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A DUALITY THEOREM FOR EXTENSIONS OF INDUCED HIGHEST WEIGHT MODULES

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We begin by recalling that homogeneous differential operators between smooth vector bundles over a real flag manifold correspond to the intertwining maps between algebraically induced highest weight modules. Within this framework we prove a duality theorem for extensions of induced highest weight modules. In particular, this leads to a duality theory for the nilpotent cohomology of any generalized Verma module.

1. Introduction. In this short note we recall (and prove) a folklore result which appeared in Boe's 1982 Yale thesis (without proof) and was attributed to G. Zuckerman. We apply the result to representations of real groups and to the theory of highest weight modules. In particular we obtain a duality theorem for extensions between parabolically induced highest weight modules, cf. Theorem 1.1 below. Nontrivial applications are discussed in §§4 and 5.

Fix a pair $(\mathfrak{g}, \mathfrak{p})$, $\mathfrak{g}$ a complex semisimple Lie algebra and $\mathfrak{p}$ a parabolic subalgebra. There exists a connected real semisimple matrix group G with a closed parabolic subgroup P so that $\mathfrak{g}$ and $\mathfrak{p}$ are the complexified Lie algebras of G and P respectively. Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ be a Levi decomposition of $\mathfrak{p}$ and $\mathfrak{h} \subseteq \mathfrak{l}$ a Cartan subalgebra of both $\mathfrak{l}$, the reductive part of $\mathfrak{p}$, and of $\mathfrak{g}$.

Recall the category $\mathscr{O}_\mathfrak{p}$ of finitely generated $\mathfrak{g}$-modules which are $\mathfrak{l}$-semisimple and $\mathfrak{p}$-locally finite. Denote by $\mathscr{A}_\mathfrak{p}$ the category of finite dimensional $\mathfrak{p}$-modules which are $\mathfrak{l}$-semisimple. Define a functor $U_\mathfrak{p} : \mathscr{A}_\mathfrak{p} \to \mathscr{O}_\mathfrak{p}$ by

$$U_\mathfrak{p}(E) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E.$$

Here $U(\mathfrak{a})$ denotes the enveloping algebra of a Lie algebra $\mathfrak{a}$. For any finite dimensional $\mathfrak{p}$-module (or P-module) E, let E^* denote the contragradient module. Our main result is then

THEOREM 1.1. *For any two $\mathfrak{p}$-modules E and F in $\mathscr{A}_\mathfrak{p}$ and any $k \geq 0$,*

$$\dim_\mathbb{C} \operatorname{Ext}^k_{\mathscr{O}_\mathfrak{p}}(U_\mathfrak{p}(E), U_\mathfrak{p}(F)) = \dim_\mathbb{C} \operatorname{Ext}^k_{\mathscr{O}_\mathfrak{p}}(U_\mathfrak{p}(F^*), U_\mathfrak{p}(E^*)).$$

Using standard homological algebra we can restate a special case of Theorem 1.1 as a duality for the nilpotent cohomology of certain standard induced modules, the generalized Verma modules.

COROLLARY 1.2. *Let E and F be irreducible finite dimensional $\mathfrak{l}$-modules and extend these to be $\mathfrak{p}$-modules by letting $\mathfrak{n}$ act trivially. Then for all $k \geq 0$,*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{l}}(E, H^k(\mathfrak{n}, U_{\mathfrak{p}}(F))) = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{l}}(F^*, H^k(\mathfrak{n}, U_{\mathfrak{p}}(E^*))).$$

In category $\mathscr{O}_{\mathfrak{p}}$ there is no natural contravariant duality functor which carries induced modules into induced modules. Thus, in order to prove Theorem 1.1 we transfer the problem into the smooth vector bundle category where there is a natural duality, the adjoint. The existence of this duality is directly related to the existence of Haar measure which has no counterpart in category $\mathscr{O}_{\mathfrak{p}}$.

2. Homogeneous differential operators. In this section we establish the folklore result mentioned above, relating homogeneous differential operators between vector bundles to $\mathfrak{g}$-homomorphisms between induced modules. This lemma seems to be known to experts; see for example the introduction to Lepowsky's paper [L] or §2 of Eastwood-Rice [**ER**]. A special case of the lemma appears in a paper of Jakobsen, [**J**]. (Jakobsen and Eastwood-Rice work in the holomorphic category. In contrast, we will work in the smooth category so as to allow applications to real noncomplex Lie groups.) However, we know of no general reference with proof in the literature. The proof below is a synthesis of conversations with B. Boe, H. Schlicktkrull and G. Zuckerman. Because we have in mind future applications to the theory of Harish-Chandra modules for real reductive Lie groups, we work in a slightly more general setting than necessary for the proof of Theorem 1.1.

Recall the Iwasawa decomposition $G = KA_mN_m$, K a maximal compact subgroup of G and θ a corresponding Cartan involution. Let $P_m = M_mA_mN_m$ be a compatible minimal parabolic subgroup. We may assume $P_m \subseteq P$. Let P have Levi decomposition $P = LN$ and recall that P is not, in general, connected. Let P^0 denote the connected component of the identity in P. Fix a closed subgroup P' with $P^0 \subseteq P' \subseteq P$ and put $L' = L \cap P'$. Since G is linear, there is a finite $\mathbb{Z}_2$-group $S \subset K$ with $P' = SP^0$, cf. [**KZ**]. Also, since G is linear and K is compact, $G/P_m^0 \cong K/M_m^0$ is a compact connected real analytic manifold. Put $X = G/P'$. Since $P_m^0 \subseteq P'$, X is a compact manifold.

Let (σ, E) be a finite dimensional (but not necessarily irreducible) representation of P'. Then there is an associated smooth homogeneous vector bundle $\mathscr{E}$ over X with fiber E at the identity coset eP'. Similarly, $\mathscr{E}^*$ is the smooth homogeneous vector bundle over X associated to the contragradient module E^*. In general, let $\Gamma_X(\mathscr{A})$ denote the smooth global sections of the smooth vector bundle $\mathscr{A}$ over X.

Since $X = G/P' = K/(K \cap P')$ as homogeneous spaces, there is a unique K-invariant volume form $d\mu_X$ on X with total volume 1. We fix this volume form on X. Each $g \in G$ determines a diffeomorphism of X via left multiplication and we obtain a corresponding modular function $c(g, x)$ defined by

$$g^* d\mu_X = c(g, x)\, du_X, \qquad g \in G,\ x \in X. \tag{2.1}$$

Following **[W]**, we can now define a smooth G-action π_E on $\Gamma_X(\mathscr{E})$ by

$$[\pi_E(g)f](x) = |c(g^{-1}, x)|^{1/2} g f(g^{-1}x), \qquad g \in G,\ x \in X,\ f \in \Gamma_X(\mathscr{E}). \tag{2.2}$$

Suppose E and F are two finite dimensional P' modules with associated G-modules $\Gamma_X(\mathscr{E})$ and $\Gamma_X(\mathscr{F})$ respectively. We denote by $\mathrm{Hom}_G(\Gamma_X(\mathscr{E}), \Gamma_X(\mathscr{F}))$ the space of continuous maps $L\colon \Gamma_X(\mathscr{E}) \to \Gamma_X(\mathscr{F})$ which intertwine π_E and π_F. The space of *homogeneous differential operators* is then

$$\mathbb{D}(\Gamma_X(\mathscr{E}), \Gamma_X(\mathscr{F})) = \{D \in \mathrm{Hom}_G(\Gamma_X(\mathscr{E}), \Gamma_X(\mathscr{F})) \mid \mathrm{supp}\, D\phi \subseteq \mathrm{supp}\, \phi \text{ for all } \phi \in \Gamma_X(\mathscr{E})\}, \tag{2.3}$$

where $\mathrm{supp}\,\phi$ is the support of the section ϕ.

For a finite dimensional P'-module E, let E also denote the $\mathfrak{p}$-module obtained by differentiation. Observe that the induced $\mathfrak{g}$-module $U_\mathfrak{p}(E^*)$ carries a compatible action of the finite group S. We can now state the desired lemma.

LEMMA 2.4. *Let E and F be finite dimensional representations of P'. Then there is a vector space isomorphism*

$$\mathbb{D}(\Gamma_X(\mathscr{E}), \Gamma_X(\mathscr{F})) \cong \mathrm{Hom}_{\mathfrak{g}, S}(U_\mathfrak{p}(F^*), U_\mathfrak{p}(E^*)).$$

Proof. Let $\mathscr{D}'(\mathscr{E}) = \mathscr{D}'(X, \mathscr{E})$ denote the continuous dual of $\Gamma_X(\mathscr{E}^*)$ (where $\Gamma_X(\mathscr{E}^*)$ is given the usual Frechet topology, **[H]**), i.e. distributions, and denote by $\mathscr{D}'_0(\mathscr{E}) = \mathscr{D}'_0(X, \mathscr{E})$ the subspace of $\mathscr{D}'(\mathscr{E})$ consisting of distributions supported at the identity coset.

The exponential map realizes $\mathfrak{n}_0^- = \theta(\mathfrak{n}_0)$ as a coordinate patch around the identity coset eP' in X. Since $\mathfrak{n}_0^- \cong \mathbb{R}^n$, the vector valued version of Schwartz's theorem then gives a vector space isomorphism $T: U(\mathfrak{n}^-)\otimes E \to \mathscr{D}'_0(\mathscr{E})$. We may describe this isomorphism explicitly as follows. We identity $\Gamma_X(\mathscr{E}^*)$ as a G-representation with the space $\mathscr{V}(\mathscr{E}^*)$ of functions on G given by

$$\begin{aligned}\Gamma_X(\mathscr{E}^*) &\cong \mathscr{V}(\mathscr{E}^*)\\ &= \{f\colon G \to E^* | f \text{ is smooth}, f(gp) = \sigma_{E^*}(p)^{-1} f(g)\\ &\qquad\qquad \text{for } g \in G, p \in P'\},\end{aligned}$$

under the left regular representation on $\mathscr{V}(\mathscr{E}^*)$. Then the isomorphism of Schwartz's theorem is given by

$$T(Y \otimes v)(\phi) = \langle R_Y \phi(e), v\rangle, \qquad \phi \in \mathscr{V}, \ Y \in U(\mathfrak{n}^-), \ v \in E.$$

Here R_Y denotes right differentiation by Y and $\langle\ ,\ \rangle$ is the natural pairing on E and E^*. Since $U_\mathfrak{p}(E) \cong U(\mathfrak{n}^-)\otimes E$, as vector spaces, a moderately tedious calculation shows that T, with the formula given above, becomes a well-defined $\mathfrak{g}$-module isomorphism

$$(2.5) \qquad T\colon U_\mathfrak{p}(E) \xrightarrow{\cong} \mathscr{D}'_0(\mathscr{E}).$$

It now suffices to give an isomorphism

$$(2.6) \qquad \Psi\colon \mathbb{D}(\Gamma_X(\mathscr{E}), \Gamma_X(\mathscr{F})) \to \mathrm{Hom}_{\mathfrak{g},S}(\mathscr{D}'_0(\mathscr{F}^*), \mathscr{D}'_0(\mathscr{E}^*)).$$

For $D \in \mathbb{D}(\Gamma_X(\mathscr{E}), \Gamma_X(\mathscr{F}))$, define $\Psi(D) \in \mathrm{Hom}_\mathbb{C}(\mathscr{D}'(\mathscr{F}^*), \mathscr{D}'(\mathscr{E}^*))$ by $\Psi(D)(\mu)(\phi) = \mu(D\phi)$ for $\phi \in \Gamma_X(\mathscr{E})$ and $\mu \in \mathscr{D}'(\mathscr{F}^*)$. The support condition on D assures that $\Psi(D)$ preserves distributions supported at the identity coset. Also, for $Y \in \mathfrak{g}$, $\phi \in \Gamma_X(\mathscr{E})$ and $\mu \in \mathscr{D}'(\mathscr{F}^*)$,

$$\begin{aligned}\Psi(D)(Y\mu)(\phi) &= Y\mu(D\phi) = \mu(d\pi_F(-Y)D\phi)\\ &= \mu(D(d\pi_E(-Y)\phi)) = (Y(\Psi(D)\mu))(\phi).\end{aligned}$$

Thus $\Psi(D)$ commutes with the action of $\mathfrak{g}$. A similar calculation shows that $\Psi(D)$ commutes with S and so we have a well-defined map Ψ for (2.6). In order to show that Ψ is an isomorphism we will construct an inverse map.

Fix a basis $v_1, \dots, v_t$ for F and let $v_1^*, \dots, v_t^*$ be the dual basis of F^*. Identify v_i^* with $1 \otimes v_i^*$ in $U_\mathfrak{p}(F^*)$. Let

$$\Lambda \in \mathrm{Hom}_{\mathfrak{g},S}(\mathscr{D}'_0(\mathscr{F}^*), \mathscr{D}'_0(\mathscr{E}^*))$$

and define

$$\Phi\colon \operatorname{Hom}_{\mathfrak{g},S}(\mathscr{D}_0'(\mathscr{F}^*), \mathscr{D}_0'(\mathscr{E}^*)) \to \operatorname{Hom}_{\mathbb{C}}(\Gamma_X(\mathscr{E}), \Gamma_X(\mathscr{F}))$$

by the formula

$$\Phi(\Lambda)(\phi)(x) = \sum_{i=1}^{t} \Lambda(T(1 \otimes v_i^*))(\pi_E(x^{-1})\phi)v_i, \qquad x \in G, \phi \in \mathscr{V}(\mathscr{E}).$$

Notice that by linearity, Φ is independent of the basis chosen for F. Since $\Phi(\Lambda)$ can be computed in terms of differentiation by elements of $U(\mathfrak{n}^-)$ translated to $x \in G$, it is clear that $\operatorname{supp}(\Phi(\Lambda)(\phi)) \subseteq \operatorname{supp}(\phi)$. Thus $\Phi(\Lambda)$ is a differential operator. Moreover, for $g \in G$, since $\pi_E(x^{-1})\pi_E(g) = \pi_E((g^{-1}x)^{-1})$, we see $\Phi(\Lambda)(\pi_E(g)\phi)(x) = \Phi(\Lambda)(\phi)(g^{-1}x) = \pi_F(g)\Phi(\Lambda)(\phi)(x)$. Thus $\Phi(\Lambda)$ is G-homogeneous. So we have

$$\Phi\colon \operatorname{Hom}_{\mathfrak{g},S}(\mathscr{D}_0'(\mathscr{F}^*), \mathscr{D}_0'(\mathscr{E}^*)) \to \mathbb{D}(\Gamma_X(\mathscr{E}), \Gamma_X(\mathscr{F})).$$

Finally, for $\phi \in \mathscr{V}(\mathscr{E})$, $D \in \mathbb{D}(\Gamma_X(\mathscr{E}), \Gamma_X(\mathscr{F}))$ and $x \in G$ we have:

$$\begin{aligned}\Phi(\Psi(D))(\phi)(x) &= \sum T(1 \otimes v_i^*)(D\pi_E(x^{-1})\phi)v_i = (D\pi_E(x^{-1})\phi)(e)\\ &= \pi_F(x^{-1})D\phi(e) = D\phi(x).\end{aligned}$$

Thus $\Phi \circ \Psi$ is the identity. A similar computation shows that $\Psi \circ \Phi$ is the identity. This proves the isomorphism in (2.6) and completes the proof of (2.4).

3. Application to highest weight modules. In this section we apply (2.4) using the pair (G, P^0), that is, we ignore the finite group S. We emphasize that the pair (G, P^0) is not unique to the data $(\mathfrak{g}, \mathfrak{p})$.

Let $\mathscr{Z}(\mathfrak{g})$ denote the center of $U(\mathfrak{g})$. If θ is a character of $\mathscr{Z}(\mathfrak{g})$ and M is an object in $\mathscr{O}_{\mathfrak{p}}$ let M_θ be the $U(\mathfrak{g})$-invariant subspace of M on which $\mathscr{Z}(\mathfrak{g})$ acts by generalized character θ. Then $M = \bigoplus M_\theta$, and although the sum is taken over all characters θ it is a finite direct sum.

LEMMA 3.1. *Let E and F be finite dimensional $\mathfrak{p}$-modules and θ a character of $\mathscr{Z}(\mathfrak{g})$. Then*

$$\operatorname{Hom}_{\mathfrak{g}}(U_{\mathfrak{p}}(E)_\theta, U_{\mathfrak{p}}(F)_\theta) \cong \operatorname{Hom}_{\mathfrak{g}}(U_{\mathfrak{p}}(F^*)_{\theta^*}, U_{\mathfrak{g}}(E^*)_{\theta^*}).$$

Proof. We begin by recalling that the function c satisfies the cocycle relation

$$c(gh, x) = c(g, hx)c(h, x), \qquad g, h \in G, x \in X.$$

Writing $e = g^{-1}g$ this formula leads us to the formula

$$|c(g, x)|\,|c(g^{-1}, gx)|^{1/2} = |c(g, x)|^{1/2}. \tag{3.2}$$

Define $A\colon \mathbb{D}(\Gamma_X(\mathscr{E}), \Gamma_X(\mathscr{F})) \to \mathbb{D}(\Gamma_X(\mathscr{F}^*), \Gamma_X(\mathscr{E}^*))$ by taking the transpose adjoint with respect to $d\mu_X$, as in **[H]**. That is, for $D \in \mathbb{D}(\Gamma_X(\mathscr{E}), \Gamma_X(\mathscr{F}))$, $A(D)$ is defined by

$$\int_X \langle \zeta, A(D)\phi\rangle\, d\mu_X = \int_X \langle D\zeta, \phi\rangle\, d\mu_X, \qquad \zeta \in \Gamma_X(\mathscr{E}),\ \phi \in \Gamma_X(\mathscr{F}^*).$$

Let g be in G. Using equation (3.2) we have for all $\zeta \in \Gamma_X(\mathscr{E})$ and $\phi \in \Gamma_X(\mathscr{E}^*)$:

$$\begin{aligned}\int_X \langle \zeta, \pi_{E^*}(g)\phi\rangle\, d\mu_X &= \int_X \langle \zeta(x), |c(g^{-1}, x)|^{1/2} g\phi(g^{-1}x)\rangle\, d\mu_X \\ &= \int_X \langle \zeta(gx), |c(g^{-1}, gx)|^{1/2} g\phi(x)\rangle |c(g, x)|\, d\mu_X \\ &= \int_X \langle |c(g, x)|^{1/2} g^{-1}\zeta(gx), \phi(x)\rangle\, d\mu_X \\ &= \int_X \langle \pi_E(g^{-1})\zeta(x), \phi(x)\rangle\, d\mu_X.\end{aligned}$$

Of course a similar equation holds if ζ is in $\Gamma_X(\mathscr{F})$ and ϕ is in $\Gamma_X(\mathscr{F}^*)$. It follows immediately that $A(D)$ is G-homogeneous whenever D is G-homogeneous. Then A is easily seen to be an isomorphism. In view of (2.4), this proves the lemma with the θ's erased. The lemma then follows easily by tracing the action of $\mathscr{Z}(\mathfrak{g})$ as it is transformed under each of the specific isomorphisms used in the proof of (2.4) and under the action of A. This completes (3.1).

LEMMA 3.3. *Let E be in $\mathscr{A}_\mathfrak{p}$ and θ a character of $\mathscr{Z}(\mathfrak{g})$. Then there is a $\mathfrak{p}$-module A in $\mathscr{A}_\mathfrak{p}$ and a $\mathfrak{p}$-module surjection $A \to E$ such that $U_\mathfrak{p}(A)_\theta$ is projective in $\mathscr{O}_\mathfrak{p}$.*

Proof. This is implicit in 4.1 and 4.2 of **[RC]**.

We can now complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Fix E, F and θ as in Lemma 3.1. Write $\mathrm{Ext}^k_\theta(U_\mathfrak{p}(E), U_\mathfrak{p}(F))$ for $\mathrm{Ext}^k_{\mathscr{O}_\mathfrak{p}}(U_\mathfrak{p}(E)_\theta, U_\mathfrak{p}(F)_\theta)$. Then we are to prove

$$\begin{aligned}&\dim \mathrm{Ext}^k_\theta(U_\mathfrak{p}(E), U_\mathfrak{p}(F)) \\ &\qquad = \dim \mathrm{Ext}^k_{\theta^*}(U_\mathfrak{p}(F^*), U_\mathfrak{p}(E^*)), \qquad k \geq 0.\end{aligned} \tag{3.4}$$

We proceed by induction on k, the case $k = 0$ being (3.1). Choose $A \to E$ as in Lemma 3.3. Let $K = \ker(A \to E)$ so that we have a short exact sequence $0 \to K \to A \to E \to 0$. Then by the projectivity of $U_\mathfrak{p}(A)_\theta$ we have $\mathrm{Ext}^1_\theta(U_\mathfrak{p}(A), U_\mathfrak{p}(F)) = 0$. There are two long exact sequences obtained by applying $\mathrm{Ext}^*_\theta(U_\mathfrak{p}(\cdot), U_\mathfrak{p}(F))$ and $\mathrm{Ext}^*_{\theta^*}(U_\mathfrak{p}(F^*), U_\mathfrak{p}(\cdot^*))$ to the short exact sequence $K \to A \to E$. Comparing the first 5 terms of these sequences and using (3.1) we see:

$$\text{(3.5)} \qquad \dim \mathrm{Ext}^1_\theta(U_\mathfrak{p}(E), U_\mathfrak{p}(F)) \le \dim \mathrm{Ext}^1_{\theta^*}(U_\mathfrak{p}(F^*), U_\mathfrak{p}(E^*)).$$

However, this argument is valid for the pair (F^*, E^*) as well as for the pair (E, F). So we see that the inequality in (3.5) is actually an equality, establishing (3.4) for $k = 1$.

For $k > 1$ the projectivity of $U_\mathfrak{p}(A)_\theta$ and comparison of the two long exact sequences gives (using induction)

$$\text{(3.6)} \quad \begin{aligned} \dim \mathrm{Ext}^k_\theta(U_\mathfrak{p}(E), U_\mathfrak{p}(F)) &= \dim \mathrm{Ext}^{k-1}_\theta(U_\mathfrak{p}(K), U_\mathfrak{p}(F)) \\ &= \dim \mathrm{Ext}^{k-1}_{\theta^*}(U_\mathfrak{p}(F^*), U_\mathfrak{p}(K^*)) \\ &\le \dim \mathrm{Ext}^k_{\theta^*}(U_\mathfrak{p}(F^*), U_\mathfrak{p}(E^*)) \end{aligned}$$

since by induction $\mathrm{Ext}^{k-1}_{\theta^*}(U_\mathfrak{p}(F^*), U_\mathfrak{p}(A^*)) = \mathrm{Ext}^{k-1}_\theta(U_\mathfrak{p}(A), U_\mathfrak{p}(F)) = 0$. Again by symmetry in the argument, the inequality in (3.6) must be an equality. This establishes (3.4) and proves Theorem 1.1.

4. Application to real groups. Consider now the case when $P = P'$; then P is not necessarily connected. As before put $X = G/P$. Given a finite dimensional P-module E we let $I_P(E)$ denote the space of K-finite vectors in $\Gamma_X(\mathscr{E})$. If E is irreducible and trivial as an N-module we refer to these as degenerate series representations of G. In case $P = P_m$, these are usually referred to as principal series representations. A general problem in the subject is to parameterize the intertwining operators between any two degenerate series representations. Since the K-finite vectors of $\Gamma_X(\mathscr{E})$ are dense in $\Gamma_X(\mathscr{E})$, we have an inclusion (via the restriction map)

$$\text{(4.1)} \qquad \mathbb{D}(\Gamma_X(\mathscr{E}), \Gamma_X(\mathscr{F})) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}, K}(I_P(E), I_P(F)).$$

Denote by $\mathrm{Diff}(I_P(E), I_P(F))$ the image of the map in (4.1). These are the *differential intertwining operators.* We may apply (2.4) and conclude that the differential intertwining operators correspond to $\mathrm{Hom}_{\mathfrak{g}, S}(U_\mathfrak{p}(F^*), U_\mathfrak{p}(E^*))$, where S is $(\mathbb{Z}/2\mathbb{Z})^m$ for some m.

As an example, let G be the real Lie group $\mathrm{SL}(2, \mathbb{R})$ and consider the category $\mathscr{HC}_F$ of admissible representations of G having the same infinitesimal character as a fixed irreducible finite dimensional G-representation F. Then $\mathscr{HC}_F$ contains two reducible principal series representations which we denote by $I_P(\mathrm{quo})$ and $I_P(\mathrm{sub})$, indicating that F is the unique irreducible quotient or subrepresentation respectively. As is well known,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}, K}(I_P(\mathrm{quo}), I_P(\mathrm{sub})) = 1 \tag{4.2}$$

and

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}, K}(I_P(\mathrm{sub}), I_P(\mathrm{quo})) = 2. \tag{4.3}$$

Using (2.4) we thus obtain

$$\dim_{\mathbb{C}} \mathrm{Diff}_{\mathfrak{g}, K}(I_P(\mathrm{sub}), I_P(\mathrm{quo})) = 1 \tag{4.4}$$

and

$$\dim_{\mathbb{C}} \mathrm{Diff}_{\mathfrak{g}, K}(I_P(\mathrm{quo}), I_P(\mathrm{sub})) = 0. \tag{4.5}$$

Similar calculations may be carried out in the case of any real rank one Lie group. This follows from **[C1]** and **[C2]**.

5. The extension problem. We were initially led to Theorem 1.1 through an interest in the *Extension Problem* for the category $\mathscr{O}_{\mathfrak{p}}$. To see the connection with our theorem let $\mathscr{O}_0$ be the subcategory of $\mathscr{O}_{\mathfrak{p}}$ consisting of modules with the same generalized infinitesimal character as the trivial $\mathfrak{g}$-module. Denote by $\mathscr{W}$ (respectively $\mathscr{W}_{\mathfrak{l}}$) the Weyl group of $\mathfrak{g}$ (respectively $\mathfrak{l}$) and recall that each coset of $\mathscr{W}_{\mathfrak{l}} \backslash \mathscr{W}$ admits a unique coset representative of minimal length. Denote by $\mathscr{W}^{\mathfrak{l}}$ the collection of these representatives. Let w_0 (resp. w_l) be the longest element of $\mathscr{W}$ (respectively $\mathscr{W}_{\mathfrak{l}}$). Let ρ be the half sum of the positive weights of $\mathfrak{h}$ in $\mathfrak{g}$. Then for each $x \in \mathscr{W}^{\mathfrak{l}}$ we denote by E_x the irreducible $\mathfrak{l}$-module of highest weight $w_l x w_0 \rho - \rho$ and consider E_x as a $\mathfrak{p}$-module by letting $\mathfrak{n}$ act trivially. Set $V_x = U_{\mathfrak{p}}(E_x)$. These are the generalized Verma modules in $\mathscr{O}_0$. We denote by L_x the unique irreducible quotient of V_x. Finally, for each $y, w \in \mathscr{W}^{\mathfrak{l}}$, define polynomials

$$E_{y,w}(q) = \sum_{k \geq 0} (-1)^k q^k \dim_{\mathbb{C}} \mathrm{Ext}^k_{\mathscr{O}_{\mathfrak{p}}}(V_y, V_w). \tag{5.1}$$

The *Extension Problem* is just the problem of computing in some fashion the polynomials $E_{y,w}(q)$. In general, this problem is as yet unsolved and there are not even conjectures for the form of the answer. However, if $\mathfrak{p}$ is a Borel subalgebra then there is a conjectured recursion formula for the polynomials, cf. **[GJ]**, and if $\mathfrak{g}$ and $\mathfrak{p}$ form an indecomposable Hermitian symmetric pair then there is a known recursion formula, cf. [S]. Theorem 1.1 imposes an interesting relation on these generating functions. To state this carefully, recall that there exists an order reversing involution $d: \mathscr{W}^{\mathfrak{l}} \to \mathscr{W}^{\mathfrak{l}}$; $d(x) = w_l x w_0$. As a corollary to (1.1) we obtain:

COROLLARY 5.2. *For all* $y, w \in \mathscr{W}^{\mathfrak{l}}$,

$$E_{y,w} = E_{d(w),d(y)}.$$

In low rank examples, (5.2) can be an effective tool in solving the extension problem. For the remainder of this section we will sketch a representative example of such a computation.

Let $\mathfrak{g}$ be $\mathfrak{sl}_4(\mathbb{C})$ and denote the simple roots of $\mathfrak{g}$ by α_1, α_2, and α_3. Choose $\mathfrak{p}$ to be the maximal parabolic subalgebra whose Levi factor $\mathfrak{l}$ has simple roots α_1 and α_3. If s_i denotes the simple reflection corresponding to α_i, then

$$\mathscr{W}^{\mathfrak{l}} = \{e, s_2, s_2s_3, s_2s_1, s_2s_3s_1, s_2s_3s_1s_2\}.$$

In this simple case, the spaces $\mathrm{Ext}^k_{\mathscr{O}_\mathfrak{p}}(V_y, L_w)$ are known explicitly for all y and w in $\mathscr{W}^{\mathfrak{l}}$. Furthermore there are explicit formulas for the (unique) Loewy series of the modules V_w. All of this information can be found in either **[CC]** or **[ES]**. By combining these two pieces of information and analyzing the collection of long exact sequences which they give rise to, one can easily compute all of the spaces $\mathrm{Ext}^k_{\mathscr{O}_\mathfrak{p}}(V_y, V_w)$ except for the single case $y = e$, $w = s_2s_3s_1$ and $k = 1$. This last case is handled by Theorem 1.1. Since $d(e) = s_2s_3s_1s_2$ and $d(s_2s_3s_1) = s_2$ we have

$$\text{(5.3)} \qquad \dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathscr{O}_\mathfrak{p}}(V_e, V_{s_2s_3s_1}) = \dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathscr{O}_\mathfrak{p}}(V_{s_2}, V_{s_2s_3s_1s_2}).$$

The right-hand side of (5.3) is easily shown (by the methods mentioned above) to be zero. This allows one to compute the following table for the polynomials $E_{y,w}(q)$.

REMARK. We must point out that in this simple example the polynomials $E_{y,w}(q)$ can also be computed by the recursion formulas given in [S].

TABLE 1
Generating polynomials $E_{y,w}$ for $\mathscr{O}_0$

$y \backslash w$	e	s_2	s_2s_3	s_2s_1	$s_2s_3s_1$	$s_2s_3s_1s_2$
e	1	$1-q$	$-q+q^2$	$-q+q^2$	q^2-q^3	$1-q-q^3+q^4$
s_2	0	1	$1-q$	$1-q$	$1-2q+q^2$	q^2-q^3
s_2s_3	0	0	1	0	$1-q$	$-q+q^2$
s_2s_1	0	0	0	1	$1-q$	$-q+q^2$
$s_2s_3s_1$	0	0	0	0	1	$1-q$
$s_2s_3s_1s_2$	0	0	0	0	0	1

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