the "γ-symbol", such that $\ker\gamma\cap\ker\sigma$ equals the compact ideal of $\mathscr{L}(\mathscr{H})$. Here σ denotes the complex-valued symbol on $\mathscr{C}_\mathscr{P}$ that arises from the Gelfand map of the commutative C^*-algebra $\mathscr{C}_\mathscr{P}/\mathscr{E}_\mathscr{P}$. We prove that $A\in\mathscr{C}_\mathscr{P}$ is Fredholm if and only if γ_A and σ_A are invertible.

The simpler case when the compact manifold reduces to a point is considered in [5]. There, a unitary map W from $L^2(\mathbb{R})$ onto $L^2(S^1)\,\overline{\otimes}\,L^2(\mathbb{Z})$ is defined, such that the conjugate $W\mathscr{E}W^{-1}$ of the commutator ideal equals $\mathscr{SL}\,\overline{\otimes}\,\mathscr{K}_\mathbb{Z}$. Here, we conjugate $\mathscr{E}_\mathscr{P}$ with $W\otimes I_\mathbb{B}$, where $I_\mathbb{B}$ denotes the identity operator on $L^2(\mathbb{B})$, and obtain $\mathscr{SL}\,\overline{\otimes}\,\mathscr{K}_\mathbb{Z}\,\overline{\otimes}\,\mathscr{K}_\mathbb{B}$.

If L is a differential operator on Ω whose coefficients are continuous and approach periodic functions at infinity, the operator $A = L\Lambda^M$ belongs to $\mathscr{C}_\mathscr{P}$, where M is the order of L. We can apply the criterion above to A and prove that L is a Fredholm operator if and only if it is uniformly elliptic and a certain family of elliptic differential operators on the compact manifold $S^1 \times \mathbb{B}$ is invertible. This applies also for matrices of such operators.

These results can be extended in a standard way to non-compact manifolds with cylindrical ends (cf. [2], VIII-3,4). Fredholm properties of elliptic-differential operators on such manifolds have been studied, for example, by Lockhart-McOwen [6] and Taubes [8]. The case where the coefficients are periodic on the ends is included in Taubes' results.

1. Definition of the algebra $\mathscr{C}_{\mathscr{P}}$ and a description of its commutator ideal. Let Ω denote the Riemannian manifold $\mathbb{R} \times \mathbb{B}$, where $\mathbb{B}$ denotes an n-dimensional compact manifold with metric tensor locally given by h_{jk}, and let $\mathscr{H}$ denote the Hilbert space $L^2(\Omega)$, with Ω being given the surface measure

$$dS = \sqrt{h}\, dt\, dx^1 \cdots dx^n ,$$

where h is the determinant of the $n \times n$-matrix $((h_{jk}))_{1 \leq j,k \leq n}$. The metric on Ω is given by $ds^2 = dt^2 + h_{jk} dx^j\, dx^k$, and the Laplace operator is locally given by

$$\Delta_\Omega = \Delta_{\mathbb{R}} + \Delta_{\mathbb{B}} = \frac{d^2}{dt^2} + \frac{1}{\sqrt{h}} \frac{\partial}{\partial x^j} \sqrt{h} h^{jk} \frac{\partial}{\partial x^k} ,$$

where $((h^{jk})) = ((h_{jk}))^{-1}$, and the summation convention from 1 to n is being used.

The symmetric operator Δ_Ω with domain $\mathbf{C}_0^\infty(\Omega)$ is essentially self-adjoint, since Ω is complete (cf. [2], IV). We denote by H the closure of $1-\Delta_\Omega$ and by Λ its inverse square root, $\Lambda = H^{-1/2}$. Since $H \geq 1$, we have $\Lambda \in \mathscr{L}(\mathscr{H})$. The algebra $\mathscr{C}_{\mathscr{P}}$ is defined as the smallest C^*-subalgebra of $\mathscr{L}(\mathscr{H})$ containing the following operators (or classes of operators):

$$(1) \qquad a \in \mathbf{C}^\infty(\mathbb{B}); \quad b \in \mathbf{CS}(\mathbb{R});$$
$$e^{ijt},\, j \in \mathbb{Z}; \quad \Lambda; \quad \frac{1}{i}\frac{\partial}{\partial t}\Lambda \quad \text{and} \quad D_x\Lambda,$$

D_x being a first order differential operator on $\mathbb{B}$, locally given by $-ib^j(x)\partial/\partial x^j$, where $b^j(x)$, $j = 1, \ldots, n$, are the components of a smooth vector field on $\mathbb{B}$. The operators $\frac{\partial}{\partial t}\Lambda$ and $D_x\Lambda$, defined on the dense subspace $\Lambda^{-1}(C_0^\infty(\Omega))$, can be extended to bounded operators of $\mathscr{L}(\mathscr{H})$ (cf. [2], for example). Bounded functions on Ω

have been identified with the corresponding multiplication operators in $\mathscr{L}(\mathscr{H})$ and $\mathbf{CS}(\mathbb{R})$ denotes the class of continuous functions on $\mathbb{R}$ with limits at $+\infty$ and $-\infty$.

Our first objective is to obtain a necessary and sufficient criterion for an operator in $\mathscr{C}_{\mathscr{P}}$ to be Fredholm. Such a criterion has been found by Cordes [3] for the algebra generated by the operators in (1) except e^{ijt}, $j \in \mathbb{Z}$.

Taking advantage of the tensor product structure of $\mathscr{H}$,

$$\mathscr{H} = L^2(\mathbb{R}) \,\overline{\otimes}\, L^2(\mathbb{B}),$$

we consider the conjugate of $\mathscr{C}_{\mathscr{P}}$ with respect to the unitary operator $F \otimes I_{\mathbb{B}}$, where $I_{\mathbb{B}}$ denotes the identity operator on $L^2(\mathbb{B})$ and F the Fourier transform on $L^2(\mathbb{R})$,

$$(Fu)(\tau) = \frac{1}{\sqrt{2\pi}} \int e^{-i\tau t} u(t)\, dt.$$

In order to simplify notation, $A \otimes I_{\mathbb{B}}$ is denoted by A and $I_{\mathbb{R}} \otimes B$ by B, whenever $A \in \mathscr{L}(L^2(\mathbb{R}))$ or $B \in \mathscr{L}(L^2(\mathbb{B}))$.

We seek to describe what are $B_k := F^{-1} A_k F$, where A_k, $k = 1, \dots, 6$, denote the operators listed in (1), in that order. The operator-valued functions $\tilde{\Lambda}(\tau) := (1+\tau^2-\Delta_{\mathbb{B}})^{-1/2}$, $\tau\tilde{\Lambda}(\tau)$ and $D_x\tilde{\Lambda}(\tau)$, $\tau \in \mathbb{R}$, are all in $\mathbf{CB}(\mathbb{R}, \mathscr{L}_{\mathbb{B}})$, as proven in [3], page 220, and thus determine operators in $\mathscr{L}(\mathscr{H})$ by multiplication in the real variable. Here $\mathscr{L}_{\mathbb{B}}$ denotes the algebra of bounded operators on $L^2(\mathbb{B})$ and $\mathbf{CB}(\mathbb{R}, \mathscr{L}_{\mathbb{B}})$ the bounded continuous $\mathscr{L}_{\mathbb{B}}$-valued functions on $\mathbb{R}$. With this interpretation, we get B_k, $k = 1, \dots, 6$, respectively given by

$$(2) \qquad a \in \mathbf{C}^\infty(\mathbb{B}); \quad b(D),\, b \in \mathrm{CS}(\mathbb{R}); \quad T_j,\, j \in \mathbb{Z};$$

$$\tilde{\Lambda}(\tau); \quad -\tau\tilde{\Lambda}(\tau) \quad \text{and} \quad D_x\tilde{\Lambda}(\tau),$$

where $b(D) := F^{-1} b F$ and T_j denotes the translation $(T_j u)(\tau) = u(\tau + j)$.

Let $\mathscr{K}_{\mathbb{B}}$ denote the ideal of compact operators on $L^2(\mathbb{B})$ and $\mathbf{CO}(\mathbb{R}, \mathscr{K}_{\mathbb{B}})$ denote the $\mathscr{K}_{\mathbb{B}}$-valued continuous functions on $\mathbb{R}$ that vanish at infinity. All commutators $[B_k, B_l]$, $k, l \neq 3$, are contained in the algebra

$$\mathscr{C}\mathscr{K} := \mathbf{CO}(\mathbb{R}, \mathscr{K}_{\mathbb{B}}) + \mathscr{K}(\mathscr{H}),$$

where $\mathscr{K}(\mathscr{H})$ denotes the ideal of compact operators of $\mathscr{L}(\mathscr{H})$, as proven in [3], Proposition 1.2. Next we investigate what are the commutators $[B_3, B_k]$, $k = 1, \ldots, 6$. We easily get $[B_3, B_1] = [B_3, B_2] = 0$. It is also clear that, for any $K(\tau) \in \mathbf{CB}(\mathbb{R}, \mathscr{L}_{\mathbb{B}})$, we have

$$(3) \qquad [T_k, K(\tau)] = (K(\tau + k) - K(\tau))T_k, \qquad k \in \mathbb{Z}.$$

PROPOSITION 1.1. *The commutators of the generators in* (2)—*and of their adjoints—of the algebra* $\widehat{C}_{\mathscr{P}} := F^{-1}\mathscr{C}_{\mathscr{P}}F$ *are contained in*

$$\mathscr{CKT} = \left\{ \sum_{j=-N}^{N} K_j(\tau)T_j + K;\, N \in \mathbb{N},\, K_j \in \mathbf{CO}(\mathbb{R}, \mathscr{K}_{\mathbb{B}}),\, K \in \mathscr{K}(\mathscr{H}) \right\}.$$

Proof. Let us first prove that $K(\tau + j) - K(\tau) \in \mathbf{CO}(\mathbb{R}, \mathscr{K}_{\mathbb{B}})$, for $K(\tau) = \tilde{\Lambda}(\tau)$, $\tau\tilde{\Lambda}(\tau)$ or $D_x\tilde{\Lambda}(\tau)$. It follows from the fact that $-\Delta_{\mathbb{B}}$ on $L^2(\mathbb{B})$ has an orthonormal basis of eigenfunctions, with eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \cdots$, $\lambda_n \to \infty$ as $n \to \infty$, that, for each $\tau \in \mathbb{R}$, $\tilde{\Lambda}(\tau)$ is unitarily equivalent to the multiplication operator $(1 + \tau^2 + \lambda_n)^{-1/2}$ on $L^2(\mathbb{N})$. Hence: $\tilde{\Lambda}(\tau) \in \mathbf{CO}(\mathbb{R}, \mathscr{K}_{\mathbb{B}})$,

$$\|\tau[\tilde{\Lambda}(\tau + j) - \tilde{\Lambda}(\tau)]\|_{L^2(\mathbb{B})} \le \max_{s \in [1, \infty)} |\tau[(s + (\tau + j)^2)^{-1/2} - (s + \tau^2)^{-1/2}]|$$

and

$$\|\tilde{\Lambda}(\tau)^{-1}\tilde{\Lambda}(\tau + j) - 1\|_{L^2(\mathbb{R})} \le \max_{s \in [1, \infty)} |(\tau^2 + s)^{1/2}((\tau + j)^2 + s)^{-1/2} - 1|.$$

Note that the right-hand sides of the two previous inequalities go to zero as $\tau \to \pm\infty$. Furthermore, as

$$\lim_{n \to \infty} (1 + \tau^2 + \lambda_n)^{1/2}(1 + (\tau + j)^2 + \lambda_n)^{-1/2} - 1 = 0,$$

we have that $\tilde{\Lambda}(\tau)^{-1}\tilde{\Lambda}(\tau + j) - 1 \in \mathscr{K}_{\mathbb{B}}$, for each $\tau \in \mathbb{R}$. We then get:

$$(\tau + j)\tilde{\Lambda}(\tau + j) - \tau\tilde{\Lambda}(\tau) = \tau(\tilde{\Lambda}(\tau + j) - \tilde{\Lambda}(\tau)) + j\tilde{\Lambda}(\tau + j) \in \mathbf{CO}(\mathbb{R}, \mathscr{K}_{\mathbb{B}})$$

and

$$D_x\tilde{\Lambda}(\tau + j) - D_x\tilde{\Lambda}(\tau) = D_x\tilde{\Lambda}(\tau)[\tilde{\Lambda}(\tau)^{-1}\tilde{\Lambda}(\tau + j) - 1] \in \mathbf{CO}(\mathbb{R}, \mathscr{K}_{\mathbb{B}}).$$

By the remarks preceding the statement of the proposition, this proves that the commutators of the generators (2) are indeed contained in $\mathscr{CKT}$. Concerning the adjoints, let us note that the classes of B_k's, $k = 1, \ldots, 5$, are self-adjoint and that, as proven in [3], $D_x\tilde{\Lambda} - \tilde{\Lambda}D_x \in \mathbf{CO}(\mathbb{R}, \mathscr{K}_{\mathbb{B}})$. Hence

$$(4) \qquad (D_x\tilde{\Lambda})^* - D_x^*\tilde{\Lambda} = \tilde{\Lambda}D_x^* - D_x^*\tilde{\Lambda} \in \mathbf{CO}(\mathbb{R}, \mathscr{K}_{\mathbb{B}}).$$

Here, D_x^* denotes the formal adjoint of D_x. The commutators of any $K(\tau) \in \mathbf{CO}(\mathbb{R}, \mathscr{K}_{\mathbb{B}})$ with the generators B_k, $k = 1, 3, 4, 5, 6$, are clearly contained in $\mathscr{CKT}$. For $K(\tau)$ of the special form $K(\tau) = a(\tau)\tilde{K}$, $a \in \mathbf{CO}(\mathbb{R})$ and $\tilde{K} \in \mathscr{K}_{\mathbb{B}}$, the commutator $[b(D), K(\tau)] = [b(D), a(\tau)] \otimes \tilde{K}$ is compact, since $[b(D), a(\tau)]$ is compact (cf. **[4]**, Chapter III, for example), for $b \in \mathbf{CS}(\mathbb{R})$. The vector space generated by all $K(\tau) = a(\tau)\tilde{K}$ as above is dense in $\mathbf{CO}(\mathbb{R}, \mathscr{K}_{\mathbb{B}})$ and thus we have

$$(5) \quad [b(D), K(\tau)] \in \mathscr{K}(\mathscr{H}), \quad \text{for } b \in \mathbf{CS}(\mathbb{R}), K(\tau) \in \mathbf{CO}(\mathbb{R}, \mathscr{K}_{\mathbb{B}}).$$

This concludes the proof. □

Denoting by $\mathscr{E}_{\mathscr{P}}$ the commutator ideal of $\mathscr{C}_{\mathscr{P}}$ and by $\mathscr{E}_{\mathscr{P}}$ the commutator ideal of $\widehat{\mathscr{C}}_{\mathscr{P}}$, it is obvious that $\widehat{\mathscr{E}}_{\mathscr{P}} = F^{-1}\mathscr{E}_{\mathscr{P}}F$.

Proposition 1.2. *The commutator ideal $\widehat{\mathscr{E}}_{\mathscr{P}}$ of the algebra $\widehat{\mathscr{C}}_{\mathscr{P}}$ is obtained by closing the set of operators*:

$$\widehat{\mathscr{E}}_{P,0} := \left\{ \sum_{j=-N}^{N} b_j(D)K_j(\tau)T_j + K; \right. \\ \left. b_j \in \mathbf{CS}(\mathbb{R}),\ N \in \mathbb{N}, K_j \in \mathbf{CO}(\mathbb{R}, \mathscr{K}_{\mathbb{B}}, K \in \mathscr{K}(\mathscr{H}) \right\}.$$

Proof. The algebra $\mathscr{C}_{\mathscr{P}}$ is a "Comparison Algebra", in the sense of **[2]**, Chapter V, with "generating classes":

$$(6) \quad \mathscr{A}^{\sharp} := \mathbf{C}_0^{\infty}(\Omega) \cup \mathbf{C}^{\infty}(\mathbb{B}) \cup \{e^{ijt}; j \in \mathbb{Z}\} \cup \{s(t) = t(1+t^2)^{-1/2}\}$$

and $\mathscr{D}^{\sharp}$ equal to the vector space generated by the first order linear partial differential expressions on $\mathbb{B}$ with smooth coefficients and by the expression $\partial/\partial t$. Indeed, $\mathscr{C}_{\mathscr{P}}$ can alternatively be defined as the C^*-algebra generated by all multiplications by functions in $\mathscr{A}^{\sharp}$ and by all $D\Lambda$, $D \in \mathscr{D}^{\sharp}$. It follows then from Lemma V-1-1 of **[2]** that $\mathscr{K}(\mathscr{H}) \subset \mathscr{E}_{\mathscr{P}}$ and therefore $\mathscr{K}(\mathscr{H}) \subset \widehat{\mathscr{E}}_{\mathscr{P}}$. Moreover, it was proven in **[3]**, Proposition 1.5, that $\mathbf{CO}(\mathbb{R}, \mathscr{K}_{\mathbb{B}})$ is contained in the commutator ideal of the C^*-algebra generated by B_4, B_5 and B_6. Thus we get $\widehat{\mathscr{E}}_{P,0} \subset \widehat{\mathscr{E}}_{\mathscr{P}}$.

All commutators of the generators (2) and their adjoints are contained in $\widehat{\mathscr{E}}_{P,0}$, by Proposition 1.1. Again using (3), (4) and (5), it is easy to verify that $\widehat{\mathscr{E}}_{P,0}$ is invariant under right or left multiplication by the operators in (2) and their adjoints. Two facts then follow:

(i) all commutators of the algebra (finitely) generated by the operators in (2) and their adjoints are contained in $\widehat{\mathscr{E}}_{P,0}$ and therefore all commutators of $\widehat{\mathscr{C}}_{\mathscr{P}}$ are contained in the closure of $\widehat{\mathscr{E}}_{P,0}$, and (ii) the closure of $\widehat{\mathscr{E}}_{P,0}$ is an ideal of $\widehat{\mathscr{C}}_{\mathscr{P}}$. By definition of commutator ideal, $\widehat{\mathscr{E}}_{\mathscr{P}}$ is contained in the closure of $\widehat{\mathscr{E}}_{P,0}$. □

Let $\mathbf{CO}(\mathbb{R})$ denote the set of continuous functions on $\mathbb{R}$ vanishing at infinity and let $\widehat{\mathscr{E}}_0$ denote the set of bounded operators on $L^2(\mathbb{R})$

$$\widehat{\mathscr{E}}_0 := \left\{ \sum_{j=-N}^{N} b_j(D)a_j(\tau)T_j + K\,;\, N \in \mathbb{N},\, b_j \in \mathbf{CS}(\mathbb{R}), \right.$$

$$\left. a_j \in \mathbf{CO}(\mathbb{R}),\, K \in \mathscr{K}_{\mathbb{R}} \right\}.$$

COROLLARY 1.3. *With $\widehat{\mathscr{E}}$ denoting the closure of $\widehat{\mathscr{E}}_0$ defined above, we have:*

$$\widehat{\mathscr{E}}_{\mathscr{P}} = \widehat{\mathscr{E}}\,\overline{\otimes}\,\mathscr{K}_{\mathbb{B}}$$

where $\overline{\otimes}$ denotes the operator-norm closure of the algebraic tensor product.

Proof. The vector-space generated by

$$\{(b(D)a(\tau)T_j + K) \otimes \widetilde{K}\,;\, b \in \mathbf{CS}(\mathbb{R}),\, a \in \mathbf{CO}(\mathbb{R}),\, j \in \mathbb{Z},$$
$$K \in \mathscr{K}_{\mathbb{R}},\, \widetilde{K} \in \mathscr{K}_{\mathbb{B}}\}$$

is dense in $\widehat{\mathscr{E}}_{P,0}$ and in $\widehat{\mathscr{E}}\,\overline{\otimes}\,\mathscr{K}_{\mathbb{B}}$. □

In the rest of this section, we define a unitary map

$$W\colon L^2(\mathbb{R}) \to L^2(\mathbb{S}^1\,;\, L^2(\mathbb{Z}))$$

and find a useful description for $(W \otimes I_{\mathbb{B}})\widehat{\mathscr{E}}_{\mathscr{P}}(W \otimes I_{\mathbb{B}})^{-1}$.

Given $u \in L^2(\mathbb{R})$, denote:

$$u^{\diamond}(\varphi) := (u(\varphi - j))_{j\in\mathbb{Z}},$$

for each $\varphi \in \mathbb{R}$. The sequence $u^{\diamond}(\varphi)$ belongs to $L^2(\mathbb{Z})$ for almost every φ, by Fubini's Theorem, since $L^2(\mathbb{R})$ can be identified with $L^2([0,1) \times \mathbb{Z})$. Let

$$F_d\colon L^2(\mathbb{S}^1, d\theta) \to L^2(\mathbb{Z}), \qquad \mathbb{S}^1 = \{e^{i\theta}\,;\, \theta \in \mathbb{R}\},$$

denote the discrete Fourier transform:

$$(F_d u)_j = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(\theta) e^{-ij\theta}\, d\theta, \qquad j \in \mathbb{Z}. \tag{7}$$

For each $\varphi \in \mathbb{R}$, define

$$Y_\varphi := F_d e^{-i\varphi\theta} F_d^{-1}. \tag{8}$$

The operators Y_φ define a smooth function on $\mathbb{R}$, taking values in the unitary operators on $L^2(\mathbb{Z})$ and satisfying $(Y_k u)_j = u_{j+k}$, for $k \in \mathbb{Z}$ and $u \in L^2(\mathbb{Z})$, and $Y_\varphi Y_\omega = Y_{\varphi+\omega}$, for $\varphi, \omega \in \mathbb{R}$.

We now define the map (with $\mathbb{S}^1 = \{e^{2\pi i\varphi}; \varphi \in \mathbb{R}\}$)

$$\begin{aligned} W\colon L^2(\mathbb{R}) &\to L^2(\mathbb{S}^1, d\varphi; L^2(\mathbb{Z})), \\ u &\mapsto (Wu)(\varphi) = Y_\varphi u^\diamond(\varphi). \end{aligned} \tag{9}$$

Let $\mathbf{CS}(\mathbb{Z})$ denote the set of sequences $b(j)$, $j \in \mathbb{Z}$, with limits as $j \to +\infty$ and $j \to -\infty$ and let $b(D_\theta)$ denote $F_d^{-1} b(M) F_d$, where $b(M)$ denotes the operator multiplication by b on $L^2(\mathbb{Z})$. We then denote by $\mathcal{SL}$ the C^*-subalgebra of $\mathcal{L}_{\mathbb{S}^1} := \mathcal{L}(L^2(\mathbb{S}^1))$ generated by $b(D_\theta)$, $b \in \mathcal{CS}(\mathbb{Z})$, and by the multiplications by smooth functions on $\mathbb{S}^1$. It is easy to check that, with $\Lambda_{\mathbb{S}^1} := (1 - \Delta_{\mathbb{S}^1})^{-1/2}$,

$$\frac{1}{i}\frac{d}{d\theta}\Lambda_{\mathbb{S}^1} = s(D_\theta), \qquad s(j) = (1 + j^2)^{-1/2}.$$

Since the polynomials in s are dense in $\mathbf{CS}(\mathbb{Z})$, $\mathcal{SL}$ coincides with the C^*-subalgebra of $\mathcal{L}_{\mathbb{S}^1}$ generated by $-i\frac{d}{d\theta}\Lambda_{\mathbb{S}^1}$ and $\mathbf{C}^\infty(\mathbb{S}^1)$. In other words, $\mathcal{SL}$ is the unique comparison algebra over $\mathbb{S}^1$. It therefore contains the compact ideal $\mathcal{K}_{\mathbb{S}^1}$ and all its commutators are compact (cf. **[2]**, Chapters V and VI).

The following theorem was proven in **[5]** (Theorem 2.6). See also [7], Theorem 1.2.

THEOREM 1.4. *With the above notation, we have:*

$$W\widehat{\mathcal{E}}W^{-1} = \mathcal{SL}\,\overline{\otimes}\,\mathcal{K}_{\mathbb{Z}}, \tag{10}$$

where $\mathcal{K}_{\mathbb{Z}}$ denotes the set of compact operators on $L^2(\mathbb{Z})$. Furthermore, for $b \in \mathbf{CS}(\mathbb{R})$, $a \in \mathbf{CO}(\mathbb{R})$ and $j \in \mathbb{Z}$, we have:

$$A^\diamond(e^{2\pi i\varphi}) := Y_\varphi a(\varphi - M) Y_{-\varphi} \in \mathbf{C}(\mathbb{S}^1, \mathcal{K}_{\mathbb{Z}})$$

and

$$W(b(D)aT_j)W^{-1} = b(D_\theta) Y_\varphi a(\varphi - M) Y_{-\varphi-j} + K, \qquad K \in \mathcal{K}_{\mathbb{S}^1\times\mathbb{Z}}. \tag{11}$$

PROPOSITION 1.5. *The map*

$$\begin{aligned} \widehat{\mathcal{E}}_{\mathcal{P}} &\to \mathcal{SL}\,\overline{\otimes}\,\mathcal{K}_{\mathbb{Z}}\,\overline{\otimes}\,\mathcal{K}_{\mathbb{B}}, \\ A &\mapsto WAW^{-1} \end{aligned}$$

is an onto $*$*-isomorphism. For* $A \in \widehat{\mathscr{E}}_{\mathscr{P}}$ *of the form* $A = b(D)K(\tau)T_j$, *with* $b \in \mathbf{CS}(\mathbb{R})$, $K(\tau) \in \mathbf{CO}(\mathbb{R}, \mathscr{K}_{\mathbb{B}})$ *and* $j \in \mathbb{Z}$, *we have*:

(12) $WAW^{-1} = b(D_\theta)Y_\varphi K(\varphi - M)Y_{-\varphi-j} + K$, *with* $K \in \mathscr{K}_{\mathbb{S}^1 \times \mathbb{Z} \times \mathbb{B}}$.

For each $\varphi \in \mathbb{R}$ *here,* $K(\varphi - M)$ *denotes the compact operator on* $L^2(\mathbb{Z})\overline{\otimes}L^2(\mathbb{B})$ *defined by the sequence* $K(\varphi - j) \in \mathscr{K}_{\mathbb{B}}$, $j \in \mathbb{Z}$. *The first term of the right-hand side of* (12) *defines therefore a* $\mathscr{K}_{\mathbb{Z}\times\mathbb{B}}$*-valued continuous function on* $\mathbb{S}^1 = \{e^{2\pi i\varphi}; \varphi \in \mathbb{R}\}$.

Proof. By Corollary 1.3 and (10),

$$W\widehat{\mathscr{E}}_{\mathscr{P}}W^{-1} = \mathscr{SL}\,\overline{\otimes}\,\mathscr{K}_{\mathbb{Z}}\,\overline{\otimes}\,\mathscr{K}_{\mathbb{B}}$$

and, by (11), formula (12) holds for $K(\tau)$ of the form $a(\tau) \otimes \widetilde{K}$, $a \in \mathbf{CO}(\mathbb{R})$ and $\widetilde{K} \in \mathscr{K}_{\mathbb{B}}$. We can then find a sequence $K_m(\tau) \in \mathbf{CO}(\mathbb{R}, \mathscr{K}_{\mathbb{B}})$ such that $K_m(\tau) \to K(\tau)$, uniformly in $\tau \in \mathbb{R}$, and (12) is valid for each $K_m(\tau)$. Then

$$Y_\varphi K_m(\varphi - M)Y_{-\varphi-j} \to Y_\varphi K(\varphi - M)Y_{-\varphi-j}$$

in $\mathscr{K}_{\mathbb{Z}\times\mathbb{B}}$, uniformly in $e^{2\pi i\varphi} \in \mathbb{S}^1$. Since the supremum-norm of a function on $\mathbb{S}^1$ taking values in $\mathscr{L}(L^2(\mathbb{Z})\overline{\otimes}L^2(\mathbb{B}))$ equals the norm of the corresponding multiplication operator on $L^2(\mathbb{S}^1)\,\overline{\otimes}\,L^2(\mathbb{Z})\,\overline{\otimes}\,L^2(\mathbb{B})$, the convergence above also holds in $\mathscr{L}(L^2(\mathbb{S}^1)\,\overline{\otimes}\,L^2(\mathbb{Z})\,\overline{\otimes}\,L^2(\mathbb{B}))$. □

Let $\mathbf{M}_{SL}$ denote the maximal-ideal space of $\mathscr{SL}/\mathscr{K}_{\mathbb{S}^1}$ and let

$$\sigma^{SL}\colon \mathscr{SL}/\mathscr{K}_{\mathbb{S}^1} \to \mathbf{C}(\mathbf{M}_{SL})$$

denote the composition of the Gelfand map with the canonical projection. We then have (cf. [2], for example): $\mathbf{M}_{SL} = \mathbb{S}^1 \times \{-1, +1\}$ and

$$\sigma_a^{SL}(\cdot, \pm 1) = a(\cdot), \quad \text{for } a \in \mathbf{C}^\infty(\mathbb{S}^1)$$

and

$$\sigma_{b(D_\theta)}^{SL}(\cdot, \pm 1) = b(\pm\infty) \quad \text{for } b \in \mathbf{CS}(\mathbb{Z}).$$

Let $\mathbf{C}(\mathbf{M}_{SL}, \mathscr{K}_{\mathbb{Z}\times\mathbb{B}})$ denote the $\mathscr{K}_{\mathbb{Z}\times\mathbb{B}}$-valued functions on $\mathbf{M}_{SL}$. Here $\mathscr{K}_{\mathbb{Z}\times\mathbb{B}}$ denotes the compact ideal of $L^2(\mathbb{Z})\,\overline{\otimes}\,L^2(\mathbb{B})$, $\mathscr{K}_{\mathbb{Z}\times\mathbb{B}} = \mathscr{K}_{\mathbb{Z}}\,\overline{\otimes}\,\mathscr{K}_{\mathbb{B}}$.

THEOREM 1.6. *There exists an onto* $*$*-isomorphism*

$$\Psi\colon \frac{\mathscr{E}_{\mathscr{P}}}{\mathscr{K}(\mathscr{H})} \to \mathbf{C}(\mathbf{M}_{SL}, \mathscr{K}_{\mathbb{Z}\times\mathbb{B}})$$

such that if $\tilde{\gamma}$ denotes the composition of Ψ with the canonical projection $\mathscr{E}_{\mathscr{P}} \to \mathscr{E}_{\mathscr{P}}/\mathscr{K}(\mathscr{H})$ and $A \in \mathscr{E}_{\mathscr{P}}$ is such that $B = F^{-1}AF$ is of the form $B = b(D)K(\tau)T_j$, where $b \in \mathbf{CS}(\mathbb{R})$, $K(\tau) \in \mathbf{CO}(\mathbb{R}, \mathscr{K}_{\mathbb{B}})$ and $j \in \mathbb{Z}$, we then have:

$$\tilde{\gamma}_A(e^{2\pi i\varphi}, \pm 1) = b(\pm\infty)Y_\varphi K(\varphi - M)Y_{-\varphi-j}.$$

Proof. Let Ψ be given by

$$\frac{\mathscr{E}_{\mathscr{P}}}{\mathscr{K}(\mathscr{H})} \to \frac{\widehat{\mathscr{E}}_{\mathscr{P}}}{\mathscr{K}(\mathscr{H})} \to \frac{\mathscr{S}\mathscr{L}\,\overline{\otimes}\,\mathscr{K}_{\mathbb{Z}}\,\overline{\otimes}\,\mathscr{K}_{\mathbb{B}}}{\mathscr{K}_{\mathbb{S}^1\times\mathbb{Z}\times\mathbb{B}}} \to \mathbf{C}(M_{SL}, \mathscr{K}_{\mathbb{Z}\times\mathbb{B}}),$$

where in the first step we take $A + \mathscr{K}(\mathscr{H}) \in \mathscr{E}_{\mathscr{P}}/\mathscr{K}(\mathscr{H})$ to $F^{-1}AF + \mathscr{K}(\mathscr{H})$, next to

$$WF^{-1}AFW^{-1} + \mathscr{K}_{\mathbb{S}^1\times\mathbb{Z}\times\mathbb{B}},$$

and in the last step we use the onto $*$-isomorphism (see **[1]**):

$$\frac{\mathscr{S}\mathscr{L}\,\overline{\otimes}\,\mathscr{K}_{\mathbb{Z}}\,\overline{\otimes}\,\mathscr{K}_{\mathbb{B}}}{\mathscr{K}_{\mathbb{S}^1\times\mathbb{Z}\times\mathbb{B}}} \to \mathbf{C}(M_{SL}, \mathscr{K}_{\mathbb{Z}\times\mathbb{B}})$$
$$A \otimes K_1 \otimes K_2 + \mathscr{K}_{\mathbb{S}^1\times\mathbb{Z}\times\mathbb{B}} \mapsto \sigma_A^{SL}(\varphi, \pm 1)K_1 \otimes K_2.$$

Defined this way, Ψ has the desired properties, by Proposition 1.5 and its proof. □

2. Definition of two symbols on $\mathscr{E}_{\mathscr{P}}$. Our first task in this section is to give a precise description of the symbol space of $\mathscr{E}_{\mathscr{P}}$, i.e., the maximal-ideal space of the commutative C^*-algebra $\mathscr{E}_{\mathscr{P}}/\mathscr{E}_{\mathscr{P}}$. The symbol space of $\mathscr{E}$, the C^*-algebra generated by the operators listed in (1) except the periodic functions e^{ijt}, was described in [3]:

THEOREM 2.1 (*Theorem* 2.3 *of* **[3]**). *The symbol space* $\mathbf{M}$ *of* $\mathscr{E}$ *can be identified with the bundle of unit spheres of the cotangent bundle of the compact manifold with boundary* $[-\infty, +\infty]\times\mathbb{B}$, *where* $[-\infty, +\infty]$ *denotes the compactification of* $\mathbb{R}$ *obtained by adding the points* $-\infty$ *and* $+\infty$. *The* σ*-symbols of the generators* A_1, A_2, A_4, A_5 *and* A_6 *are given below as functions of the local coordinates* $(t, x; \tau, \xi)$, *where* $(t, \tau) \in [-\infty, +\infty] \times \mathbb{R}^*$, $(x, \xi) \in T^*\mathbb{B}$ *and* $\tau^2 + h^{jk}\xi_j\xi_k = 1$:

$$\sigma_{A_1} = a(x), \quad \sigma_{A_2} = b(t), \quad \sigma_{A_4} = 0, \quad \sigma_{A_5} = \tau, \quad \sigma_{A_6} = b^j(x)\xi_j.$$

When periodic functions are adjoined to the algebra, the points over $|t| = \infty$ become circles. More precisely, we have:

THEOREM 2.2. *The symbol space* $\mathbf{M}_P$ *of* $\mathscr{C}_{\mathscr{P}}$ *is homeomorphic to the closed subset of* $\mathbf{M} \times \mathbb{S}^1$ *described in local coordinates by*

$$\{((t, x; \tau, \xi), e^{i\theta}); (t, x; \tau, \xi) \in \mathbf{M}, \theta \in \mathbb{R} \text{ and } \theta = t \text{ if } |t| < \infty\}.$$

Using this description of $\mathbf{M}_P$, *the* σ*-symbols of the generators in* (1) *are respectively given by*

$$a(x), \quad b(t), \quad e^{ij\theta}, \quad 0, \quad \tau \text{ and } b^j(x)\xi_j.$$

Proof. Let $\mathbf{P}_{2\pi}$ denote the closed algebra generated by $\{e^{ijt}; j \in \mathbb{Z}\}$, i.e., the 2π-periodic continuous functions on $\mathbb{R}$. Its maximal-ideal space is $\mathbb{S}^1$, with $e^{i\theta} \in \mathbb{S}^1$ defining the multiplicative linear functional $f \to f(\theta)$.

With $\mathscr{E}$ denoting the commutator ideal of $\mathscr{C}$, the maximal-ideal space of $\mathscr{C}/\mathscr{E}$ is $\mathbf{M}$, as described in Theorem 2.1. By definition of the Gelfand map, a point $(t, x; \tau, \xi)$ defines the multiplicative linear functional

$$A + \mathscr{E} \to \sigma_A(t, x; \tau, \xi).$$

The following maps are canonically defined:

$$i_1 : \frac{\mathscr{C}}{\mathscr{E}} \to \frac{\mathscr{C}_{\mathscr{P}}}{\mathscr{E}_{\mathscr{P}}} \tag{13}$$

and

$$i_2 : \mathbf{P}_{2\pi} \to \frac{\mathscr{C}_{\mathscr{P}}}{\mathscr{E}_{\mathscr{P}}}. \tag{14}$$

(It is obvious that $\mathscr{E} \subseteq \mathscr{E}_{\mathscr{P}}$)

Let us denote by $\imath$ the product of the dual maps of i_1 and i_2, i.e.,

$$\begin{aligned} \imath : \mathbf{M}_P &\to \mathbf{M} \times \mathbb{S}^1, \\ w &\mapsto (w \circ i_1, , w \circ i_2), \end{aligned} \tag{15}$$

where w denotes a multiplicative linear functional on $\mathscr{C}_{\mathscr{P}}/\mathscr{E}_{\mathscr{P}}$.

As the images of i_1 and i_2 generate $\mathscr{C}_{\mathscr{P}}/\mathscr{E}_{\mathscr{P}}$, $\imath$ is an injective map, clearly continuous, which proves that $\mathbf{M}_P$ is homeomorphic to a compact subset of $\mathbf{M} \times \mathbb{S}^1$. Now we proceed to investigate which points of $\mathbf{M} \times \mathbb{S}^1$ belong to the image of $\imath$. This dual-map argument is essentially "Herman's Lemma" (cf. [**4**]).

As in the proof of Proposition 1.2, here again we use general results on comparison algebras. It follows from Theorem VII-1-5 of [**2**] that for every point of the cosphere-bundle of Ω, $(t, x; \tau, \xi) \in S^*\Omega$, there is a multiplicative linear functional on $\mathscr{C}_{\mathscr{P}}/\mathscr{E}_{\mathscr{P}}$ that takes any function a, belonging to the closed algebra generated by $A^\sharp$ in (6), to $a(x, t)$ and $D\Lambda$,

$$D = \frac{1}{i}\frac{\partial}{\partial t} + \frac{1}{i}b^j(x)\frac{\partial}{\partial x^j} + q(x) \in \mathscr{D}^\sharp,$$

to $\tau + b^j(x)\xi_j$. This multiplicative linear functional must correspond to the point

$$((t, x; \tau, \xi), e^{it}) \in \mathbf{M} \times \mathbb{S}^1,$$

with $|t| < \infty$.

Suppose now that $((t, x; \tau, \xi), e^{i\theta})$ is in the image of $\imath$ and that $|t| < \infty$. Let ω denote the corresponding multiplicative linear functional on $\mathscr{C}_{\mathscr{P}}/\mathscr{E}_{\mathscr{P}}$ and χ denote a function in $\mathbf{C}_0^\infty(\Omega)$ with $\chi(t) = 1$. It is clear that $\chi(\cdot)e^{i(\cdot)} + \mathscr{E}_{\mathscr{P}}$ belongs to the image of i_1 and thus, by (15),

$$\omega(\chi(\cdot)e^{i(\cdot)} + \mathscr{E}_{\mathscr{P}}) = e^{it}.$$

On the other hand, since $e^{i(\cdot)} + \mathscr{E}_{\mathscr{P}}$ belongs to the image of i_2, we get:

$$\omega(\chi(\cdot)e^{i(\cdot)} + \mathscr{E}_{\mathscr{P}}) = \omega(\chi(\cdot) + \mathscr{E}_{\mathscr{P}})\omega(e^{i(\cdot)} + \mathscr{E}_{\mathscr{P}}) = e^{i\theta}.$$

We then obtain $e^{i\theta} = e^{it}$.

For $t = \pm\infty$ and any $e^{i\theta} \in \mathbb{S}^1$, let us consider the sequence $t_m = \theta \pm 2\pi m$. Since $\mathbf{M}_P$ is closed and

$$((t_m, x; \tau, \xi), e^{it_m}) \to ((t, x; \tau, \xi), e^{i\theta}) \quad \text{as } m \to \infty,$$

we conclude that $((t, x; \tau, \xi), e^{i\theta}) \in \mathbf{M}_P$. □

REMARK 2.3. We have just proved above that

$$\mathbf{W}_P := \{((t, x; \tau, \xi), e^{i\theta}) \in \mathbf{M}_P; |t| < \infty\}$$

is dense in $\mathbf{M}_P$.

Next we define the γ-symbol.

The $\mathbf{C}^*$-algebra $\mathscr{C}_{\mathscr{P}}/\mathscr{K}(\mathscr{H})$ has the closed two-sided ideal $\mathscr{E}_{\mathscr{P}}/\mathscr{K}(\mathscr{H})$, which was proven to be $*$-isomorphic to $\mathbf{C}(M_{SL}, \mathscr{K}_{\mathbb{Z}\times\mathbb{B}})$ in Theorem 1.6. Every $A \in \mathscr{C}_{\mathscr{P}}$ determines a bounded operator of $\mathscr{L}(\mathscr{E}_{\mathscr{P}}/\mathscr{K}(\mathscr{H}))$ by $E + \mathscr{K}(\mathscr{H}) \to AE + \mathscr{K}(\mathscr{H})$, thus defining

$$T: \mathscr{C}_{\mathscr{P}} \to \mathscr{L}(\mathscr{E}_{\mathscr{P}}/\mathscr{K}(\mathscr{H})).$$

Let us define

$$(16)\qquad \begin{aligned} \gamma\colon \mathscr{E}_{\mathscr{P}} &\to \mathscr{L}(\mathbf{C}(\mathbf{M}_{SL}, \mathscr{K}_{\mathbb{Z}\times\mathbb{B}})) \\ A &\mapsto \gamma_A = \Psi T_A \Psi^{-1}, \end{aligned}$$

for Ψ defined in Theorem 1.6.

For $E \in \mathscr{E}_{\mathscr{P}}$, γ_E is the operator multiplication by $\tilde{\gamma}_E \in \mathbf{C}(\mathbf{M}_{SL}, \mathscr{K}_{\mathbb{Z}\times\mathbb{B}})$ (see Theorem 1.6). Let $\mathbf{C}(\mathbf{M}_{SL}, \mathscr{L}_{\mathbb{Z}\times\mathbb{B}})$ denote the continuous functions on $\mathbf{M}_{SL}$ taking values in $\mathscr{L}_{\mathbb{Z}\times\mathbb{B}} := \mathscr{L}(L^2(\mathbb{Z}\times\mathbb{B}))$. Identifying functions in $\mathbf{C}(\mathbf{M}_{SL}, \mathscr{L}_{\mathbb{Z}\times\mathbb{B}})$ with the corresponding multiplication operator of $\mathscr{L}(\mathbf{C}(\mathscr{M}_{SL}, \mathscr{K}_{\mathbb{Z}\times\mathbb{B}}))$, we can say then that γ is an extension of $\tilde{\gamma}$.

PROPOSITION 2.4. *There exists a $*$-homomorphism*

$$\gamma\colon \mathscr{E}_{\mathscr{P}} \to \mathbf{C}(\mathbf{M}_{SL}, \mathscr{L}_{\mathbb{Z}\times\mathbb{B}}),$$

where

$$\mathbf{M}_{SL} = \{e^{2\pi i\varphi}\,;\ \varphi\in\mathbb{R}\}\times\{+1, -1\},$$

given on the generators (1), *according to notation established in* §1 *and in Theorem* 1.6, *by*:

$$(17)\qquad \begin{gathered} \gamma_{A_1} = a(x)\,;\quad \gamma_{A_2} = b(\pm\infty)\,; \\ \gamma_{A_3} = Y_{-j}\,;\quad \gamma_{A_4} = Y_\varphi\widetilde{\Lambda}(\varphi - M)Y_{-\varphi}\,; \\ \gamma_{A_5} = Y_\varphi K(\varphi - M)Y_{-\varphi},\quad \text{where } K(\tau) = -\tau\widetilde{\Lambda}(\tau),\ \tau\in\mathbb{R} \text{ and} \\ \gamma_{A_6} = Y_\varphi L(\varphi - M)Y_{-\varphi},\quad \text{where } L(\tau) = D_x\widetilde{\Lambda}(\tau),\ \tau\in\mathbb{R}. \end{gathered}$$

Furthermore, γ restricted to the C^*-algebra $C^\diamond_{\mathscr{P}}$, generated by the operators in (1) *except $b\in\mathbf{CS}(\mathbb{R})$, is an isometry.*

Proof. Let us calculate γ, defined in (16), for the generators $A_1, \dots, A_6$ of (1). By Proposition 1.2, it is enough to calculate the result of a left multiplication by A_p, $p = 1, \dots, 6$, on operators $E\in\mathscr{E}_{\mathscr{P}}$ such that $F^{-1}EF$ are of the form $c(D)K(\tau)T_l$, $c\in\mathbf{CS}(\mathbb{R})$, $K\in\mathbf{CO}(\mathbb{R}, \mathscr{K}_{\mathbb{B}})$ and $l\in\mathbb{Z}$. For such an E, we get $F^{-1}(A_pE)F$, $p = 1, \dots, 6$, equal to, modulo compact operators,

$$c(D)a(x)K(\tau)T_l,\qquad (cb)(D)K(\tau)T_l,\qquad c(D)K(\tau+j)T_{j+l},$$

$$c(D)\widetilde{\Lambda}(\tau)K(\tau)T_l,\qquad -c(D)\tau\widetilde{\Lambda}(\tau)K(\tau)T_l \ \text{ and }\ c(D)D_x\widetilde{\Lambda}(\tau)K(\tau)T_l,$$

respectively. Here we have used (3) and

$$[c(D), B_k]\in\mathscr{K}(\mathscr{H}),\qquad k = 4, 5, 6$$

(cf. [3], Proposition 1.2). By Theorem 1.6, we get:

$$\begin{aligned}
\gamma_{A_1E}(\varphi, \pm 1) &= c(\pm\infty)Y_\varphi \widetilde{K}(\varphi - M)Y_{-\varphi-1} \\
&= a(x)\gamma_E(\varphi, \pm 1) \quad (\widetilde{K} = aK), \\
\gamma_{A_2E}(\varphi, \pm 1) &= (cb)(\pm\infty)Y_\varphi K(\varphi - M)Y_{-\varphi-1} = b(\pm\infty)\gamma_E(\varphi, \pm\infty), \\
\gamma_{A_3E}(\varphi, \pm 1) &= c(\pm\infty)Y_\varphi K(\varphi + j - M)Y_{-\varphi-j-l} = Y_j\gamma_E(\varphi, \pm 1), \\
\gamma_{A_4E}(\varphi, \pm 1) &= c(\pm\infty)Y_\varphi(\widetilde{\Lambda}K)(\varphi - M)Y_{-\varphi-l} \\
&= Y_\varphi\widetilde{\Lambda}(\varphi - M)Y_{-\varphi}\gamma_E(\varphi, \pm 1)
\end{aligned}$$

and analogously for $p = 5$ and 6. This proves formulas (17).

For any $A \in \mathscr{C}_{\mathscr{P}}$ such that $F^{-1}AF = J(\tau) \in \mathbf{CO}(\mathbb{R}, \mathscr{K}_{\mathbb{B}})$, it is also clear, using (5), that

$$\gamma_A(\varphi, \pm 1) = Y_\varphi J(\varphi - M)Y_{-\varphi}.$$

Hence, by (4), $\gamma_{A_6^*}$ also belongs to $\mathbf{C}(\mathbf{M}_{SL}, \mathscr{L}_{\mathbb{Z}\times\mathbb{B}})$.

The norm of the operator of $\mathscr{L}(\mathbf{C}(\mathbf{M}_{SL}, \mathscr{K}_{\mathbb{Z}\times\mathbb{B}}))$ given by multiplication by a function in $\mathbf{C}(\mathbf{M}_{SL}, \mathscr{L}_{\mathbb{Z}\times\mathbb{B}})$ is equal to the sup-norm of this function. In other words, the C^*-algebra $\mathbf{C}(\mathbf{M}_{SL}, \mathscr{L}_{\mathbb{Z}\times\mathbb{B}})$ is isometrically imbedded in $\mathscr{L}(\mathbf{C}(\mathbf{M}_{SL}, \mathscr{K}_{\mathbb{Z}\times\mathbb{B}}))$. As the image of a dense subalgebra of $\mathscr{C}_{\mathbb{P}}$ is contained in $\mathbf{C}(\mathbf{M}_{SL}, \mathscr{L}_{\mathbb{Z}\times\mathbb{B}})$, we conclude that γ maps $\mathscr{C}_{\mathscr{P}}$ into $\mathbf{C}(\mathbf{M}_{SL}, \mathscr{L}_{\mathbb{Z}\times\mathbb{B}})$.

Using the identification

$$L^2(\mathbb{S}^1)\,\overline{\otimes}\,L^2(\mathbb{Z})\,\overline{\otimes}\,L^2(\mathbb{B}) = L^2(\mathbb{S}^1, L^2(\mathbb{Z}\times\mathbb{B})),$$

it can be straightforwardly verified that, for $A(\tau) \in \mathbf{CB}(\mathbb{R}, \mathscr{L}_{\mathbb{B}})$, $WA(\tau)W^{-1} \in \mathbf{C}(\mathbb{S}^1, \mathscr{L}_{\mathbb{Z}\times\mathbb{B}})$ and it is given by $Y_\varphi A(\varphi - M)Y_{-\varphi}$. This means that for $k = 1, 4, 5, 6$, we have

$$\gamma_{A_k} = WF^{-1}A_kFW^{-1} \quad \text{and} \quad \gamma_{A_k^*} = WF^{-1}A_k^*FW^{-1}.$$

It is also clear that $WT_jW^{-1} = Y_{-j}$ and, hence,

$$\gamma_A = WF^{-1}A(WF^{-1})^{-1}, \quad \text{for } A \in \mathscr{C}_{\mathscr{P}}^{\diamond},$$

proving that

$$\|\gamma_A\|_{\mathbf{C}(\mathbf{M}_{SL}, \mathscr{L}_{\mathbb{Z}\times\mathbb{B}})} = \|A\|_{\mathscr{L}(\mathscr{H})} \quad \text{and} \quad \gamma_{A^*} = (\gamma A)^* \quad \text{for } A \in \mathscr{C}_{\mathscr{P}}^{\diamond}.$$

This finishes the proof, since it is obvious that $\gamma_{A_2^*} = (\gamma_{A_2})^*$. □

The σ-symbol and the γ-symbol, defined in Theorem 2.2 and Proposition 2.4 respectively, are related by:

PROPOSITION 2.5. *For every* $A \in \mathscr{C}_{\mathscr{P}}$, $\|\sigma_A|_{\mathbf{M}_P \setminus \mathbf{W}_P}\| \leq \|\gamma_A\|$, *i.e.,*

$$\sup\{|\sigma_A((t, x; \tau, \xi), e^{i\theta})|; |t| = \infty\} \leq \sup\{\|\gamma_A(m)\|_{\mathscr{L}_{\mathbb{Z}\times\mathbb{B}}}; m \in \mathbf{M}_{SL}\}.$$

Proof. Since the commutators of A_2 with the other generators in (1) and their adjoints are compact (cf. [3], Proposition 1.2), the set of operators of the form

$$(18)\quad A = \sum_{j=1}^{N} b_j(t)A_j + K,$$

$$b_j \in \mathbf{CS}(\mathbb{R}),\ A_j \in \mathscr{C}_{\mathscr{P}}^{\diamond},\ K \in \mathscr{K}(\mathscr{H}),\ N \in \mathbb{N},$$

is dense in $\mathscr{C}_{\mathscr{P}}$. As $\sigma_K = 0$ and $\gamma_K = 0$ for $K \in \mathscr{K}(\mathscr{H})$, it suffices to assume A of the form (18) with $K = 0$.

For such an A, Theorem 2.2 implies:

$$\sigma_A((t, x; \tau, \xi), e^{i\theta}) = \sum_{j=1}^{N} b_j(t)\sigma_{A_j}((t, x; \tau, \xi), e^{i\theta}).$$

Letting $A^{\pm}$ denote the operators $\sum_{j=1}^{N} b_j(\pm\infty)A_j$, it is clear then that

$$\sigma_A((+\infty, x; \tau, \xi), e^{i\theta}) = \sigma_{A^+}((\pm\infty, x; \tau, \xi), e^{i\theta}) \quad \text{and}$$
$$\sigma_A((-\infty, x; \tau, \xi), e^{i\theta}) = \sigma_{A_-}((\pm\infty, x; \tau, \xi), e^{i\theta});$$

hence:

$$(19)\qquad \|\sigma_A|_{\mathbf{M}_P \setminus \mathbf{W}_P}\| \leq \max\{\|\sigma_{A^+}\|, \|\sigma_{A^-}\|\}.$$

The map $\sigma\colon \mathscr{C}_{\mathscr{P}} \to \mathbf{C}(\mathbf{M}_P)$ was defined as the composition of the Gelfand map (an isometry) with the canonical projection $\mathscr{C}_{\mathscr{P}} \to \mathscr{C}_{\mathscr{P}}/\mathscr{K}(\mathscr{H})$. It then follows that

$$\|\sigma_{A^\pm}\| \leq \|A^{\pm}\|.$$

As $A^{\pm} \in \mathscr{C}_{\mathscr{P}}^{\diamond}$, where γ is an isometry,

$$(20)\qquad \|\sigma_{A^\pm}\|_{\mathbf{C}(\mathbf{M}_P)} \leq \|\gamma_{A^\pm}\|_{\mathbf{C}(\mathbf{M}_{SL}, \mathscr{L}_{\mathbb{Z}\times\mathbb{B}})}.$$

By Proposition 2.4,

$$\gamma_A(\varphi, +1) = \sum_{j=1}^{N} b_j(+\infty)\gamma_{A_j}(\varphi, +1) = \gamma_{A^+}(\varphi, +1)$$

and

$$\gamma_A(\varphi, -1) = \gamma_{A^-}(\varphi, -1).$$

Furthermore, for any $A \in \mathscr{C}_{\mathscr{P}}^{\diamond}$, it is clear from (17) that $\gamma_A(\varphi, +1) = \gamma_A(\varphi, -1)$ and, therefore,

$$\|\gamma_A\| = \max\{\|\gamma_{A^+}\|, \|\gamma_{A^-}\|\} \tag{21}$$

We are finished by (19), (20) and (21). □

If $\gamma_A = 0$, then, $\sigma_A|_{\mathbf{M}_P \setminus \mathbf{W}_P} = 0$. The converse is also true:

PROPOSITION 2.6. *An operator $A \in \mathscr{C}_{\mathscr{P}}$ belongs to the kernel of γ if and only if σ_A vanishes on $\mathbf{M}_P \setminus \mathbf{W}_P$. Furthermore, we have*:

$$\ker \gamma \cap \ker \sigma = \mathscr{K}(\mathscr{H}). \tag{22}$$

Proof. Let $\mathscr{T}_0$ denote the C^*-algebra generated by multiplications by functions in $\mathbf{C}_0^\infty(\Omega)$ and by the operators of the form $D\Lambda$, where D is a first order linear differential operator on Ω with smooth coefficients of compact support. Given A_0, one of these generators just described, we can find $\chi \in \mathbf{C}_0^\infty(\mathbb{R})$ such that $\chi A_0 = A_0$ and then $\gamma_{A_0} = \gamma_\chi \gamma_{A_0} = 0$, by Proposition 2.4. So, we have $\mathscr{T}_0 \subseteq \ker \gamma$.

Using the nomenclature of [2], $\mathscr{T}_0$ is the minimal comparison algebra associated with the triple $\{\Omega, dS, H\}$. It can be easily concluded from [2], Lemma VII-1-2, that $A \in \mathscr{C}_{\mathscr{P}}$ belongs to $\mathscr{T}_0$ if and only if σ_A vanishes on $\mathbf{M}_P \setminus \mathbf{W}_P$, proving that $\mathscr{T}_0 \subseteq \ker \gamma$, by Proposition 2.5.

Since $\ker \sigma = \mathscr{E}_{\mathscr{P}}$ and $\ker \gamma = \mathscr{T}_0$, the equality in (22) follows from [2], Theorem VII-1-3. □

3. A Fredholm criterion and an application to differential operators. We will now give a necessary and sufficient criterion for an $N \times N$-matrix whose entries are operators in $\mathscr{C}_{\mathscr{P}}$, regarded as a bounded operator on $L^2(\Omega, \mathbb{C}^N)$, $N \geq 1$, to be Fredholm. Let us denote $L^2(\Omega, \mathbb{C}^N)$ by $\mathscr{H}^N$ and by $\mathscr{C}_{\mathscr{P}}^N$ the C^*-subalgebra of $\mathscr{L}(\mathscr{H}^N)$

$$\mathscr{C}_{\mathscr{P}}^N := \{((A_{jk}));\, A_{jk} \in \mathscr{C}_{\mathscr{P}},\, 1 \leq j, k \leq N\}.$$

It is easy to see that the compact ideal of $\mathscr{L}(\mathscr{H}^N)$ coincides with the matrices with entries in $\mathscr{K}(\mathscr{H})$, i.e.,

$$\mathscr{K}(\mathscr{H}^N) = \mathscr{K}^N := \{((K_{jk}));\, K_{jk} \in \mathscr{K}(\mathscr{H}),\, 1 \leq j, k \leq N\}.$$

Let us define two symbols on $\mathscr{C}_{\mathscr{P}}^N$:

$$\sigma_A^N = ((\sigma_{A_{jk}}))_{1 \leq j, k \leq N} \quad \text{and} \quad \gamma_A^N = ((\gamma_{A_{jk}}))_{1 \leq j, k \leq N},$$

where $A = ((A_{jk}))_{1\le j,k\le N} \in \mathscr{C}_{\mathscr{P}}^N$. The following proposition follows immediately from the definitions above and Proposition 2.6.

PROPOSITION 3.1. *The γ^N-symbol of an operator $A \in \mathscr{C}_{\mathscr{P}}^N$ is identically zero if and only if its σ^N-symbol vanishes on $\mathbf{M}_P \backslash \mathbf{W}_P$. Furthermore, we have:*

$$\ker \sigma^N \cap \ker \gamma^N = \mathscr{K}^N. \tag{23}$$

THEOREM 3.2. *For an operator $A = ((A_{jk}))_{1\le j,k\le N} \in \mathscr{C}_{\mathscr{P}}^N$ to be Fredholm, it is necessary and sufficient that:*

(i) *σ_A^N be invertible, i.e., the $N\times N$-matrix $((\sigma_{A_{jk}}(m)))$ be invertible for all $m \in \mathbf{M}_P$, and*

(ii) *γ_A^N be invertible, i.e., the $N \times N$-matrix, with entries in $\mathbf{C}(\mathbf{M}_{SL}, \mathscr{L}_{\mathbb{Z}\times\mathbb{B}})$, $((\gamma_{A_{jk}}(m)))$ be invertible for all $m \in \mathbf{M}_{SL}$.*

Proof. Suppose that A is Fredholm and let B be such that $1 - AB$ and $1 - BA$ are compact. We have $B \in \mathscr{C}_{\mathscr{P}}^N$, since $\mathscr{C}_{\mathscr{P}}^N/\mathscr{K}^N$ is a C^*-subalgebra of $\mathscr{L}(\mathscr{H}^N)/\mathscr{K}^N$. We then get

$$\sigma_{1-AB}^N = \sigma_{1-BA}^N = 0 \quad \text{and} \quad \gamma_{1-AB}^N = \gamma_{1-BA}^N = 0$$

and, hence,

$$1 = \sigma_A^N \sigma_B^N = \sigma_B^N \sigma_A^N \quad \text{and} \quad 1 = \gamma_A^N \gamma_B^N = \gamma_B^N \gamma_A^N.$$

Conversely, suppose that (i) and (ii) above are satisfied. Since $\gamma^N : \mathscr{C}_{\mathscr{P}}^N \to \mathbf{C}(\mathbf{M}_{SL}, N\times N\text{-matrices with entries in } \mathscr{L}(L^2(\mathbb{Z})\bar{\otimes} L^2(\mathbb{B})))$ is a $*$-homomorphism (by Proposition 2.4), its range is a C^*-algebra. There must be then a $B \in \mathscr{C}_{\mathscr{P}}^N$ such that $\gamma_B^N = (\gamma_A^N)^{-1}$. Since $1 - AB \in \ker \gamma^N$, $1 - \sigma_A^N \sigma_B^N$ vanishes on $\mathbf{M}_P \backslash \mathbf{W}_P$, by Proposition 3.1. As the map σ is surjective, so is σ^N. An operator $Q \in \mathscr{C}_{\mathscr{P}}^N$ can therefore be found such that its symbol σ_Q^N equals the continuous function vanishing on $\mathbf{M}_P \backslash \mathbf{W}_P$

$$(\sigma_A^N)^{-1} - \sigma_B^N.$$

By Proposition 3.1 again, $Q \in \ker \gamma^N$ and, then,

$$\gamma_{1-A(B+Q)}^N = \gamma_{1-(B+Q)A}^N = 0.$$

Since we also have

$$\sigma_{1-A(B+Q)}^N = 1 - \sigma_A^N \sigma_B^N - \sigma_A^N \sigma_Q^N = 0 = \sigma_{1-(B+Q)A}^N,$$

the operator $B + Q$ is an inverse for A, modulo a compact operator, by equation (23). □

In order to apply this result to differential operators, it is convenient to conjugate the γ-symbol with the discrete Fourier transform. We define:

$$(24)\qquad \begin{aligned}\Gamma\colon \mathscr{C}_{\mathscr{P}} &\to \mathbf{C}(\mathbf{M}_{SL}, \mathscr{L}_{\mathbb{S}^1\times\mathbb{B}})\\ A &\mapsto \Gamma_A(m) = F_d^{-1}\gamma_A(m)F_d, \qquad m \in \mathbf{M}_{SL},\end{aligned}$$

where $F_d\colon L^2(\mathbb{S}^1) \to L^2(\mathbb{Z})$, $\mathbb{S}^1 = \{e^{i\theta};\, \theta \in \mathbb{R}\}$, was defined in (7), and, as usual, F_d also denotes $F_d \otimes I_{\mathbb{B}}$.

Next we calculate Γ_A for the generators of $\mathscr{C}_{\mathscr{P}}$. It is obvious that, for $a \in \mathbb{C}^\infty(\mathbb{B})$,

$$(25)\qquad \Gamma_a(\varphi, \pm 1) = a, \qquad (e^{2\pi i\varphi}, \pm 1) \in \mathbf{M}_{SL},$$

and, for $b \in \mathbf{CS}(\mathbb{R})$,

$$(26)\qquad \Gamma_b(\varphi, \pm 1) = b(\pm\infty), \quad \text{independent of } \varphi.$$

For $j \in \mathbb{Z}$, $F_d^{-1}Y_{-j}F_d$ equals the operator multiplication by $e^{ij\theta}$ on $\mathbb{S}^1 = \{e^{i\theta}, \theta \in \mathbb{R}\}$, and then, by (24) and (17),

$$(27)\qquad \Gamma_{e^{+ijt}}(\varphi, \pm 1) = e^{ij\theta}, \quad \text{for all } (e^{2\pi i\varphi}, \pm 1) \in \mathbf{M}_{SL}.$$

Let $a \in \mathbf{C}(\Omega)$ be of the form

$$(28)\qquad a(t, x) = a_+(t, x)\chi_+(t, x) + a_-(t, x)\chi_-(t, x) + a_0(t, x),$$

where $a_\pm$ are continuous and 2π-periodic in t, $a_0 \in \mathbf{CO}(\Omega)$ and $\chi_\pm \in \mathbf{CS}(\mathbb{R})$ satisfy $\chi_\pm(\pm\infty) = 1$, $\chi_+ + \chi_- = 1$. By the continuity of Γ, (25), (26) and (27), it follows that

$$(29)\qquad \Gamma_a(\varphi, \pm 1) = a_\pm(\theta, x), \quad \text{for } (e^{2\pi i\varphi}, \pm 1) \in \mathbf{M}_{SL}.$$

Note that (28) gives Γ_{A_1}, Γ_{A_2} and Γ_{A_3}, for A_p as defined on page 283.

Now we calculate $F_d^{-1}K(\varphi - M)F_d$, for $\varphi \in \mathbb{R}$ and $K(\tau) = \widetilde{\Lambda}(\tau)$, $-\tau\widetilde{\Lambda}(\tau)$ or $D_x\widetilde{\Lambda}(\tau)$, which is needed for obtaining Γ_{A_p}, $p = 4, 5, 6$. Let us use again that $-\Delta_{\mathbb{B}}$ has an orthonormal basis of eigenfunctions w_m, $m \in \mathbb{N}$, with eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$, $\lambda_m \to \infty$ as $m \to \infty$, and define the unitary map

$$\begin{aligned}U\colon L^2(\mathbb{B}) &\to L^2(\mathbb{N}),\\ u &\mapsto (w_m, u)_{m\in\mathbb{N}}.\end{aligned}$$

By the spectral theorem, the conjugate $U(1 + (\varphi - j)^2 - \Delta_{\mathbb{B}})^{-1/2}U^{-1}$ equals the operator multiplication by $(1+(\varphi-j)^2+\lambda_m)^{-1/2}$ on $L^2(\mathbb{N})$, for each $j \in \mathbb{Z}$, $\varphi \in \mathbb{R}$. The operator $\widetilde{\Lambda}(\varphi - M) \in \mathscr{L}_{\mathbb{Z}\times\mathbb{B}}$ acts on

$$\mathbf{u} = (u_j)_{j\in\mathbb{Z}} \in L^2(\mathbb{Z}; L^2(\mathbb{B}))$$

by

$$\widetilde{\Lambda}(\varphi - M)\mathbf{u} = ((1 + (\varphi - j)^2 + \Delta_{\mathbb{B}})^{-1/2} u_j)_{j\in\mathbb{Z}}$$

and, thus,

$$(30) \qquad (I_{\mathbb{Z}} \otimes U)\widetilde{\Lambda}(\varphi - M)(I_{\mathbb{Z}} \otimes U)^{-1} = (1 + (\varphi - j)^2 + \lambda_m)^{-1/2},$$

where, by $(1 + (\varphi - j)^2 + \lambda_m)^{-1/2}$, we now mean the corresponding multiplication operator on $L^2(\mathbb{Z})\overline{\otimes}L^2(\mathbb{N})$.

Let us adopt the notation:

$$(31) \quad 1 + (\varphi - D_\theta)^2 - \Delta_{\mathbb{B}} := (F_d \otimes U)^{-1}(1 + (\varphi - j)^2 + \lambda_m)(F_d \otimes U).$$

It is easy to see that $1 + (\varphi - D_\theta)^2 - \Delta_{\mathbb{B}}$ is the unique self-adjoint realization of the differential expression $1 + (\varphi + i\frac{\partial}{\partial\theta})^2 - \Delta_{\mathbb{B}}$ on $\mathbb{S}^1 \times \mathbb{B}$ (see Lemma 3.3). By (30) and (31) then, we obtain:

$$(32) \qquad (F_d \otimes I_{\mathbb{B}})^{-1}\widetilde{\Lambda}(\varphi - M)(F_d \otimes I_{\mathbb{B}}) = (1 + (\varphi - D_\theta)^2 - \Delta_{\mathbb{B}})^{-1/2},$$

for every $\varphi \in \mathbb{R}$. Using that $Y_\varphi = F_d^{-1} e^{-i\varphi\theta} F_d$, $\varphi \in \mathbb{R}$ and (17), it follows that:

$$(33) \qquad \Gamma_\Lambda(\varphi, \pm 1) = e^{-i\varphi\theta}(1 + (D_\theta - \varphi)^2 - \Delta_{\mathbb{B}})^{-1/2} e^{i\varphi\theta}, \quad (e^{2\pi i\varphi}, \pm 1) \in \mathbf{M}_{SL}.$$

Since, for each $j \in \mathbb{Z}$ and each $\varphi \in \mathbb{R}$,

$$U(\varphi - j)(1 + (\varphi - j)^2 - \Delta_{\mathbb{B}})^{-1/2} U^{-1}$$

equals the operator multiplication by

$$(\varphi - j)(1 + (\varphi - j)^2 + \lambda_m)^{-1/2}$$

on $L^2(\mathbb{N})$, we obtain, in a way analogous to how (33) was obtained:

$$(34) \qquad \Gamma_{A_4}(\varphi, \pm 1) = e^{-i\varphi\theta}(D_\theta - \varphi)(1 + (D_\theta - \varphi)^2 - \Delta_{\mathbb{B}})^{-1/2} e^{i\varphi\theta}, \quad (e^{2\pi i\varphi}, \pm 1) \in \mathbf{M}_{SL}.$$

Here we have assumed the notation:

$$(\varphi - D_\theta)(1 + (\varphi - D_\theta)^2 - \Delta_{\mathbb{B}})^{-1/2} := (F_d \otimes U)^{-1}(\varphi - j)(1 + (\varphi - j)^2 + \lambda_m)^{-1/2}(F_d \otimes U).$$

For the last type of generator, we need the following lemma.

LEMMA 3.3. *The subspace*

$$\{u \in L^2(\mathbb{S}^1 \times \mathbb{B})\,;\,(1+(\varphi - D_\theta)^2 - \Delta_{\mathbb{B}})^{-1/2} u \in \mathbf{C}^\infty(\mathbb{S}^1 \times \mathbb{B})\}$$

is dense in $L^2(\mathbb{S}^1 \times \mathbb{B})$, *for every* $\varphi \in \mathbb{R}$.

Proof. The statement is true for $\varphi = 0$, since

$$1 + D_\theta^2 - \Delta_{\mathbb{B}} = 1 - \Delta_{\mathbb{S}^1 \times \mathbb{B}}$$

is essentially self-adjoint on $\mathbf{C}^\infty(\mathbb{S}^1 \times \mathbb{B})$, by [2], Theorem IV-1-8, for example. For $\varphi \in \mathbb{R}$,

$$(1+(\varphi - D_\theta)^2 - \Delta_{\mathbb{B}})^{-1/2}(1 + D_\theta^2 - \Delta_{\mathbb{B}})^{1/2}$$

is a Banach-space isomorphism, since it is unitarily equivalent to the multiplication by the function on $\mathbb{Z} \times \mathbb{N}$

$$(1+(\varphi - j)^2 + \lambda_m)^{-1/2}(1 + j^2 + \lambda_m)^{-1/2},$$

which is bounded and bounded away from zero. □

For every $v \in \mathbf{C}^\infty(\mathbb{S}^1 \times \mathbb{B})$, it is clear that

$$D_x F_d v = F_d D_x v,$$

where, on the right-hand side, D_x is regarded as a differential expression on $\mathbb{S}^1 \times \mathbb{B}$ and, on the left-hand side, D_x acts, as a differential operator on $\mathbb{B}$, on each component $w_j \in \mathbf{C}^\infty(\mathbb{B})$ of

$$w = (w_j)_{j \in \mathbb{Z}} = F_d v \in L^2(\mathbb{Z}; L^2(\mathbb{B})).$$

By Lemma 3.3, it therefore follows that

$$(35)\qquad F_d[D_x(1+(\varphi - D_\theta)^2 - \Delta_{\mathbb{B}})^{-1/2}]F_d^{-1} = D_x[F_d(1+(\varphi - D_\theta)^2 - \Delta_{\mathbb{B}})^{-1/2}F_d^{-1}].$$

The right-hand side of (35) equals $D_z\widetilde{\Lambda}(\varphi - M)$, by (32). We have, hence:

$$(36)\qquad \Gamma_{A_6}(\varphi\,,\pm 1) = e^{-i\varphi\theta}[D_x(1+(\varphi - D_\theta)^2 - \Delta_{\mathbb{B}})^{-1/2}]e^{i\varphi\theta}.$$

Equations (29), (31), (32), (33), (34) and (36) prove:

PROPOSITION 3.4. *The map* Γ *defined in* (24) *is given on the generators of* $\mathscr{C}_{\mathscr{P}}$ (*with* $m = (e^{2\pi i\varphi}, \pm 1) \in \mathbf{M}_{SL}$ *and* $\Gamma_A(\varphi\,,\pm 1) \in \mathscr{L}_{\mathbb{S}^1 \times \mathbb{B}}$, $\mathbb{S}^1 = \{e^{i\theta}\,;\,\theta \in \mathbb{R}\}$) *by*:

$$\Gamma_a(\varphi\,,\pm 1) = a_\pm(\theta\,,x), \quad \textit{for } a \textit{ as in } (28)$$

$$\Gamma_\Lambda(\varphi\,,\pm 1) = e^{-i\varphi\theta}(1 + (D_\theta - \varphi)^2 - \Delta_{\mathbb{B}})^{-1/2}e^{i\varphi\theta}$$

$$\Gamma_{-i\frac{\partial}{\partial t}\Lambda}(\varphi\,,\pm 1) = e^{-i\varphi\theta}(D_\theta - \varphi)(1 + (D_\theta - \varphi)^2 - \Delta_{\mathbb{B}})^{-1/2}e^{i\varphi\theta}$$

$$\Gamma_{D_x\Lambda}(\varphi\,,\pm 1) = e^{-i\varphi\theta}D_x(1 + (D_\theta - \varphi)^2 - \Delta_{\mathbb{B}})^{-1/2}e^{i\varphi\theta}.$$

REMARK 3.5. Because of the way Γ was defined, it is obvious that condition (ii) of Theorem 3.2 can be replaced by

(ii′) The matrix $\Gamma_A^N(m) := ((\Gamma_{A_{jk}}(m)))_{1\leq j,k\leq N}$ is invertible for all $m \in \mathbf{M}_{SL}$.

Our next and final objective is to find necessary and sufficient conditions for a differential operator with semi-periodic coefficients on Ω to be Fredholm. Most of the ideas and proofs in what follows are borrowed from [2], §§VII.3 and IX.3, where the more general problem of finding differential expressions within reach of a Comparison Algebra is addressed.

PROPOSITION 3.6. *Let L be an Mth order differential expression on $\mathbb{B}$, with smooth coefficients. The operator $L\Lambda^M$, defined initially on the dense subspace $\Lambda^{-M}(\mathbf{C}_0^\infty(\Omega))$, can be extended to a bounded operator A in $\mathscr{L}(\mathscr{H})$. Moreover, we have that $A \in \mathscr{C}_{\mathscr{P}}$, σ_A coincides with the principal symbol of L on $\mathbf{W}_P$ (points of $\mathbf{M}_P$ over $|t| < \infty$) and*

$$\Gamma_A(\varphi, \pm 1) = e^{-i\varphi\theta} L(1 + (D_\theta - \varphi)^2 - \Delta_{\mathbb{B}})^{-1/2} e^{i\varphi\theta}, \qquad (e^{2\pi i\varphi}, \pm 1) \in \mathbf{M}_{SL}.$$

Proof. It is easy to see that any Mth order differential expression on a compact manifold equals a sum of products of at most M first-order differential expressions. (See, for example, the proof of Proposition VI-3-1 of [2].) It is therefore enough to consider L of the form

$$L = D_1 D_2 \cdots D_M,$$

where D_j, $j = 1, \ldots, M$, are first order expressions. For $M = 1$, the proposition is true by Theorem 2.2 and Proposition 3.4.

Using that $\Lambda^2 = H^{-1}$, $H = 1 - \Delta_{\mathbb{R}} - \Delta_{\mathbb{B}}$, it is easy to see that, for $u \in \Lambda^{-2}(\mathbf{C}_0^\infty(\Omega))$, and D_1 and D_2 first order expressions, we have:

$$D_1 D_2 \Lambda^2 u = D_1 \Lambda^2 D_2 u + D_1 \Lambda^2 [H, D_2] \Lambda^2 u. \tag{37}$$

The commutator $[H, D_2]$ is a second order expression on $\mathbb{B}$ and can therefore be expressed as a sum of products of at most two first order differential expressions:

$$[H, D_2] = \sum_{j=1}^{p} F_j G_j.$$

This shows that, on the dense subspace $\Lambda^{-2}(C_0^\infty(\Omega))$, $D_1D_2\Lambda^2$ equals the operator

$$(D_1\Lambda)(D_2^*\Lambda)^* + (D_1\Lambda)\sum_{j=1}^{p}(F_j^*\Lambda)^*(G_j\Lambda)\Lambda \in \mathscr{C}_{\mathscr{P}},$$

where D^* denotes the formal adjoint of a differential expression D.

Since $\sigma_\Lambda = 0$, we get:

$$\sigma_{D_1D_2\Lambda^2} = \sigma_{D_1\Lambda}\sigma_{D_2^*\Lambda},$$

which, restricted to $\mathbf{W}_P$, coincides with the principal symbol of D_1, D_2, by Theorem 2.2. It also follows that:

$$\Gamma_{D_1D_2\Lambda^2} = \Gamma_{D_1\Lambda}\Gamma^*_{D_2^*\Lambda} + \Gamma_{D_1\Lambda}\sum_{j=1}^{p}\Gamma^*_{F_j^*\Lambda}\Gamma_{G_j\Lambda}\Gamma_\Lambda.$$

By Proposition 3.4, we get:

$$\begin{aligned} e^{i\varphi\theta}&\Gamma_{D_1D_2\Lambda^2}(\varphi, \pm 1)e^{-i\varphi\theta} \\ &= (D_1\Lambda_\varphi)(D_2^*\Lambda_\varphi)^* + D_1\Lambda_\varphi\sum_{j=1}^{p}(F_j^*\Lambda_\varphi)^*(G_j\Lambda_\varphi)\Lambda_\varphi \\ &= D_1\Lambda_\varphi^2 D_2 + D_1\Lambda_\varphi^2\sum_{j=1}^{p}F_jG_j\Lambda_\varphi^2, \end{aligned}$$

where $\Lambda_\varphi = H_\varphi^{-1/2}$, $H_\varphi = 1 + (D_\theta - \varphi)^2 - \Delta_{\mathbb{B}}$. Since $[H, D_2]$ and $[H_\varphi, D_2]$ are equal (as expressions on $\mathbb{B}$), we get:

$$e^{i\varphi\theta}\Gamma_{D_1D_2\Lambda^2}(\varphi, \pm 1)e^{-i\varphi\theta} = D_1\Lambda_\varphi^2 D_2 + D_1\Lambda_\varphi^2[H_\varphi, D_2]\Lambda_\varphi^2 = D_1D_2\Lambda_\varphi^2$$

proving the proposition for $L = D_1D_2$.

Suppose now that the proposition is true for sums of products of at most M first order differential expressions and let $L = D_1D_2\cdots D_{M+1}$ be a product of first order expressions. Define: $F = D_1D_2$ and $G = D_3\cdots D_{M+1}$. Using the formula

$$L\Lambda^{M+1}u = F\Lambda^2G\Lambda^{M-1}u + F\Lambda^2[H, G]\Lambda^{M+1}u, \qquad u \in \Lambda^{-M-1}(\mathbf{C}_0^\infty(\Omega)),$$

the proposition follows for this L, by the same argument as above. □

Let $\{U_\beta\}$ be a finite atlas on $\mathbb{B}$ and $\{\varphi_\beta\}$ a subordinate partition of unity, i.e. support $\varphi_\beta \subset U_\beta$. Let L be a differential operator on Ω, acting on $\mathbb{C}^N$-valued functions, locally given on U_β by

$$(38) \qquad L = \sum_{j=0}^{\widetilde{M}} \sum_{|\alpha| \le M_j} A_{\beta,j,\alpha}(t, x) \left(\frac{1}{i}\frac{\partial}{\partial x}\right)^\alpha \left(\frac{1}{i}\frac{\partial}{\partial t}\right)^j,$$

where

$$\left(\frac{1}{i}\frac{\partial}{\partial x}\right)^\alpha := \left(-i\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(-i\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \quad \text{for } \alpha \in \mathbb{N}^n$$

and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We will say that L has *semi-periodic coefficients* if the matrices

$$\widetilde{A}_{\beta,j,\alpha}(t, x) := \varphi_\beta(x) A_{\beta,j,\alpha}(t, x),$$

regarded as functions on Ω, have as entries functions of the type (28). It is easy to see that this definition is independent of the choice of atlas on $\mathbb{B}$. We want to decide when

$$L\colon H^M(\Omega, \mathbb{C}^N) \to L^2(\Omega, \mathbb{C}^N)$$

is a Fredholm operator, assuming that L has semi-periodic coefficients. Here M denotes the order of L, $M = \max\{M_j + j, j = 1, \ldots, \widetilde{M}\}$.

We also denote by Λ the operator $\Lambda \otimes I_N$ on $\mathscr{L}(L^2(\Omega, \mathbb{C}^N))$, where I_N denotes the $N \times N$ identity matrix. Since Λ commutes with $\frac{\partial}{\partial t}$ and $L = \sum L_\beta$, for $L_\beta := \varphi_\beta L$, we get:

$$L\Lambda^M = \sum_{\beta,j,\alpha} (t, x) \left(\frac{1}{i}\frac{\partial}{\partial x}\right)^\alpha \Lambda^{|\alpha|} \left(\frac{1}{i}\frac{\partial}{\partial t}\right)^j \Lambda^j \Lambda^{M-|\alpha|-j}.$$

After multiplying $(\frac{1}{i}\frac{\partial}{\partial x})^\alpha$ above by $\chi_{\beta,j,\alpha} \in \mathbf{C}_0^\infty(U_\beta)$, $\chi_{\beta,j,\alpha}(x) = 1$ for x in the support of $\widetilde{A}_{\beta,j,\alpha}$, we still get the same operator and $\chi_{\beta,j,\alpha}(x)(\frac{1}{i}\frac{\partial}{\partial x})^\alpha$ is now a differential expression defined on $\mathbb{B}$. We can therefore apply Proposition 3.6 and conclude that $L\Lambda^M \in \mathscr{C}_{\mathscr{P}}^N$. Using, moreover, that $\sigma_{\Lambda^{M-|\alpha|-j}} = 0$ for $|\alpha| + j < M$, we get:

$$\sigma_{L\Lambda^M}(t, x; \tau, \xi) = \sum_\beta \sum_{|\alpha|+j=M} \widetilde{A}_{\beta,j,\alpha}(t, x)\xi^\alpha \tau^j, \qquad |t| < \infty.$$

The right-hand side of the previous equation coincides with the principal symbol of L restricted to the co-sphere bundle of Ω. Invertibility of the σ-symbol is therefore equivalent to uniform ellipticity of L, by Remark 2.3.

The operator-valued symbol $\Gamma_{L\Lambda^M}$ is also given by Proposition 3.6 (and Proposition 3.4):

$$e^{-i\varphi\theta}\Gamma_{L\Lambda^M}(\varphi, \pm 1)e^{i\varphi\theta} = \sum_{\beta, j, \alpha} \widetilde{A}^{\pm}_{\beta, j, \alpha}(\theta, x)\left(\frac{1}{i}\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{1}{i}\frac{\partial}{\partial\theta} - \varphi\right)^{j}\Lambda^M_{\varphi},$$

where we have used that Λ_φ and $\frac{\partial}{\partial\theta}$ commute. We have denoted by $\widetilde{A}^{\pm}_{\beta, j, \alpha}$ the 2π-periodic continuous functions such that

$$\widetilde{A}_{\beta, j, \alpha}(t, x) - \chi_+(t)\widetilde{A}^{+}_{\beta, j, \alpha}(t, x) - \chi_-(t)\widetilde{A}^{-}_{\beta, j, \alpha}(t, x) \in \mathbf{CO}(\Omega).$$

(See (28).)

Let $L^{\pm}_{\beta}(\varphi)$ denote the differential expressions on $\mathbb{S}^1 \times \mathbb{B}$

$$L^{\pm}_{\beta}(\varphi) := \sum_{j=0}^{\widetilde{M}} \sum_{|\alpha| \le M_j} +\widetilde{A}^{\pm}_{\beta, j, \alpha}(\theta, x)\left(\frac{1}{i}\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{1}{i}\frac{\partial}{\partial\theta} - \varphi\right)^{j},$$

and define the operator

$$(39) \qquad L^{\pm}(\varphi) := \sum_{\beta} L^{\pm}_{\beta}(\varphi) \colon H^M(\mathbb{S}^1 \times \mathbb{B}, \mathbb{C}^N) \to L^2(\mathbb{S}^1 \times \mathbb{B}, \mathbb{C}^N).$$

Since Λ_φ is an isomorphism from

$$L^2(\mathbb{S}^1 \times \mathbb{B}, \mathbb{C}^N) \quad \text{onto } H^M(\mathbb{S}^1 \times \mathbb{B}, \mathbb{C}^N),$$

the above considerations, together with Theorem 3.2 and Remark 3.5 prove the following theorem.

THEOREM 3.7. *Let L denote an Mth order differential operator on Ω of the form (38), with continuous semi-periodic coefficients, and let $L^{\pm}(\varphi)$ denote the differential operators on $S^1 \times \mathbb{B}$ defined in (39). Then*

$$L \colon H^M(\Omega, \mathbb{C}^N) \to L^2(\Omega, \mathbb{C}^N)$$

is Fredholm if and only if L is uniformly elliptic and $L^{\pm}(\varphi)$ are invertible for all $\varphi \in [0, 1]$.

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