PACIFIC JOURNAL OF MATHEMATICS
Vol. 146, No. 2, 1990

# SEQUENCE TRANSFORMATIONS THAT GUARANTEE A GIVEN RATE OF CONVERGENCE

G. H. FRICKE AND J. A. FRIDY

**Let $t$ be a positive number sequence and define the sequence space $\Omega(t) := \{x : x_k = O(t_k)\}$. Characterizations are given for matrices that map the spaces $l^1$, $l^\infty$, $c$, or $c_0$ into $\Omega(t)$, thus ensuring that the transformed sequence converges at least as fast as $t$. These results yield information about matrices that map $l^1$, $l^\infty$, $c$, or $c_0$ into $G := \bigcup_{r\in(0,1)} \Omega(r^n)$, the set of geometrically dominated sequences.**

**1. Introduction**. For each $r$ in the interval $(0, 1)$ let

$$G(r) = \{\text{complex sequences } x : x_k = O(r^k)\}$$

and define the set of *geometrically dominated sequences* as

$$G = \bigcup_{r\in(0,1)} G(r).$$

The *analytic sequences* are defined by

$$\mathscr{A} = \left\{\text{complex sequences } x : \limsup_n |x_n|^{1/n} < \infty\right\}.$$

Obviously $G \subseteq \mathscr{A}$. In **[2, 6, 9, 10]** the various authors studied matrix transformations from $\mathscr{A}$ or $G$ into $l^1$, $c$, or $l^\infty$, but the question of mapping *from* $l^1$, $c$, or $l^\infty$ *into* $\mathscr{A}$ or $G$ was not considered. We shall use the customary notation for a matrix transformation: if $A$ is an infinite matrix with complex entries and $x$ is a complex number sequence, then $A$ transforms $x$ into the sequence $Ax$ whose $n$th term is given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k.$$

The present work began as a study of $l^1 - G$ and $c - G$ matrices, but it was found that such results are merely special cases of a more general theory. To set the stage for the general theory we replace the geometric sequence $\{r^k\}$ with a nonnegative sequence $t$ and define

$$\Omega(t) = \{x : x_k = O(t_k)\}.$$

Throughout the paper $T$ will denote a sequence $\{t^{(m)}\}_{n=1}^{\infty}$ of nonnegative number sequences such that $t^{(m)} \in \Omega(t^{(m+1)})$ for each $m$; this ensures that $\Omega(t^{(m)}) \subseteq \Omega(t^{(m+1)})$, and we define

$$D(T) = \bigcup_{m=0}^{\infty} \Omega(t^{(m)}).$$

EXAMPLE 1. If for each $m$, $t^{(m)}$ is the geometric sequence $\{r_m^k\}$, where $0 < r_m < 1$ and $r_m \uparrow 1$, then $D(T)$ is $G$.

For a given summability matrix $A$, the sequences $\mu$ and $\sigma$ are defined by

$$\mu_n = \sup_k |a_{nk}|$$

and

$$\sigma_n = \sum_{k=0}^{\infty} |a_{nk}|.$$

The main results of this paper state that for $A$ to map $l^1$ [respectively, $c$] into a "big-oh space" such as $\Omega(t)$, it is necessary and sufficient that $\mu$ be in $\Omega(t)$ [respectively, $\sigma \in \Omega(t)$]. Moreover, in order for $A$ to map $l^1$ [respectively, $c$] into $D(T)$ it is necessary that $A$ map into a particular $\Omega(t^{(m)})$ for some $t^{(m)}$ in $T$. Thus the characterizations of $A: l^1 \to G$ and $A: c \to G$ are obtained as special cases of the general theory.

The final section of the paper contains a brief discussion of some classical matrix methods as mappings into $\Omega(t)$ spaces.

**2. The main results.** Using the notation as given above, we proceed to our first general result.

THEOREM 1. *If $A$ is a summability matrix and $T$, $D(T)$, and $\mu$ are as given above, then the following are equivalent*:

(i) $\mu \in D(T)$;
(ii) *there exists a $t^{(m)}$ in $T$ such that $\mu \in \Omega(t^{(m)})$*;
(iii) *there exists a $t^{(m)}$ in $T$ such that $A: l^1 \to \Omega(t^{(m)})$*;
(iv) $A: l^1 \to D(T)$.

*Proof*. Implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious. To prove that (iv) $\Rightarrow$ (i), first note that (iv) implies that

$$\mu_n < \infty \quad \text{for each } n \tag{1}$$

and

$$\text{each column of } A \text{ is in } D(T). \tag{2}$$

It follows from (2) that the sum of any finite number of columns of $A$ is in $D(T)$, so there exists a sequence $\{t^{(m^*)}\}$ in $T$ and a positive number sequence $\{B_{m^*}\}$ that increases to $\infty$ such that

$$(3) \qquad \sum_{i \le m^*} |a_{ni}| \le B_{m^*} t_n^{(m^*)} \quad \text{for each } n.$$

To simplify the notation we will write $m$ in place of $m^*$ since the nestedness of the sets $\{\Omega(t^{(m)})\}$ ensures that nothing is lost by considering the subsequence $\{t^{(m^*)}\}$ as the entire sequence $\{t^{(m)}\}$.

Now assume (1), (2), and (3) hold, but $\mu \notin D(T)$; we wish to construct an $x$ in $l^1$ such that $Ax \notin D(T)$. From each row of $A$ select an entry satisfying

$$|a_{n,k^*(m)}| \ge \tfrac{1}{2}\mu_n.$$

Since $\mu \notin D(T)$, (2) allows us to choose a subsequence of these entries, say $\{a_{n'(i),k'(i)}\}_{k=1}^{\infty}$ such that $n'(i)$ and $k'(i)$ increase with $i$; also, $\mu \notin D(T)$ allows us to choose the $n'(i)$ so that

$$(4) \qquad \sup_i \{\mu_{n'(i)}/t_{n'(i)}^{(m)}\} = \infty \quad \text{for all } m.$$

For each $i$ we also have

$$(5) \qquad |a_{n'(i),k'(i)}| \ge \tfrac{1}{2}\mu_{n'(i)} = \tfrac{1}{2}\sup_k\{|a_{n'(i),k}|\}.$$

Finally, choose a further subsequence of these entries so that for each $j$

$$(6) \qquad |a_{n(j),k(j)}| > 12 B_{k(j-1)} t_{n(j)}^{(k(j-1))} 4^j.$$

Now define $x$ in $l^1$ by

$$x_k = \begin{cases} 4^{-j}, & \text{if } k = k(j) \text{ and } a_{n(j),k(j)} \ne 0, \\ 0, & \text{otherwise.} \end{cases}$$

For each $j$ this yields

$$\begin{aligned} |(Ax)_{n(j)}| &\ge -\sum_{i<j} |a_{n(j),k(i)}| 4^{-i} + |a_{n(j),k(j)}| 4^{-j} + \sum_{i>j} |a_{n(j),k(i)}| 4^{-i} \\ &\ge -B_{k(j-1)} t_{n(j)}^{(k(j-1))} + \tfrac{1}{2}\mu_{n(j)} 4^{-j} - \mu_{n(j)} \sum_{i>j} 4^{-j} \\ &\ge -B_{k(j-1)} t_{n(j)}^{(k(j-1))} + \tfrac{1}{6}\mu_{n(j)} 4^{-j} \\ &> -B_{k(j-1)} t_{n(j)}^{(k(j-1))} + \tfrac{1}{6}[12 B_{k(j-1)} t_{n(j)}^{(k(j-1))} 4^j] 4^{-j} \\ &= B_{k(j-1)} t_{n(j)}^{(k(j-1))}. \end{aligned}$$

Hence, $Ax \notin \Omega(t^{(k(j-1))})$, and it follows that $Ax \notin D(T)$.

By taking $T$ as in Example 1 we get the following corollary as an immediate consequence of Theorem 1.

COROLLARY 1A. *If $A$ is a summability matrix and $\mu_n = \sup_k |a_{nk}|$, then $A: l^1 \to G$ if and only if $\mu \in G$.*

Another consequence of Theorem 1 can be obtained by replacing $T$ with a single sequence, i.e., $t^{(m)} = t$ for every $m$. Thus $D(T) = \Omega(t)$, and we get the following result.

COROLLARY 1B. *If $A$ is a summability matrix and $t$ is a nonnegative number sequence, then $A = l^1 \to \Omega(t)$ if and only if $\mu \in \Omega(t)$.*

This corollary gives rise to the title of the paper, because it characterizes those matrices $A$ that will transform every absolutely convergent series $\sum x_k$ into a series $\sum(Ax)_n$ that converges at least as fast as a given series $\sum t_n$.

Now we turn our attention to matrix mappings from $l^\infty$ into $D(T)$, which, as we shall see, subsumes the cases of mappings from $c$ or $c_0$ into $D(T)$.

THEOREM 2. *If $A$ is a summability matrix and $T$, $D(T)$, and $\sigma$ are as given above, then the following are equivalent:*

(i) $\sigma \in D(T)$;
(ii) *there exists a $t^{(m)}$ in $T$ such that $\sigma \in \Omega(t^{(m)})$;*
(iii) *there exists a $t^{(m)}$ in $T$ such that $A: l^\infty \to \Omega(t^{(m)})$;*
(iv) *there exists a $t^{(m)}$ in $T$ such that $A: c \to \Omega(t^{(m)})$;*
(v) *there exists a $t^{(m)}$ in $T$ such that $A: c_0 \to \Omega(t^{(m)})$;*
(vi) $A: c_0 \to D(T)$.

*Proof.* As in the proof of Theorem 1, most of the implications are obvious and we prove here only that (vi) $\Rightarrow$ (i). Note that (vi) implies that each row of $A$ is in $l^1$ (i.e., $\sigma_n < \infty$) and each column of $A$ is in some $\Omega(t^{(m)})$. Therefore finite sums of the column sequences are in $D(T)$, and we can choose a sequence of $t^{(j)}$'s with constants $B_j > 0$ such that for each $j$

$$\sum_{k \leq j} |a_{nk}| \leq B_j t_n^{(j)}. \tag{7}$$

Suppose that $\sigma \notin D(T)$. Choose increasing sequences $\{k(j)\}$ and $\{n(j)\}$ of column and row indices as follows: $k(1)$ and $n(0)$ are chosen arbitrarily, and after $k(j)$ and $n(j-1)$ are selected choose $n(j) > n(j-1)$ so that

$$\sigma_{n(j)} \geq [j(2B_{k(j)} + j) + 2B_{k(j)}]t_{n(j)}^{(k(j))}. \tag{8}$$

Next choose $k(j+1) > k(j)$ so that

$$\sum_{k>k(j+1)} |a_{n(j),k}| < B_{k(j)} t_{n(j)}^{(k(j))}. \tag{9}$$

Thus (7), (8), and (9) together yield

$$\sum_{k=1+k(j)}^{k(j+1)} |a_{n(j),k}| \geq j(2B_{k(j)} + j)t_{n(j)}^{(k(j))}. \tag{10}$$

Now define the sequence $x$ by

$$x_k = \frac{\overline{a}_{n(j),k}}{|a_{n(j),k}|j}, \quad \text{if } k(j) < k \leq k(j+1) \text{ and } a_{n(j),k} \neq 0,$$

and $x_k = 0$ otherwise. It is clear that $x \in c_0$, but for each $j$ we have

$$\begin{aligned} |(Ax)_{n(j)}| &\geq -\sum_{k\leq k(j)} |a_{n(j),k}| + \left| \sum_{k=1+k(j)}^{k(j+1)} a_{n(j),k} x_k \right| - \sum_{k>k(j+1)} |a_{n(j),k}| \\ &\geq -2B_{k(j)} t_{n(j)}^{(k(j))} + \sum_{k=1+k(j)}^{k(j+1)} |a_{n(j),k}| \frac{1}{j} \geq j t_{n(j)}^{(k(j))}. \end{aligned}$$

Hence, $Ax \notin \Omega(t^{(k(j))})$ for $j = 1, 2, \ldots$, so $Ax \notin D(T)$. Thus we have shown that if (i) is false then (vi) does not hold, which completes the proof.

As with Theorem 1, we can state two immediate corollaries to Theorem 2. The first is the case in which $D(T) = G$ as in Example 1, and the second is the case in which $T$ consists of a single sequence.

Corollary 2A. *If $A$ is a summability matrix and $\sigma_n = \sum_{k=1}^{n} |a_{nk}|$, then $A$ maps $l^\infty$, $c$, and $c_0$ into $G$ if and only if $\sigma \in G$.*

Corollary 2B. *If $A$ is a summability matrix and $t$ is a nonnegative number sequence, then $A$ maps $l^\infty$, $c$, and $c_0$ into $\Omega(t)$ if and only if $\sigma \in \Omega(t)$.*

Once again, it is the latter corollary that is described in the title of the paper: for, if one wishes to have a matrix $A$ that transforms every null sequence into a sequence that converges at least as rapidly as some $t_n \downarrow 0$, then $A$ must satisfy $\sigma \in \Omega(t)$. Similarly, if $t$ is a nonzero constant sequence, then $\Omega(t) = l^\infty$, and in this case Corollary 2B reduces to the well-known result that $A$ preserves boundedness if and only if $\sigma$ is bounded.

Another observation should be made about obtaining a "given rate of convergence" by mapping $c_0$ into $\Omega(t)$. Recent work **[1, 7]** has shown that regular matrices cannot *accelerate* the rate of convergence of *every* null sequence. Therefore, we emphasize that having $A$ map $c_0$ into $\Omega(t)$ does not say that every sequence in $c_0$ is accelerated, even if $t_n \downarrow 0$ very rapidly; some sequences that are already in $\Omega(t)$ may map into other members of $\Omega(t)$ that converge at the same rate or slower.

**3. Examples involving classical matrices.** If $A$ is regular then $\sigma \in l^\infty$ but $\sigma \notin c_0$. Therefore Theorem 2 and its corollaries do not yield much information about regular matrices, and this includes most of the classical methods. We can, however, draw some conclusions about mapping $l^1$ into $\Omega(t)$ by finding $\mu$ and applying Corollary 1B.

EXAMPLE 2. The Cesàro matrix of order $j$ is given by

$$C_j[n, k] = \frac{\binom{n+j-1-k}{j-1}}{\binom{n+j}{j}}$$

(see, e.g., **[8**, p. 46], so it is plain that

$$\mu_n = C_j[n, 0] = \frac{j}{n+j},$$

whence $C_j\colon l^1 \to \Omega(1/n)$.

EXAMPLE 3. For the Euler-Knopp mean of order $r$ it is known **[3**, Theorem 9] that $\mu_n \sim [2\pi r(1-r)n]^{-1/2}$; so by Corollary 1B, $E_r\colon l^1 \to \Omega(n^{-1/2})$.

EXAMPLE 4. For the Taylor matrix $T_r$ **[3**, Theorem 11] we have $\mu_n \sim (1-r)[2\pi r n]^{-1/2}$, and therefore

$$T_r\colon l^1 \to \Omega(n^{-1/2}).$$

EXAMPLE 5. The Borel matrix **[8**, p. 53] is given by $B[n, k] = e^{-n} n^k / k!$, and it is not hard to show that

$$\mu_n = B[n, n] = \left(\frac{n}{e}\right)^n \frac{1}{n!}.$$

From Stirling's Formula it follows that

$$\mu_n \sim \frac{1}{\sqrt{r\pi n}},$$

and so $B$, like $E_r$ and $T_r$, maps $l^1$ into $\Omega(n^{-1/2})$.

EXAMPLE 6. The Hausdorff means [5, Chapter 5] can be defined by

$$H_\varphi[n, k] = \int_0^1 E_r[n, k]\, d\varphi,$$

where $E_r$ is the Euler-Knopp mean and $\int_0^1 |d\varphi| < \infty$. Thus from Example 3 we infer that

$$|H_\varphi[n, k]| \leq Kn^{-1/2} \int_{0+}^{1-} [r(1-r)]^{-1/2} |d\varphi(r)|,$$

where $K$ is a constant. Therefore we conclude that if

$$\int_{0+}^{1-} [r(1-r)]^{-1/2} |d\varphi| < \infty,$$

then $\mu_n = O(n^{-1/2})$ and $H_\varphi\colon l^1 \to \Omega(n^{-1/2})$.

EXAMPLE 7. The Nörlund mean generated by the nonnegative sequence $p$ with $p_0 > 0$ is given by

$$N_p[n, k] = \begin{cases} p_{n-k}/P_n, & \text{if } k \leq n, \\ 0, & \text{if } k > n. \end{cases}$$

In general,

$$\mu_n = \frac{1}{P_n} \max\{p_k\}_{k=0}^n,$$

and this is somewhat awkward in Corollary 1B. In case $p$ is monotonic the $\mu$ formula becomes quite simple, and we can state the following easy conclusions:

(i) if $p$ is nonincreasing, then $\mu_n = p_0/P_n$ and $N_p\colon l^1 \to \Omega(1/P_n)$;

(ii) if $p$ is nondecreasing then $\mu_n = p_n/P_n$ and $N_p = l^1 \to \Omega(p_n/P_n)$.

It should be noted that these results do not always give much information. For example, in Case (i) if $p \in l^1$ then $1/P \notin c_0$ and (i) asserts only that $N_p$ maps $l^1$ into $l^\infty$. (Every Nörlund matrix maps $l^\infty$ into $l^\infty$ because $\sigma_n := 1$.) Similarly, if $p_n = R^n$, where $R > 1$, then $p_n/P_n \sim (R-1)/R$, and (ii) tells us only that $N_p$ maps $l^1$ into $l^\infty$.

EXAMPLE 8. An Abel matrix [4] is given by

$$A_t[n, k] = t_n(1 - t_n)^k,$$

where $0 < t_n < 1$ and $\lim_n t_n = 0$. For $A_t$ it is obvious that $\mu_n = t_n$, and therefore by Corollary 1B, $A_t\colon l^1 \to \Omega(t)$.

The authors would like to thank Professor James DeFranza for some very helpful conversations.

## REFERENCES


[1] J. P. Delahaye and B. Germain-Bonne, *The set of logarithmically convergent sequences cannot be accelerated*, SIAM J. Numer. Anal., **19** (1982), 840–844.

[2] G. H. Fricke and J. A. Fridy, *Matrix summability of geometrically dominated series*, Canad. J. Math., **39** (1987), 568–582.

[3] J. A. Fridy, *Matrix summability and a generalized Gibbs phenomenon*, Aequat. Math., **14** (1976), 405–412.

[4] ——, *Abel transformations into $l^1$*, Canad. Math. Bull., **25** (1982), 421–427.

[5] G. H. Hardy, *Divergent Series*, Clarendon Press, Oxford, 1949.

[6] R. T. Jacob, Jr., *Matrix transformations involving simple sequence spaces*, Pacific J. Math., **70** (1977), 179–187.

[7] T. A. Keagy and W. F. Ford, *Acceleration by subsequence transformations*, Pacific J. Math., **132** (1988), 357–362.

[8] R. E. Powell and S. M. Shah, *Summability Theory and Applications*, Revised Ed., Prentice Hall of India, New Delhi, 1988.

[9] P. C. Tonne, *Matrix transformations on the power-series convergent on the unit disc*, J. London Math. Soc. (2), **4** (1972), 667–670.

[10] ——, *Matrix representations for linear transformations on series analytic in the unit disc*, Pacific J. Math., **44** (1973), 385–392.


Received March 20, 1989.

WRIGHT STATE UNIVERSITY
DAYTON, OH 45435

AND

KENT STATE UNIVERSITY
KENT, OH 44242