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EXPLICIT $\overline{\partial}$-PRIMITIVES OF HENKIN-LEITERER KERNELS ON STEIN MANIFOLDS

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In this paper we construct explicitly $\overline{\partial}$-primitives and use them to obtain a representation formula for holomorphic functions and a theorem on extendability of CR-functions.

1. Introduction. Let X be a Stein manifold of dimension n, $h : X \to \mathbb{C}^p$ $(p \leq n-1)$ a holomorphic map and let $Z(h) = \{\zeta \in X : h(\zeta) = 0\}$. If $K(\zeta, z) = K^{(s^*, \nu)}(\zeta, z)$ is a Henkin-Leiterer type kernel on X (see §2 for notation) then $K(\zeta, z)$ is a $\overline{\partial}$-closed $(n, n-1)$-form in ζ, for a fixed z, i.e., $\overline{\partial}_\zeta K(\zeta, z) = 0$, whose singularity occurs at $\zeta = z$. On the other hand, since $X - Z(h)$ is $(n-2)$-complete (see Sorani and Villani [**8**, p. 435]), it follows that the cohomology group

$$H^{n-1}(X - Z(h), \mathscr{O}^n) \cong H^{(n, n-1)}_{\overline{\partial}}(X - Z(h))$$

vanishes (see Andreotti and Grauert [**1**, p. 250]). Therefore, for a fixed $z \in Z(h)$, there exists an $(n, n-2)$-form $\eta(\zeta, z)$, in $X - Z(h)$, so that

$$\overline{\partial}_\zeta \eta(\zeta, z) = K(\zeta, z).$$

For some problems, however, it is important to have explicit formulas for such $\overline{\partial}$-primitives, η, of K; the problems we have in mind are related to integral representations (see for example Stout [**9**] and Hatziafratis [**2**]) and extendability of CR-functions (see for example Lupacciolu [**6**], Tomassini [**11**] and Stout [**10**]). Since such forms $\eta(\zeta, z)$ are not unique, their dependence on z, for example, may be difficult to control with cohomological arguments.

In this paper we construct explicitly such $\overline{\partial}$-primitives and use them to obtain a representation formula for holomorphic functions and a theorem on extendability of CR-functions.

The arrangement of the paper is as follows. First in §2 we review the main points of the Henkin-Leiterer construction; with X and h as above we consider a domain $D \subset X$, a Stein neighborhood W of $\overline{D}$ and we briefly discuss what a Leray section $s^* = s^*(\zeta, z)$ and the associated Henkin-Leiterer kernel $K(\zeta, z) = K^{(s^*, \nu)}(\zeta, z)$ are.

Then in §3 we carry out the construction of the $\overline{\partial}$-primitives $\eta_h(\zeta, z)$ and in Theorem 3.1 we prove that indeed $\overline{\partial}_\zeta \eta_h(\zeta, z) = K(\zeta, z)$ for $\zeta \in W - Z(h - h(z))$, ζ, z being always so that $s^*(\zeta, z)$ is defined. (At this point we would like to point out that we were led to consider this construction by the paper of Laurent-Thiebaut [**5**] in which the case $p = 1$ is studied.)

Our main application of this construction is a Cauchy type integral representation formula for holomorphic functions. Fix a $z \in D$, we consider an open set $\Gamma \subset \partial D$ (open in ∂D) with $\partial\Gamma$ smooth so that $\Gamma \supset (\partial D) \cap Z(h - h(z))$ and we prove (Theorem 3.2) that for $f \in C(\overline{\Gamma} \cup D) \cap \mathscr{O}(D)$ we have

$$f(z) = \int_{\zeta\in\Gamma} f(\zeta)K(\zeta, z) - \int_{\zeta\in\partial\Gamma} f(\zeta)\eta_h(\zeta, z).$$

This integral formula expresses the value of f at z in terms of its values on a part of the boundary of D namely $\overline{\Gamma}$. In particular it provides a formula for extending CR-functions from parts of the boundary (if such extensions exist); this is the point of Theorem 4.1 in §4. This theorem gives a necessary and sufficient condition for the extendability of a CR-function f from a part of the boundary of D to a holomorphic function in D; roughly speaking the condition says that certain integrals involving the CR-function and taken over certain cycles which lie in the domain (on ∂D) of f should agree.

Finally with regards to the Theorem 3.2 we mention the work of Patil [7] where a different method was devised for recovering, in some cases, an H^2-function from its boundary values on a set of positive measure.

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2. Henkin-Leiterer type kernels. In this section we will establish notation and recall the main points of the Henkin-Leiterer construction on Stein manifolds.

Let X be a Stein manifold of dimension n and let $T(X)$ denote its holomorphic tangent bundle with the fiber above z $(z \in X)$ denoted by $T_z(X)$. Then, following Henkin and Leiterer [**4**, Ch. 4], there exists a holomorphic map $s : X \times X \to T(X)$ and a holomorphic function $\varphi : X \times X \to \mathbb{C}$ so that

(i) $s(\zeta, z) \in T_z(X)$ for $(\zeta, z) \in X \times X$,

(ii) $s(z, z) = 0$ and $s(\cdot, z)$ is a biholomorphic map from a neighborhood of $z \in X$ to a neighborhood of $0 \in T_z(X) \cong \mathbb{C}^n$,

(iii) $\varphi(z, z) = 1$ and there exists a positive integer ν_0 so that $\varphi^{\nu_0}(\zeta, z)\|s(\zeta, z)\|^{-2}$ is a C^2-function on $X\times X-\Delta = X\times X-\{(z, z) : z\in X\}$, for any norm $\|\cdot\|$ on $T(X)$; in particular $\varphi^{\nu}\|s\|^{-2}$ is of class C^r on $X\times X-\Delta$ provided that $\nu \geq \nu_1(r)$ for some integer $\nu_1(r)$.

Now fix $D\subset X$, a relatively compact domain in X with smooth boundary. Recall that a Leray section for (D, s, φ) is a C^1-map $s^* = s^*(\zeta, z)$ defined for $z\in D$ and for ζ in a neighborhood of ∂D, denoted by $\mathrm{Dom}(s^*(\cdot, z))$ and depending on z, with values in $T^*(X)$, the holomorphic cotangent bundle of X, so that:

(i) $s^*(\zeta, z)\in T_z^*(X)$ ($T_z^*(X)$ denotes the fiber of $T^*(X)$ above z),
(ii) $\langle s^*(\zeta, z), s(\zeta, z)\rangle \neq 0$ whenever $\varphi(\zeta, z)\neq 0$ and
(iii) there is an integer ν^* so that the function

$$\varphi^{\nu^*}(\zeta, z)(\langle s^*(\zeta, z), s(\zeta, z)\rangle)^{-1}$$

is of class C^1 for $(\zeta, z)\in V\times L$, for each compact subset L of D and where V is a neighborhood of ∂D, depending on L. Here $\langle\cdot, \cdot\rangle$ denotes the pairing of cotangent vectors with tangent vectors.

For examples of Leray sections, which always exist in the above setting, see [**4**, p. 165].

To a Leray section s^*, Henkin and Leiterer associate an $(n, n-1)$-form in the following way:

$$K^{(s^*, \nu)}(\zeta, z) = \varphi^{\nu}(\zeta, z)\frac{\omega'_\zeta(s^*(\zeta, z))\wedge\omega_\zeta(s(\zeta, z))}{\langle s^*(\zeta, z), s(\zeta, z)\rangle^n}$$

where ν is assumed to be large enough so that $K^{(s^*, \nu)}(\zeta, z)$ is continuous in each $V\times L$ ($\nu\geq n\nu^*$ is enough); the differential forms $\omega'_\zeta(s^*(\zeta, z))$ are defined in terms of local coordinates (U, χ) at z; let $(s_1, \dots, s_n)$ and $(s_1^*, \dots, s_n^*)$ be the expressions of s and s^* in terms of the local coordinate system (U, χ), i.e.,

$$s(\zeta, z) = \sum_{j=1}^n s_j(\zeta, z)\left(\frac{\partial}{\partial\chi_j}\right)_z \quad\text{and}\quad s^*(\zeta, z) = \sum_{j=1}^n s_j^*(\zeta, z)(d\chi_j)_z;$$

here $\{(\partial/\partial\chi_j)_z\}_{j=1}^n$ is the usual basis of $T_z(X)$ with respect to (U, χ) and $\{(d\chi_j)_z\}_{j=1}^n$ is the corresponding basis for $T_z^*(X)$.

Then

$$\omega_\zeta(s(\zeta, z)) = d_\zeta s_1(\zeta, z)\wedge\cdots\wedge d_\zeta s_n(\zeta, z)$$

and

$$\omega'_\zeta(s(\zeta, z)) = c_n\sum_{j=1}^n(-1)^{j-1}s_j^*(\zeta, z)\bigwedge_{k\neq j}d_\zeta s_k^*(\zeta, z)$$

where $c_n = (-1)^{n(n-1)/2}(n-1)!/(2\pi i)^n$.

Of course by the way $\omega_\zeta(s(\zeta, z))$ and $\omega'_\zeta(s^*(\zeta, z))$ are defined, they depend on the choice of the local coordinates (U, χ). It turns out, however, that their wedge product and therefore $K^{(s^*, \nu)}(\zeta, z)$ are independent of the choice of local coordinates, i.e., $K^{(s^*, \nu)}(\zeta, z)$ is a globally defined $(n, n-1)$-form, see [**4**, p. 166].

REMARK. The discussion, given in §1, in which we justify by a cohomological argument the existence of $\bar\partial$-primitives, $\eta(\zeta, z)$, of $K(\zeta, z) = K^{(s^*, \nu)}(\zeta, z)$, applies for a particular class of Leray sections, the ones which are defined for $(\zeta, z) \in X \times X$, i.e., $D = X$ and $\mathrm{Dom}(s^*(\cdot, z)) = X$; the point here is that, in the general case, $\mathrm{Dom}(s^*(\cdot, z)) - Z(h)$ is not $(n-2)$-complete; however it is possible to give a cohomological argument to prove existence of the $\bar\partial$-primitives in the general case too; this argument amounts to modifying, in a way, $s^*(\zeta, z)$ so that the argument given in §1 applies (see also the remark following the proof of Theorem 3.1 below).

3. Construction of the $\bar\partial$-primitives. With the notation of §2, let us consider a holomorphic map $h : W \to \mathbb{C}^p$, $p \le n-1$, where W is a Stein neighborhood of $\overline{D}$; let $Z(h - h(z))$ denote the zero-set of $h - h(z)$, i.e.,

$$Z(h - h(z)) = \{\zeta \in W : h(\zeta) = h(z)\}.$$

In this section we will construct a $\bar\partial$-primitive of $K^{(s^*, \nu)}(\zeta, z)$ in $W \cap \mathrm{Dom}(s^*(\cdot, z)) - Z(h - h(z))$; in this construction, z is a fixed point of D; the dependence of the construction on z, however, will be immediately clear, because of the explicit way the construction is carried out.

According to [**4**, Lemma 4.7.2] there exist holomorphic maps h_i^* : $W \times W \to T^*(X)$, $i = 1, \dots, p$, so that $h_i^*(\zeta, z) \in T_z^*(X)$ and

$$\langle h_i^*(\zeta, z), s(\zeta, z)\rangle = \varphi(\zeta, z) \cdot (h_i(\zeta) - h_i(z))$$

for $(\zeta, z) \in W \times W$ and $i = 1, \dots, p$. Using such holomorphic maps h_i^* we now define a C^∞-map $t^* : W \times W \to T^*(X)$ in the following way:

$$t^*(\zeta, z) = \sum_{i=1}^{p} (\overline{h}_i(\zeta) - \overline{h}_i(z)) h_i^*(\zeta, z);$$

then it is clear that t^* is a well-defined C^∞-map with $t^*(\zeta, z) \in T_z^*(X)$.

Also notice that

$$\text{(I)} \qquad \langle t^*(\zeta, z), s(\zeta, z)\rangle = \varphi(\zeta, z)\sum_{i=1}^{p}|h_i(\zeta) - h_i(z)|^2.$$

Let

$$\eta_h^{(s^*, \nu)}(\zeta, z)$$
$$= -c_n'\varphi^{\nu-n+1}\sum_{l=0}^{n-2}\varphi^l \frac{\det[s_j^*, t_j^*, \overbrace{\overline{\partial}_\zeta s_j^*}^{l}, \overbrace{\overline{\partial}_\zeta t_j^*}^{n-l-2}]\wedge \omega_\zeta(s(\zeta, z))}{(\langle s^*, s\rangle^{l+1}(\sum_{i=1}^{p}|h_i(\zeta) - h_i(z)|^2)^{n-l-1}}$$

where $c_n' = (-1)^{n(n-1)/2}(2\pi i)^{-n}$; $(s_1^*, \dots, s_n^*)$ and $(t_1^*, \dots, t_n^*)$ are the expressions of $s^*(\zeta, z)$ and $t^*(\zeta, z)$, respectively, with respect to the local coordinates (U, χ) considered in §2; let us point out that $\omega_\zeta(s(\zeta, z))$, in the definition of $\eta_h^{(s^*, \nu)}(\zeta, z)$ above, is computed with respect to the same coordinates (U, χ); thus if $(s_1, \dots, s_n)$ are the expressions of $s(\zeta, z)$ with respect to (U, χ) then $\omega_\zeta(s(\zeta, z)) = \partial_\zeta s_1 \wedge \cdots \wedge \partial_\zeta s_n$. In the determinants which appear in the definition of $\eta_h^{(s^*, \nu)}$, j runs from $j = 1$ to $j = n$ forming the n rows of them.

Although the differential form $\eta_h^{(s^*, \nu)}(\zeta, z)$ is introduced locally, it turns out that it is invariantly defined since we have

LEMMA 3.1. *$\eta_h^{(s^*, \nu)}(\zeta, z)$ is a globally defined $(n, n-1)$-form, i.e., it is independent of the choice of local coordinates, with $\zeta \in W \cap \mathrm{Dom}(s^*(\cdot, z)) - Z(h - h(z))$ and a fixed $z \in D$.*

Proof. Let $(\widetilde{U}, \tilde{\chi})$ be another coordinate system at z; let $(\tilde{s}_1^*, \dots, \tilde{s}_n^*)$, $(\tilde{t}_1^*, \dots, \tilde{t}_n^*)$ and $(\tilde{s}_1, \dots, \tilde{s}_n)$ be the expressions of s^*, t^* and s, respectively, with respect to $(\widetilde{U}, \tilde{\chi})$. Then

$$(\tilde{s}_j) = G\cdot(s_j),$$
$$(\tilde{s}_j^*) = (G')^{-1}\cdot(s_j^*),$$
$$(\tilde{t}_j^*) = (G')^{-1}\cdot(t_j^*),$$

where $G = G(z)$ is the transition matrix from (U, χ) to $(\widetilde{U}, \tilde{\chi})$ for the holomorphic vector bundle $T(X)$, in which case $(G')^{-1}$, the inverse of the transpose of G, is the transition matrix from (U, χ) to $(\widetilde{U}, \tilde{\chi})$ for the bundle $T^*(X)$; of course $G = G(z)$ depends only on z; here (s_j) denotes the transpose of $(s_1, \dots, s_n)$ and similarly for the others; the dot denotes matrix multiplication. Therefore,

$$(\partial_\zeta \tilde{s}_j) = G \cdot (\partial_\zeta s_j),$$
$$(\overline{\partial}_\zeta \hat{s}_j^*) = (G')^{-1} \cdot (\overline{\partial}_\zeta s_j^*),$$
$$(\overline{\partial}_\zeta \tilde{t}_j^*) = (G')^{-1} \cdot (\overline{\partial}_\zeta t_j^*).$$

It follows from the above relations and properties of determinants with entries differential forms (see [3, p. 94]) that

$$\det[\tilde{s}_j^*, \tilde{t}_j^*, \overbrace{\overline{\partial}_\zeta \tilde{s}_j^*}^{l}, \overbrace{\overline{\partial}_\zeta \tilde{t}_j^*}^{n-l-2}] = \det[(G')^{-1}] \det[s_j^*, t_j^*, \overbrace{\overline{\partial} s_j^*}^{l}, \overbrace{\overline{\partial} t_j^*}^{n-l-2}]$$

and

$$\partial_\zeta \tilde{s}_1 \wedge \cdots \wedge \partial_\zeta \tilde{s}_n = \det(G) \partial_\zeta s_1 \wedge \cdots \wedge \partial_\zeta s_n.$$

Since $\det[(G')^{-1}] = [\det(G)]^{-1}$, it follows that $\eta_h^{(s^*, \nu)}(\zeta, z)$ is, indeed, independent of local coordinates. This completes the proof of the lemma.

REMARK. The holomorphic maps h_i^* $(i = 1, \ldots, p)$ are by no means unique; thus the differential form $\eta_h^{(s^*, \nu)}$ depends on the choice of h_i^*. We will come back to this point later.

LEMMA 3.2. *Let σ^* and τ^* be defined, for (ζ, z) with $\varphi(\zeta, z) \neq 0$ and $\zeta \in W \cap \mathrm{Dom}(s^*(\cdot, z)) - Z(h - h(z))$, as follows:*

$$\sigma^*(\zeta, z) = (\langle s^*(\zeta, z), s(\zeta, z)\rangle)^{-1} \cdot s^*(\zeta, z) \quad \textit{and}$$
$$\tau^*(\zeta, z) = (\langle t^*(\zeta, z), s(\zeta, z)\rangle)^{-1} \cdot t^*(\zeta, z).$$

Then

$$\eta_h^{(s^*, \nu)}(\zeta, z) = -c_n' \cdot \varphi^\nu \cdot \sum_{l=0}^{n-2} \det[\sigma_j^*, \tau_j^*, \overbrace{\overline{\partial}_\zeta \sigma_j^*}^{l}, \overbrace{\overline{\partial}_\zeta \tau_j^*}^{n-l-2}] \wedge \omega_\zeta(s)$$

where σ_j^* and τ_j^* are the expressions of σ^* and τ^* with respect to the local coordinates (U, χ) and $\omega_\zeta(s) = \omega_\zeta(s(\zeta, z))$ is the differential form as in the definition of $\eta_h^{(s^*, \nu)}$ with respect to the same coordinates (U, χ).

Proof. First notice that σ^* and τ^* are well-defined since $\varphi(\zeta, z) \neq 0$ implies $\langle s^*(\zeta, z), s(\zeta, z)\rangle \neq 0$ and together with $\zeta \notin Z(h - h(z))$, they imply also that $\langle t^*(\zeta, z), s(\zeta, z)\rangle \neq 0$; this is because of (I). It follows from the definition of σ^* and τ^* that

$$\overline{\partial}_\zeta \sigma_j^* = (\langle s^*, s\rangle)^{-1} \overline{\partial} s_j^* + s_j^* \overline{\partial}_\zeta [(\langle s^*, s\rangle)^{-1}] \quad \text{and}$$
$$\overline{\partial}_\zeta \tau_j^* = (\langle t^*, s\rangle)^{-1} \overline{\partial} t_j^* + t_j^* \overline{\partial}_\zeta [(\langle t^*, s\rangle)^{-1}].$$

Now the lemma follows from the above equations, from (I) and properties of determinants.

We are ready now to prove that $\eta_h^{(s^*,\nu)}$ is a $\overline{\partial}_\zeta$-primitive of $K^{(s^*,\nu)}$. More precisely we have

THEOREM 3.1. *Let D be a domain on the Stein manifold X, $\dim_{\mathbb{C}} X = n$, and $h : W \to \mathbb{C}^p$ a holomorphic map, $p \leq n-1$, where W is a Stein neighborhood of $\overline{D}$. Let $s^* = s^*(\zeta, z)$ and $K^{(s^*,\nu)}$ be as in § 2 and let $\eta_h^{(s^*,\nu)}$ be the above constructed differential form. Then, for a fixed $z \in D$, we have*

$$\overline{\partial}_\zeta \eta_h^{(s^*,\nu)}(\zeta, z) = d_\zeta \eta_h^{(s^*,\nu)}(\zeta, z) = K^{(s^*,\nu)}(\zeta, z)$$

for $\zeta \in W \cap \mathrm{Dom}(s^*(\cdot, z)) - Z(h - h(z))$.

Proof. Let us consider first (ζ, z) with $\varphi(\zeta, z) \neq 0$. Then, by the definition of σ^* and τ^*,

(1) $$\langle \sigma^*, s\rangle = 1 \quad \text{and} \quad \langle \tau^*, s\rangle = 1.$$

Working always with a fixed coordinate system (U, χ) at z, (1) can be written as

(2) $$\sum_{j=1}^{n} \sigma_j^* s_j = 1 \quad \text{and} \quad \sum_{j=1}^{n} \tau_j^* s_j = 1.$$

It follows from (2) that $s_j \neq 0$ for at least one $j \in \{1, \dots, n\}$. We may assume, without loss of generality, that $s_1 \neq 0$. Then, by Lemma 3.2,

(3)
$$\eta_h^{(s^*,\nu)} = -\frac{c_n'}{s_1}\varphi^\nu \sum_{l=0}^{n-2} \det \begin{bmatrix} \sigma_1^* s_1 & \tau_1^* s_1 & \overbrace{\overline{\partial}(\sigma_1^* s_1)}^{l} & \overbrace{\overline{\partial}(\tau_1^* s_1)}^{n-l-2} \\ \sigma_j^* & \tau_j^* & \overline{\partial}\sigma_j^* & \overline{\partial}\tau_j^* \end{bmatrix} \wedge \omega_\zeta(s);$$

in the determinants in (3) j runs from $j = 2$ to $j = n$ forming the 2nd up to the nth row of them. In obtaining (3) we also used the fact that $s_1 = s_1(\zeta, z)$ is holomorphic in ζ (throughout this proof $\overline{\partial} = \overline{\partial}_\zeta$). Next, multiplying the jth-rows of each determinant in (3) $(2 \leq j \leq n)$ by s_j and adding them to the first row of it we obtain, in view of (2),

(4) $$\eta_h^{(s^*,\nu)} = -\frac{c_n'}{s_1}\varphi^\nu \sum_{l=0}^{n-2} \det \begin{bmatrix} 1 & 1 & \overbrace{0}^{l} & \overbrace{0}^{n-l-2} \\ \sigma_j^* & \tau_j^* & \overline{\partial}\sigma_j^* & \overline{\partial}\tau_j^* \end{bmatrix} \wedge \omega_\zeta(s).$$

Applying $\overline{\partial} = \overline{\partial}_\zeta$ to both sides of (4) and using the fact that φ is holomorphic in ζ, we obtain

$$\overline{\partial}\eta_h^{(s^*,\nu)} = -\frac{c_n'}{s_1}\varphi^\nu \sum_{l=0}^{n-2}\left(\det\begin{bmatrix} 0 & 1 & \overbrace{0}^{l} & \overbrace{0}^{n-l-2} \\ \overline{\partial}\sigma_j^* & \tau_j^* & \overline{\partial}\sigma_j^* & \overline{\partial}\tau_j^* \end{bmatrix}\wedge\omega_\zeta(s)\right.$$
$$\left. + \det\begin{bmatrix} 1 & 0 & \overbrace{0}^{l} & \overbrace{0}^{n-l-2} \\ \sigma_j^* & \overline{\partial}\tau_j^* & \overline{\partial}\sigma_j^* & \overline{\partial}\tau_j^* \end{bmatrix}\wedge\omega_\zeta(s)\right)$$

or, after a computation,

$$(5)\quad \overline{\partial}\eta_h^{(s^*,\nu)} = -\frac{c_n'}{s_1}\varphi^\nu\sum_{l=0}^{n-2}(\det[\overbrace{\overline{\partial}\sigma_j^*}^{l},\ \overbrace{\overline{\partial}\tau_j^*}^{n-l-1}] - \det[\overbrace{\overline{\partial}\sigma_j^*}^{l+1},\ \overbrace{\overline{\partial}\tau_j^*}^{n-l-2}])\wedge\omega_\zeta(s)$$
$$= \frac{c_n'}{s_1}\varphi^\nu \det[\overbrace{\overline{\partial}\sigma_j^*}^{n-1}]\wedge\omega_\zeta(s) - \frac{c_n'}{s_1}\varphi^\nu\det[\overbrace{\overline{\partial}\tau_j^*}^{n-1}]\wedge\omega_\zeta(s);$$

all the determinants in (5) are $(n-1)\times(n-1)$ and j runs from $j=2$ to $j=n$ forming their $(n-1)$ rows. Now we claim that

$$(6)\qquad \frac{c_n'}{s_1}\varphi^\nu\det\left(\left[\overbrace{\overline{\partial}\sigma_j^*}^{n-1}\right]_{j=2}^{n}\right)\wedge\omega_\zeta(s) = K^{(s^*,\nu)}$$

and

$$(7)\qquad \frac{1}{s_1}\det\left[\overbrace{\overline{\partial}\tau_j^*}^{n-1}\right]_{j=2}^{n} = 0.$$

First let us prove (6). It follows from the definition of $K^{(s^*,\nu)}$ and the relations between s_j^* and σ_j^* (exactly as in the proof of Lemma 2.2) that

$$K^{(s*,\nu)} = \frac{c_n'\varphi^\nu}{s_1}\det\left(\begin{bmatrix} \sigma_1^* s_1 & \overbrace{\overline{\partial}(\sigma_1^* s_1)}^{n-1} \\ \sigma_j^* & \overline{\partial}\sigma_j^* \end{bmatrix}_{j=2}^{n}\right)\wedge\omega_\zeta(s).$$

Therefore, in view of (2),

$$K^{(s^*,\nu)} = \frac{c_n'\varphi^\nu}{s_1}\det\left(\begin{bmatrix} 1 & \overbrace{0}^{n-1} \\ \sigma_j^* & \overline{\partial}\sigma_j^* \end{bmatrix}_{j=2}^{n}\right)\wedge\omega_\zeta(s)$$

which immediately implies (6).

Similarly, to prove (7) we write its left-hand side (in view of the relation between τ^* and t^*) as follows:

$$\frac{1}{s_1}\det[\overbrace{\overline{\partial}\tau_j^*}^{n-1}] = (\langle t^*, s\rangle)^{-n}\det([t_j^*, \overbrace{\overline{\partial}t_j^*}^{n-1}]_{j=1}^n). \tag{8}$$

Let h_{ij}^* $(1 \le i \le p,\ 1 \le j \le n)$ be the expressions of h_i^* with respect to the local coordinates (U, χ), i.e.,

$$h_i^*(\zeta, z) = \sum_{j=1}^n h_{ij}^*(\zeta, z)(d\chi_j)_z.$$

Recalling that $t^* = \sum_{i=1}^p(\overline{h}_i - \overline{h}_i(z))h_i^*$ we obtain

$$t_j^* = \sum_{i=1}^p(\overline{h}_i - \overline{h}_i(z))h_{ij}^* \quad\text{and}\quad \overline{\partial}t_j^* = \sum_{i=1}^p h_{ij}^*\overline{\partial h_i}, \tag{9}$$

since h_{ij}^* are holomorphic in ζ. Now to prove (7) we distinguish two cases:

1st case: $p \le n-2$; in this case

$$\overline{\partial}t_{j_1}\wedge\cdots\wedge\overline{\partial}t_{j_{n-1}} = 0 \tag{10}$$

for $1 \le j_1 < \cdots < j_{n-1} \le n$; this follows from (9); but (10) and (8) imply (7) in this case.

2nd case: $p = n-1$; in this case, substituting (9) into the right-hand side of (8), we obtain

$$\begin{aligned}\det[t_j^*, \overbrace{\overline{\partial}t_j^*}^{n-1}]_{j=1}^n &\\ = p!\det&\left(\left[\sum_{i=1}^p(h_i - h_i(z))h_{ij}^*, h_{1j}^*, \dots h_{pj}^*\right]_{j=1}^n\right)\\ &\times\overline{\partial h_1}\wedge\cdots\wedge\overline{\partial h_p} = 0;\end{aligned} \tag{11}$$

since (11) and (8) imply (7), the proof of (7) is complete. Finally (7), (6) and (5) imply the formula of the theorem in the case $\varphi(\zeta, z) \ne 0$ and, since the set $\{\varphi(\zeta, z) \ne 0\}$ is dense, this completes the proof of the theorem.

Remark. As we pointed out before, $\eta_h^{(s^*, \nu)}$ depends on the choice of $\{h_i^*\}_{i=1}^p$; in the case $p \le n-2$, however, this dependence is not essential in a sense which we will make precise now.

Let $[\eta_h^{(s^*,\nu)}]$ denote the cohomology class of $\eta_h^{(s^*,\nu)}$ in the Dolbault cohomology group $H_{\overline{\partial}}^{(n,n-2)}(V_z - Z(h-h(z)))$ where V_z is an open neighborhood of ∂D with $\overline{V}_z \subset W \cap \operatorname{Dom}(s^*(\cdot, z))$ (here z is fixed, as usual, and ζ is the variable).

Let $(h_i^*)' : W \times W \to T^*(X)$, $i = 1, \dots, p$, be holomorphic maps, with $(h_i^*)'(\zeta, z) \in T_z^*(X)$ and $\langle (h_i^*)', s\rangle = \varphi \cdot (h_i - h_i(z))$, i.e., another choice for h_i^* and let $(\eta_h^{(s^*,\nu)})'$ denote the $\overline{\partial}$-primitive of $K^{(s^*,\nu)}$ in $W \cap \operatorname{Dom}(s^*(\cdot, z)) - Z(h-h(z))$ associated to $(h_i^*)'$. We claim that

$$[\eta_h^{(s^*,\nu)}] = [(\eta_h^{(s^*,\nu)})'];$$

in other words, the cohomology class $[\eta_h^{(s^*,\nu)}]$ is independent of the choice of h_i^*. To prove this we argue as follows. Let $\psi(\zeta, z)$ be a C^∞ function with $0 \le \psi(\zeta, z) \le 1$, having compact support contained in $W \cap \operatorname{Dom}(s^*(\cdot, z))$, which is identically one in a neighborhood of $\overline{V}_z$. Let $\overline{s}(\zeta, z)$ denote a Leray section for (D, s, φ) with $\operatorname{Dom}(\overline{s}(\cdot, z)) = W$ and defined for $z \in W$; such a Leray section always exists (see [**4**, p. 164]; let us point out that $\overline{s}(\zeta, z)$ is not the complex conjugate of $s(\zeta, z)$). Define

$$\lambda^*(\zeta, z) = \varphi^{\nu_1}(\zeta, z)\left[\frac{\psi(\zeta, z)}{\langle s^*(\zeta, z), s(\zeta, z)\rangle} s^*(\zeta, z) + \frac{1-\psi(\zeta, z)}{\langle \overline{s}(\zeta, z), s(\zeta, z)\rangle}\overline{s}(\zeta, z)\right]$$

where $\nu_1 = \max(\nu_0, \nu^*)$. Since

$$\langle \lambda^*(\zeta, z), s(\zeta, z)\rangle = \varphi^{\nu_1}(\zeta, z),$$

it follows that λ^* is a Leray section for (D, s, φ); thus we may associate to λ^* the $\overline{\partial}$-primitives $\eta_h^{(\lambda^*,\nu)}$ and $(\eta_h^{(\lambda^*,\nu)})'$ of $K^{(\lambda^*,\nu)}$, in $W - Z(h-h(z))$, corresponding to h_i^* and $(h_i^*)'$. It follows from Theorem 3.1 that

$$\overline{\partial}(\eta_h^{(\lambda^*,\nu)} - (\eta_h^{(\lambda^*,\nu)})') = K^{(\lambda^*,\nu)} - K^{(\lambda^*,\nu)} = 0 \quad \text{in } W - Z(h-h(z));$$

but $W - Z(h-h(z))$ is $(n-3)$-complete (here $p \le n-2$; see [**7**, p. 435]) whence $H^{n-2}(W - Z(h-h(z)), \mathcal{O}^n) = 0$ (see [**1**, p. 250]); therefore, from Dolbault's theorem, there exists an $(n, n-3)$-form θ in $W - Z(h-h(z))$ with

$$\eta_h^{(\lambda^*,\nu)} - (\eta_h^{(\lambda^*,\nu)})' = \overline{\partial}\theta.$$

Since $\psi \equiv 1$ in a neighborhood of $\overline{V}_z$, it follows from the proof of Lemma 3.2 that

$$\eta_h^{(\lambda^*,\nu)} = \eta_h^{(s^*,\nu)} \quad \text{and}$$
$$(\eta_h^{(\lambda^*,\nu)})' = (\eta_h^{(s^*,\nu)})' \quad \text{in } V_z - Z(h-h(z))$$

whence

$$\eta_h^{(s^*,\nu)} - (\eta_h^{(s^*,\nu)})' = \bar{\partial}\theta \quad \text{in } V_z - Z(h-h(z)).$$

This proves the claim that the cohomology class $[\eta_h^{(s^*,\nu)}]$ does not depend on the choice of h_i^*. Notice also that if Γ is an open subset of ∂D with $\overline{\Gamma} \subset \partial D - Z(h-h(z))$ and f is a smooth CR-function on Γ then

$$f\eta_h^{(s^*,\nu)} - f(\eta_h^{(s^*,\nu)})' = \bar{\partial}(f\theta) = d(f\theta)$$

whence we obtain

$$\int_c f\eta_h^{(s^*,\nu)}(\cdot\,, z) = \int_c f(\eta_h^{(s^*,\nu)}(\cdot\,, z))'$$

for every $(2n-2)$-dimensional cycle c in Γ.

The following theorem is a generalization of the Henkin-Leiterer version of the Cauchy-Fantappiè formula; its proof is similar to the proof of Proposition 2.4 in [**6**, p. 185].

THEOREM 3.2. *Let D be a domain on the Stein manifold X and let h, W, $K^{(s^*,\nu)}$ and $\eta_h^{(s^*,\nu)}$ be as in Theorem* 3.1. *Let $z \in D$ and let $\Gamma \subset \partial D$ be an open subset of ∂D with $\partial\Gamma$ smooth and so that $\Gamma \supset (\partial D) \cap Z(h-h(z))$. Then for $f \in C(\overline{\Gamma} \cup D) \cap \mathscr{O}(D)$, i.e., continuous on $\overline{\Gamma} \cup D$ and holomorphic in D, we have the following representation formula*:

$$f(z) = \int_{\zeta\in\Gamma} f(\zeta)K^{(s^*,\nu)}(\zeta\,, z) - \int_{\zeta\in\partial\Gamma} f(\zeta)\eta_h^{(s^*,\nu)}(\zeta\,, z).$$

Proof. Let $G \subset D$ be an open subset of D so that $\partial G \cap \partial D = \Gamma$ and $D \cap Z(h-h(z)) \subset G$. We also assume that $\partial G = \Gamma \cup \Gamma_0$ where $\Gamma_0 = \partial G \cap \overline{D} \subset W$. Then, by [**4**, Theorem 4.3.4], we have

$$(1) \qquad f(z) = \int_{\partial G} fK^{(s^*,\nu)}(\cdot\,, z)$$
$$= \int_{\Gamma} fK^{(s^*,\nu)}(\cdot\,, z) + \int_{\Gamma_0} fK^{(s^*,\nu)}(\cdot\,, z).$$

Since $\Gamma_0 \subset W - Z(h - h(z))$ it follows from Theorem 3.1, Stokes's theorem and the fact that f is holomorphic in D that

$$\text{(2)} \qquad \int_{\Gamma_0} f K^{(s^*,\nu)}(\cdot, z) = \int_{\partial\Gamma_0} f \eta_h^{(s^*,\nu)}(\cdot, z) = -\int_{\partial\Gamma} f \eta_h^{(s^*,\nu)}(\cdot, z).$$

Now the formula of the theorem follows from (1) and (2).

REMARK. If $\Gamma = \partial D$ then $\partial\Gamma = \emptyset$ and the formula of Theorem 3.2 reduces to that of [**4**, Theorem 4.3.4].

4. Extending CR-functions. Let (D, s, φ), W and s^* be as in §3; we assume furthermore that $s^*(\zeta, z)$ is defined, as a Leray section, for all $(\zeta, z) \in W \times W$. Let E be a closed subset of ∂D so that each connected component of $\partial D - E$ contains a peak point for $\mathscr{O}(\overline{D})$, i.e., a point ζ_0 for which there exists a $g \in \mathscr{O}(\overline{D})$ with $|g(\zeta_0)| > |g(\zeta)|$ for $\zeta \in \overline{D} - \{\zeta_0\}$. For each $z \in W - E$ let

$$\mathscr{P}_z = \{h : W \to \mathbb{C}^{n-2} : h \text{ holomorphic, } z \in Z(h) \text{ and } Z(h) \cap E = \emptyset\}.$$

We can now state a criterion for extendability of CR-functions defined on $\partial D - E$; a version of it in $\mathbb{C}^n$, with the Bochner-Martinelli kernel in place of the Henkin-Leiterer type kernel, is in [**2**]; its proof is based on ideas from [**6**] and [**5**].

THEOREM 4.1. *With notation as above, suppose that* $\mathscr{P}_z \neq \emptyset$ *for each* $z \in W - E$ *and let* f *be a smooth CR-function on* $\partial D - E$. *Then a necessary and sufficient condition that* f *extends to a holomorphic function in* D *is*

$$\text{(1)} \qquad \int_{\zeta\in\partial\Gamma} f(\zeta)\eta_h^{(s^*,\nu)}(\zeta, z) = \int_{\zeta\in\partial\Gamma} f(\zeta)\eta_g^{(s^*,\nu)}(\zeta, z)$$

for $h, g \in \mathscr{P}_z$, $\Gamma \supset (\partial D) \cap (Z(h) \cup Z(g))$ *open (in* ∂D*) with* $\overline{\Gamma} \subset \partial D - E$ *and* $\partial\Gamma$ *smooth and* $z \in W - E$.

REMARKS. (i) Of course $\eta_h^{(s^*,\nu)}$ and $\eta_g^{(s^*,\nu)}$ are $\overline{\partial}$-primitives of $K^{(s^*,\nu)}$ in $W \cap \mathrm{Dom}(s^*(\cdot, z)) - Z(h - h(z))$ and $W \cap \mathrm{Dom}(s^*(\cdot, z)) - Z(g - g(z))$ respectively. As we pointed out before, in the remark following the proof of Theorem 3.1, given Γ, the value of the integral

$$\int_{\zeta\in\partial\Gamma} f(\zeta)\eta_h^{(s^*,\nu)}(\zeta, z)$$

is uniquely determined by h, i.e., it is independent of the choice of h_i^*.

(ii) Observe that if Γ' has the properties required for Γ then, by Theorem 3.1 and Stokes's theorem,

$$\int_{\zeta\in\partial\Gamma'} f(\zeta)\eta_h^{(s^*,\nu)}(\zeta, z) - \int_{\zeta\in\partial\Gamma} f(\zeta)\eta_h^{(s^*,\nu)}(\zeta, z) = \int_{\zeta\in(\Gamma'-\Gamma)\cup(\Gamma-\Gamma')} f(\zeta)K^{(s^*,\nu)}(\zeta, z)$$

with the various parts of $(\Gamma'-\Gamma)\cup(\Gamma-\Gamma')$ appropriately oriented; therefore if (1) holds for Γ it will also hold for Γ'.

Proof of Theorem 4.1. First the necessity of (1) follows immediately from Theorem 3.2.

Now we prove sufficiency of (1), i.e., we assume that (1) holds and we prove that f extends to a holomorphic function in D. To this end let $z\in W-\partial D$ and let $h\in\mathscr{P}_z$; choose Γ and define

$$F(z) = \int_{\zeta\in\Gamma} f(\zeta)K^{(s^*,\nu)}(\zeta, z) - \int_{\zeta\in\partial\Gamma} f(\zeta)\eta_h^{(s^*,\nu)}(\zeta, z). \tag{2}$$

Condition (1) now guarantees that $F(z)$ is well-defined, i.e., it is independent of the various choices (basically of the choice of h, in view of the previous remarks). Next we prove that F is holomorphic; for this we compute $\overline{\partial}_z F$.

$$\overline{\partial}_z F(z) = \int_{\zeta\in\Gamma} f(\zeta)\overline{\partial}_z K^{(s^*,\nu)}(\zeta, z) - \int_{\zeta\in\partial\Gamma} f(\zeta)\overline{\partial}_z\eta_h^{(s^*,\nu)}(\zeta, z). \tag{3}$$

This computation is justified, in part, by the explicit formula for $\eta_h^{(s^*,\nu)}$; notice that if $h\in\mathscr{P}_z$ then $h-h(z')\in\mathscr{P}_{z'}$ for z' close to z; thus, in (3),

$$\overline{\partial}_z\eta_h^{(s^*,\nu)}(\zeta, z) = \overline{\partial}_{z'}\eta_h^{(s^*,\nu)}(\zeta, z')|_{z'=z};$$

the point here is that h, too, depends on z. But

$$\overline{\partial}_z K^{(s^*,\nu)}(\zeta, z) = \overline{\partial}_\zeta\widetilde{K}(\zeta, z) \tag{4}$$

where

$$\widetilde{K}(\zeta, z) = -(n-1)c_n'\varphi^\nu\frac{\det[s_j^*, \overline{\partial}_z s_j^*, \overbrace{\overline{\partial}_\zeta s_j^*}^{n-2}]\wedge\omega_\zeta(s)}{\langle s^*, s\rangle^n}$$

(with the notation of §3; in particular we make use of a local coordinate system (U, χ) as in §3; the independence of $\widetilde{K}(\zeta, z)$ of the choice of (U, χ) is proved exactly as Lemma 3.1); this is proved in [**3**, p. 107]; $\widetilde{K}(\zeta, z)$ is defined for $z \in W$ and $\zeta \in W - \{z\}$.

But, by Theorem 3.1,

$$\overline{\partial}_\zeta(\overline{\partial}_z \eta_h^{(s^*, \nu)}) = \overline{\partial}_z K^{(s^*, \nu)}(\zeta, z). \tag{5}$$

It follows from (4) and (5) that

$$\overline{\partial}_\zeta[\overline{\partial}_z \eta_h^{(s^*, \nu)} - \widetilde{K}] = 0 \quad \text{in } W - Z(h).$$

Since $W - Z(h)$ is $(n-3)$-complete, it follows that there exists an $(n, n-3)$-form $\mu(\zeta, z)$ in ζ, whose coefficients are $(0, 1)$-forms in z (locally in (U, χ)), so that

$$\overline{\partial}_z \eta_h^{(s^*, \nu)} - \widetilde{K} = \overline{\partial}_\zeta \mu \quad \text{in } W - Z(h) \tag{6}$$

(this argument is similar to the remark following the proof of Theorem 3.1).

But (3), in view of (4) and (6), becomes:

$$\overline{\partial}_z F = \int_\Gamma f \overline{\partial}_\zeta \widetilde{K} - \int_{\partial\Gamma} f\widetilde{K} - \int_{\partial\Gamma} f \overline{\partial}_\zeta \mu,$$

from which, by Stokes's theorem and the fact that f is a CR-function we obtain $\overline{\partial}_z F = 0$; thus F is holomorphic in $W - \partial D$. An argument similar to that in [**6**, pp. 188–190] proves that $F = 0$ in $W - \overline{D}$ and that $F|_D$ is indeed a holomorphic extension of f. For the Plemelj type formula in the setting of Stein manifolds, which is required here, see [**5**].

This completes the proof of Theorem 4.1.

Comments. (i) The point of using the differential form $\widetilde{K}$ in the proof of Theorem 4.1 is that, although $\eta_h^{(s^*, \nu)}$ is not defined on $Z(h)$, $\overline{\partial}_z \eta_h^{(s^*, \nu)}$ is $\overline{\partial}_\zeta$-cohomologous to $\widetilde{K}$ in a neighborhood of $\partial\Gamma$, and $\widetilde{K}$ is defined in $W - \{z\}$.

(ii) A point which may be investigated further is to find geometric conditions under which equality (1) holds; for example, if $\dim_{\mathbb{C}}(Z(h) \cap Z(g)) \geq 1$, does it follow that (1) holds?

(iii) If $h, q \in \mathscr{P}_z$ and $h_1 = \cdots = h_{n-2}$ and $g_1 = \cdots = g_{n-2}$ $(n \geq 3)$ then the difference $\eta_h^{(s^*, \nu)} - \eta_g^{(s^*, \nu)}$ is $\overline{\partial}$-exact in $W - (Z(h) \cup Z(g))$ (this is proved in [**5**]), which implies that (1) holds in this case.

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