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FINITE WEIGHT PROJECTIONS IN VON NEUMANN ALGEBRAS

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The ideal of definition of a faithful semifinite normal weight on a countably decomposable von Neumann algebra is the set generated by all positive elements of finite weight. The set is a hereditary left ideal and therefore contains projections. In this paper the family of weights whose ideals of definition form projection lattices is completely characterized. These weights are the ones that are comparable to a combination of traces and normal functionals. A central spectral resolution is introduced and used to analyze the Radon-Nikodym derivatives of a weight with regard to a trace. Also introduced are two parameters that measure whether the ideal of definition contains two projections of least upper bound 1 and how close the weight is to being a trace respectively.

1. Introduction. The set of the projections of a two-sided ideal in a von Neumann algebra is a lattice because the set of projections is hereditary and closed under the Murray-von Neumann equivalence relation (if it contains a projection, it contains also all the projections in the algebra majorized by or equivalent to the projection [**16**]). The situation is quite different if we take a one-sided ideal. While still hereditary [**10**, §1.5.2], a one-sided ideal that is closed under the equivalence relation for projections is a two-sided ideal. However, there are interesting cases of one-sided ideals where the set of projections is nevertheless a lattice, e.g. the right ideal of "finite rank" operators in a type III_λ factor (cf. [**5**, §3]).

There are one-sided ideals whose set of projections is not a lattice. For example, if φ is a faithful semifinite normal weight (henceforth f.s.n. for short) on a von Neumann algebra R, then

$$N_\varphi = \{x \in R \mid \varphi(x^*x) < \infty\}$$

is a left ideal and

$$M_\varphi = \operatorname{span}\{x \in R^+ \mid \varphi(x) < \infty\} = N_\varphi^* N_\varphi$$

is a hereditary algebra [**15**, 10.14]. In [**5**, §7] the first two named authors applied a result by U. Haagerup ([**2**], cf. [**13**, 30.12]) relating to infinite weights in type III_λ factors to construct two projections

p, q in M_φ whose least upper bound $p \vee q$ was not in M_φ. In [**6**, Example 5.4], the same authors used a different technique employing the discrete crossed product decomposition of a type III_λ factor R to construct two projections p, q in M_φ with $p \vee q = 1$. Both constructions made use of the fact that the least upper bound of two distinct rank-one projections in $M_2(\mathbb{C})$ (the 2×2 complex matrix algebra) is the identity. A similar result, also depending on a 2×2 matrix construction, was obtained by A. Amann and the third named author in [**1**] for the null ideal

$$L_\omega = \{x \in R \mid \omega(x^*x) = 0\}$$

of a singular state $\omega \in R^*$ from quantum mechanics.

Motivated by the analogy between the ideals L_ω for a singular state ω and N_φ for an infinite weight φ, we determine in this paper necessary and sufficient conditions under which the set $P(M_\varphi)$ of the projections of the ideal of definition M_φ of an f.s.n. weight φ on a von Neumann algebra R is a lattice. Without loss of generality we may always assume that the weights that we consider are faithful (otherwise we could pass to the algebra reduced to the support of the weight). Since all the projections in M_φ are σ-finite and since the least upper bound of two σ-finite projections is also σ-finite, we reduce our considerations to σ-finite von Neumann algebras.

Our main result is:

THEOREM 1. *Let R be a σ-finite von Neumann algebra and let φ be a f.s.n. weight on R. Then $P(M_\varphi)$ is a lattice if and only if there is a decomposition of the identity into mutually orthogonal central projections $e + f + g = 1$ such that R_f is a semifinite algebra and R_g is a direct sum of type I_∞ factors equipped with the f.s.n. trace* Tr *(the direct sum of the canonical traces on the factors) so that*

(a) *φ restricted to R_e is a finite functional,*

(b) *$P(M_{\varphi(f\cdot)}) = P(M_\tau)$ for some f.s.n trace τ on R_f, and*

(c) $P(M_{\varphi(g\cdot)}) \subset P(M_{\mathrm{Tr}})$.

We define two parameters

$$I(\varphi) = \inf\{\varphi(p+q) \mid p \vee q = 1\,,\ p\,,\ q \text{ projections in } R\}$$

and

$$J(\varphi) = \sup\left\{\frac{\varphi(p \vee q)}{\varphi(p+q)} \mid 0 \neq p\,,\ q \in P(M_\varphi)\right\}$$

to study the lattice properties of $P(M_\varphi)$. The first $I(\varphi)$ measures how close the identity 1 is to being the least upper bound of projections in $P(M_\varphi)$ while the second $J(\varphi)$ measures how close φ is to being a trace.

For a type III algebra we see from Theorem 1 that $P(M_\varphi)$ is a lattice if and only if φ is a finite functional. For semifinite algebras we see from Theorem 1 that the situation is more complex. Here we exploit the properties of the Radon-Nikodym derivative h of φ with respect to a f.s.n. trace τ (cf. **[11]**). The Radon-Nikodym derivative h is a positive selfadjoint (possibly unbounded) operator affiliated with the centralizer algebra

$$R^\varphi = \{x \in R \mid \sigma_t^\varphi(x) \equiv x\}$$

such that $\varphi(x) = \tau(hx)$ for all $x \in R^+$, where

$$\tau(hx) = \tau(x^{1/2}hx^{1/2}) = \lim_n \tau(x^{1/2}h\chi(-\infty, n)(h)x^{1/2}).$$

Here $\chi E(h)$ denotes the spectral projection of h corresponding to the Borel set E. (When there is no possibility of confusion, we shall drop the reference to h and just write χE. We generally use the whole real line as the domain of the spectral resolution even for positive operators since it allows us to unify the notation when we analyze the essential central spectrum of an unbounded operator. The essential central spectrum $Z - \sigma^e(x)$ for a bounded operator x has been developed in **[4]** and **[14]**. Here we extend the concept of essential central spectrum to an unbounded selfadjoint operator h via the spectral resolution and arrive at a concept of central intervals. We calculate the parameters $I(\varphi)$ and $J(\varphi)$ and show the former is related to the central essential spectrum of the Radon-Nikodym derivative while the latter is related to the spread in the essential spectrum of the Radon-Nikodym derivative. In particular, we have that $J(\varphi) = 1$ if and only if φ is a trace.

We analyze $P(M_\varphi)$ for finite algebras separately. We find a canonical trace τ_φ associated to a f.s.n. weight φ and show that $P(M_\varphi)$ is a lattice if and only if $P(M_\varphi)$ and $P(M_{\tau_\varphi})$ coincide.

One of the tools used throughout this paper is the notion of φ-semifinite projection, i.e., a projection p such that the restriction of φ to R_p is still semifinite. We also use a 2×2 matrix construction to obtain the sum of two orthogonal projections as a least upper bound of two projections only one of which has to be controlled.

A few remarks about our notations: the algebra R operates on the Hilbert space H and has identity 1; Z denotes the center of R, $\widehat{Z}^+$ denotes the extended positive part of Z, and $\widetilde{Z}$ denotes the selfadjoint operators on H affiliated with Z. Recall that under the identification of Z with $L^\infty\{\Gamma, \mu\}$, where Γ is a locally compact space and μ is a positive Radon measure, the set $\widehat{Z}^+$ coincides with the set of μ-measurable extended real valued nonnegative functions and it is closed under least upper bounds. On the other hand, the set $\widetilde{Z}$ coincides with the set of all real valued measurable functions that are finite almost everywhere. For every projection p in R, R_p is the algebra pRp restricted to the space pH and $c(p)$ is the central support of p; $p \vee q$ and $p \wedge q$ are the least upper bound and the greatest lower bound respectively of the projections p and q; $R(x)$ and $N(x)$ are respectively the left support (i.e. the range projection) and the null projection of an operator x; k^+ and k^- are the positive part and the negative part respectively of a selfadjoint (possibly unbounded) operator k. For the rest of our notations we refer the reader to [**15**].

2. φ-semifinite projections. Let R be a σ-finite (i.e. countably decomposable) von Neumann algebra and let φ be a faithful semifinite normal weight on R (f.s.n for short). Let

$$M_\varphi = \operatorname{span}\{x \in R^+ \mid \varphi(x) < \infty\}.$$

DEFINITION 2.1. A projection $p \in R$ is said to be *φ-semifinite* (φ-s. for short) if the restriction of φ to R_p is semifinite. The projection p is said to be *φ-purely infinite* if the restriction of φ to R_p assumes only the values $\{0, \infty\}$.

Notice that the restriction of φ to R_p is always a faithful and normal weight, and it is semifinite if and only if $M_\varphi \cap R_p$ is σ-weakly dense in R_p.

We shall often use the following criterions for a projection to be φ-s.

LEMMA 2.2. *Let $p \in R$ be a projection; then the following conditions are equivalent.*

(i) *p is φ-s.*

(ii) *There is an $x \in M_\varphi^+$ such that $R(x) = p$.*

(iii) *There is a sequence of mutually orthogonal projections p_n in M_φ such that $p = \sum p_n$.*

Proof. Let p be φ-s. We can find a countable strongly dense subset $\{x_n\}$ in the unit ball of $M_\varphi \cap R_p^+$ because the unit ball of R_p^+ is metrizable in the strong operator topology [**8**, 5.7.46]. Then the series

$\sum 2^{-n}(1+\varphi(x_n))^{-1}x_n$ converges uniformly to an operator $x \in R_p^+$. We see that $R(x) = p$ due to the density of the set $\{x_n\}$. By the normality of φ, we get that

$$\varphi(x) = \sum 2^{-n}(1+\varphi(x_n))^{-1}\varphi(x_n) < \infty.$$

Assume now that there is an $x \in M_\varphi^+$ such that $R(x) = p$. The spectral projections $p_n = \chi[n^{-1}, (n+1)^{-1})(x)$ of x corresponding to the intervals $[n^{-1}, (n+1)^{-1})$ are mutually orthogonal with sum equal to $R(x)$. Moreover, each projection p_n is in M_φ since

$$\varphi(p_n) \le n\varphi(p_n x) \le n\varphi(x) < \infty.$$

Thus, the projection p is the sum of the sequence of mutually orthogonal projections $\{p_n\}$ in M_φ.

Finally, if p is the sum of a sequence of mutually orthogonal projections $\{p_n\}$ in M_φ, let $q_n = p_1 + \cdots + p_n$. The subset $\bigcup(R_{q_n})$ of M_φ is σ-weakly dense in R_p. □

Given a projection p in R, we can find a maximal sequence of mutually orthogonal φ-s. subprojections $\{p_n\}$ of p. By maximality, the projection $p - \sum p_n$ is φ-purely infinite. So p can be decomposed into the sum of a φ-s. and a φ-purely infinite projection. This decomposition is in general not unique. Indeed, the identity operator is φ-s. by definition but may be decomposed as a nontrivial sum of a φ-s. and a φ-purely infinite projection (see remarks after Proposition 2.4).

In finite algebras there are no φ-purely infinite projections.

PROPOSITION 2.3. *Every projection in a finite von Neumann algebra is φ-semifinite.*

Proof. Let $p \ne 1$ be an arbitrary nonzero projection in the finite von Neumann algebra R. Let τ be a f.n. finite trace with $\tau(1) = 1$. Let φ be a f.s.n. weight on R and let h be the Radon-Nikodym derivative of φ with respect to τ. By the normality of the trace there is some $n > 0$ and a spectral projection $q = \chi[0, n)(h)$ such that $\tau(q) > 1 - \tau(p)$. By the Parallelogram Law we have that

$$p \vee q - p \sim q - p \wedge q.$$

Then we have that

$$1 - \tau(p) \ge \tau(p \vee q) - \tau(p) = \tau(q) - \tau(p \wedge q) > 1 - \tau(p) - \tau(p \wedge q).$$

This shows that the projection $p \wedge q$ is not 0. Moreover

$$\varphi(p \wedge q) \leq \varphi(q) = \tau(hq) \leq n\tau(q) < \infty.$$

Thus, we have shown that every nonzero projection in R majorizes a nonzero projection in M_φ. By a maximality argument, we have that every projection can be written as the sum of a sequence of mutually orthogonal projections in M_φ. Thus, every projection is φ-s. by Lemma 2.2. □

If φ is a trace, then clearly there are also no φ-purely infinite projections. In the case of $B(H)$, there are φ-purely infinite projections if and only if the Radon-Nikodym derivative h of φ with respect to the canonical trace tr is unbounded. Indeed, on the one hand, if h is bounded then any finite projection is in M_φ. On the other hand, if h is unbounded, then there is a unit vector ξ not in the domain of $h^{1/2}$. Setting p equal to the one dimensional projection with range span ξ, and $h_n = h\chi[0, n)(h)$ we get that

$$\begin{aligned} \operatorname{tr}(hp) &= \lim_n \operatorname{tr}(ph_np) \\ &= \lim_n (h_n\xi, \xi) \\ &= \lim_n \|h_n^{1/2}\xi\|^2 = \infty. \end{aligned}$$

More generally, let $\varphi = \psi \otimes \operatorname{tr}(h\cdot)$ be a f.s.n weight on a von Neumann algebra of the form $R \otimes B(H)$. Again if h is unbounded, then we have

$$\varphi(q \otimes p) = \psi(q)\operatorname{tr}(hp) = \infty$$

for any nonzero projection $q \in R$. From this we conclude that $1 \otimes p$ is φ-purely infinite.

The φ-semifinite projections exhibit some of the standard properties associated with semifinite projections.

Proposition 2.4. *Let φ be f.s.n. weight on the von Neumann algebra R. Then*

(i) *The supremum of a countable set of φ-s. projections is φ-s.*

(ii) *Let p be a φ-s. projection and let $a \in R^\varphi$; then $R(apa^*)$ is φ-s.*

(iii) *Let p, q be two φ-s. projections with $p \geq q$. If $q \in M_\varphi$, then $p - q$ is φ-s.*

(iv) *Let R be semifinite, let τ be a f.s.n. trace on R and let h be the Radon-Nikodym derivative of φ with respect to τ; then a projection p is φ-s. if and only if php is selfadjoint.*

Proof (i). Let $\{p_n\}$ be a sequence of φ-s. projections. Let $\{p_{nm}\}$ be sequences of projections in M_φ with $p_n = \sum_m p_{nm}$. Then

$$x = \sum_{m,n} 2^{-m-n}(1+\varphi(p_{nm}))^{-1}p_{nm}$$

is an element in M_φ and has range projection equal to $\sup p_n$. By Lemma 2.2, this implies that $\sup p_n$ is φ-s.

Proof (ii). By Lemma 2.2 we can find an $x \in M_\varphi^+$ with $R(x) = p$. Since $axa^* \in M_\varphi$ because M_φ is an R^φ-module, and since $R(apa^*) = R(axa^*)$, we see that $R(apa^*)$ is φ-s.

Proof (iii). By Lemma 2.2 we can find an $x \in M_\varphi^+$ with $R(x) = p$. Then

$$(p-q)x(p-q) \leq 2pxp + 2qxq \leq 2x + 2\|x\|q,$$

whence $(p-q)x(p-q) \in M_\varphi^+$. Since $p - q = R((p-q)x(p-q))$, we conclude, again by Lemma 2.2, that $p - q$ is φ-s.

Proof (iv). Assume first that $p \in M_\varphi$ and let $h_n = h\chi[0, n)(h)$. The sequence ph_np increases monotonically and hence it has a limit k belonging to the extended positive part $\widehat{M}^+$ of M, and $\hat{k}$ has a unique representation $\hat{k} = k + \infty q$ where $k = q^\perp k q^\perp$ is a positive selfadjoint operator affiliated with M and $q \in M$ is a projection [2, Remarks after Definition 1.3, Lemma 1.4 and Theorem 1.5]. Then τ has an extension $\hat{\tau}$ to $\widehat{M}^+$ and

$$\varphi(p) = \lim_n \tau(ph_np) = \hat{\tau}(\hat{k}) = \tau(k) + \infty\tau(q),$$

whence $q = 0$ [2, Proposition 1.10]. Thus $ph_np \uparrow k$ in the sense that

$$(ph_np\xi, \xi) \uparrow \begin{cases} \|k^{1/2}\xi\|^2 & \text{if } \xi \in D(k^{1/2}), \\ \infty & \text{if } \xi \notin D(k^{1/2}). \end{cases}$$

Since we also have

$$(ph_np\xi, \xi) \uparrow \begin{cases} \|h^{1/2}p\xi\|^2 & \text{if } \xi \in D(h^{1/2}p), \\ \infty & \text{if } \xi \notin D(h^{1/2}p), \end{cases}$$

we obtain that $D(k^{1/2}) = D(h^{1/2}p)$ and hence that $h^{1/2}p$ is densely defined. (We can actually show that $k = php$, but we don't need this fact here.)

Assume now that p is φ-s., so that by Lemma 2.2 there is a sequence of mutually orthogonal projections $p_k \in M_\varphi$ such that $p = \sum p_k$. We have just proven that $h^{1/2}p_k$ is densely defined for each k, and a routine argument shows that then $h^{1/2}p$ too is densely defined. Clearly, $h^{1/2}p$ is closed and since $(h^{1/2}p)^* \supseteq ph^{1/2}$, we see that $ph^{1/2}$ too is closed. It is easy to verify that $(ph^{1/2})^* = h^{1/2}p$, and since $ph^{1/2} = (ph^{1/2})^{**}$, we have also $(h^{1/2}p)^* = ph^{1/2}$. Therefore, by **[8**, Theorem 2.7.8 (v)], we obtain that $(h^{1/2}p)^*ph^{1/2} = php$ is selfadjoint.

Conversely, if php is selfadjoint then $\tau(php)$ is an s.n. weight on R which coincides with φ on R_p. □

In general the condition $q \in M_\varphi$ in (iii) cannot be relaxed. We can show this by refining the example after Proposition 2.3. Let $R = B(H)$, let h be a positive injective selfadjoint unbounded operator on H, and let φ be the f.s.n. weight defined by $\varphi = \mathrm{tr}(h\cdot)$. Working with the spectral resolution of h, we can find an orthonormal basis $\{\xi_n\}$ for H such that $\xi_1 \notin D(h^{1/2})$ while $\{\xi_n \mid n \geq 2\}$ is contained in $D(h^{1/2})$. Setting p_n equal to the one dimensional projection of H on the subspace generated by ξ_n, we get that the φ-purely infinite projection p_1 can be written as $p_1 = 1 - \sum\{p_n \mid n \geq 2\}$ whereas 1 and $\sum\{p_n \mid n \geq 2\}$ are φ-s.

Notice also that by (ii) every projection in R^φ and in particular every central projection is φ-s.

LEMMA 2.5. *Let φ be a f.s.n. weight on R; then, for every projection p in R and every φ-s. projection q with $p \prec q$, there is a φ-s. projection q' with $q' \leq q$ and $p \sim q'$.*

Proof. There is a projection $p' \sim p$ with $p' \leq q$. So there is no loss of generality in the assumption that $p \leq q$. Since the weight φ restricted to the algebra R_q is f.s.n., we may assume also that $q = 1$. If p were properly infinite, then it would be equivalent to its central support $c(p)$, which is φ-s. by Proposition 2.4(i), and if R were finite, then p would be φ-s. by Proposition 2.3. Thus we can assume that p is a finite projection of central support 1 and that R is a properly infinite semifinite algebra. Let Φ be a faithful normal operator valued trace on R with $\Phi(p) = 1$, let ω be a f.n. state on Z, let $\tau = \omega \circ \Phi$

be the corresponding f.s.n. (scalar) trace, and let h be the Radon-Nikodym derivative of φ with respect to τ. We may find a sequence $\{n_i\}$ of integers and a sequence $\{e_i\}$ of orthogonal central projections of sum 1 with $e_i \leq \Phi(\chi(-\infty, n_i)(h))e_i$. Then we have that

$$\Phi(p) = 1 \leq \sum \Phi(\chi(-\infty, n_i)(h))e_i = \Phi\left(\sum \chi(-\infty, n_i)(h)e_i\right).$$

This proves that

$$p \sim q' \leq \sum \chi(-\infty, n_i)(h)e_i.$$

Since

$$\varphi(q'e_i) = \tau(hq'e_i) \leq n_i\tau(q'e_i) \leq n_i\tau(p) < \infty,$$

we see that $q' = \sum q'e_i$ is φ-s. by Lemma 2.2. □

3. A 2×2 matrix construction. Now we can start to investigate the condition on a f.s.n. weight φ under which the set $P(M_\varphi)$ of projections of M_φ is a lattice.

LEMMA 3.1. *Let p, s be two equivalent and orthogonal φ-s. projections in R with $p \in M_\varphi$. Then for every $\varepsilon > 0$ there is a projection $q \in M_\varphi$ such that $p \vee q = p + s$ and $\varphi(q) < \varphi(p) + \varepsilon$.*

Proof. We actually obtain a projection q with $q \sim p \sim s$. By Lemma 2.2 we can decompose s into a sum $\sum s_n$ of mutually orthogonal projections s_n in M_φ. This decomposition induces a corresponding partition of p into the sum of mutually orthogonal projections $p = \sum p_n$ with $p_n \sim s_n$. There are partial isometries $u_n \in R$ implementing this equivalence, i.e.,

$$u_n^* u_n = p_n \quad \text{and} \quad u_n u_n^* = s_n.$$

Since p_n and s_n are in M_φ, so are also u_n and u_n^*, and by the Cauchy-Schwarz inequality $|\varphi(u_n)|$ and $|\varphi(u_n^*)|$ are both bounded by $\sqrt{\varphi(p_n)\varphi(s_n)}$. Choose also a sequence $\delta_n \in (0, 1)$ such that

$$\sum \left\{\delta_n\varphi(s_n) + 2\sqrt{\delta_n\varphi(p_n)\varphi(s_n)}\right\} < \varepsilon.$$

Let $R_n = \operatorname{span}\{p_n, s_n, u_n, u_n^*\}$; then R_n is a subalgebra of R naturally isomorphic to $M_2(\mathbb{C})$. Let $q_n \in R_n$ be the projection corresponding to

$$\begin{bmatrix} 1-\delta_n & \sqrt{\delta_n(1-\delta_n)} \\ \sqrt{\delta_n(1-\delta_n)} & \delta_n \end{bmatrix}$$

i.e.,

$$q_n = (1-\delta_n)p_n + \delta_n s_n + \sqrt{\delta_n(1-\delta_n)}(u_n + u_n^*).$$

Then $p_n \neq q_n$ because $\delta_n \neq 0$, and hence $p_n \vee q_n = p_n + s_n$. Let $q = \sum q_n$; then it is easy to verify that

$$p \vee q = \sum (p_n \vee q_n) = \sum (p_n + s_n) = p + s.$$

On the other hand,

$$\begin{aligned}\varphi(q_n) &\leq (1-\delta_n)\varphi(p_n) + \delta_n\varphi(s_n) + \sqrt{\delta_n(1-\delta_n)}(|\varphi(u_n)| + |\varphi(u_n^*)|)\\ &\leq \varphi(p_n) + \delta_n\varphi(s_n) + 2\sqrt{\delta_n\varphi(p_n)\varphi(s_n)},\end{aligned}$$

so that

$$\varphi(q) \leq \sum \varphi(p_n) + \sum \left\{\delta_n\varphi(s_n) + 2\sqrt{\delta_n\varphi(p_n)\varphi(s_n)}\right\} \leq \varphi(p) + \varepsilon. \quad \square$$

Since $\varphi(p) < \infty$ and q is φ-s. imply that $p \vee q - p$ is φ-s. due to Proposition 2.4(i), the hypothesis that s is φ-s. cannot be avoided in Lemma 3.1.

PROPOSITION 3.2. *The set of projections $P(M_\varphi)$ is not a lattice if and only if there are two equivalent and orthogonal φ-s. projections r and s in R such that $r \in M_\varphi$ and $s \notin M_\varphi$.*

Proof. First suppose that $P(M_\varphi)$ is not a lattice. Let p and q be two projections in M_φ such that $p \vee q$ is not in M_φ. Since

$$(p - p\wedge q)\vee q = p \vee q \quad \text{and} \quad (p - p \wedge q)\wedge q = 0,$$

by passing to subprojections if necessary, we may assume that $p \wedge q = 0$. By the Comparison Theorem there is a central projection e such that

$$pe \sim q_1 \leq qe$$

and

$$q(1-e) \sim p_1 \leq p(1-e).$$

The projections qe and $p(1-e)$ are in M_φ and so the projection $r = p_1 + q_1$ is in M_φ too. However, the projection

$$s = (p \vee q - q)e + (p \vee q - p)(1-e)$$

is φ-s. (Proposition 2.4(i) and (iii)) but it is not in M_φ. Also, s is orthogonal to r and it is equivalent to it via the Parallelogram Law

$$(p \vee q - q)e + (p \vee q - p)(1-e) \sim pe + q(1-e) \sim p_1 + q_1$$

due to the assumption that $p \wedge q = 0$.

Now suppose that r and s are orthogonal equivalent φ-s. projections with $r \in M_\varphi$ and $s \notin M_\varphi$. By Lemma 3.1 there is a projection q in R with $r \vee q = r + s$ and with $\varphi(q) \leq \varphi(r) + 1 < \infty$. Since $r + s \notin M_\varphi$, we see that $P(M_\varphi)$ is not a lattice. □

PROPOSITION 3.3. *Let R be a type* III *algebra and let φ be a f.s.n. weight on R. Then the set $P(M_\varphi)$ is a lattice if and only if φ is finite.*

Proof. The condition is clearly sufficient. Assume now that $P(M_\varphi)$ is a lattice. Let $\{p_n\}$ be a maximal set (necessarily countable since R is σ-finite) of nonzero projections in $P(M_\varphi)$ with mutually orthogonal central supports. Since $e = 1 - c(\sum p_n)$ is φ-s. by Proposition 2.4(ii), by the maximality of the family we see that $e = 0$. Since we may write each p_n as the sum of a sequence of mutually orthogonal equivalent projections, we may assume without loss of generality that $\varphi(p_n) < 2^{-n}$ for every $n = 1, 2, \ldots$. The projection $p = \sum p_n$ is then a projection in M_φ of central support 1. Since R is type III and σ-finite, by passing to a subprojection of p, we may assume that $p \sim 1 - p \sim 1$. Since $1 - p$ is φ-s. due to Proposition 2.4, by Lemma 3.1 we can find a projection q such that

$$p \vee q = p + (1 - p) = 1 \quad \text{and} \quad \varphi(q) \leq \varphi(p) + 1 < \infty.$$

By assumption the set $P(M_\varphi)$ is a lattice, and therefore, $\varphi(1) < \infty$. Thus, the weight φ is a finite normal functional. □

We now introduce two numbers associated with every f.s.n weight.

DEFINITION 3.4. Let φ be a f.s.n. weight on the von Neumann algebra R. Then let

$$I(\varphi) = \inf\{\varphi(p + q) \mid p \vee q = 1, p, q \text{ projections in } R\},$$

and

$$J(\varphi) = \sup\left\{\frac{\varphi(p \vee q)}{\varphi(p + q)} \mid p, q \in M_\varphi \text{ and } p + q \neq 0\right\}.$$

We have already seen that

$$I(\varphi) = \inf\{\varphi(p + q) \mid p \vee q = 1, p \wedge q = 0, p, q \text{ projections in } R\}$$

since $(p - p \wedge q) \vee q = p \vee q$ and $(p - p \wedge q) \wedge q = 0$.

We use the numbers $I(\varphi)$ and $J(\varphi)$ to determine when $P(M_\varphi)$ is a lattice.

PROPOSITION 3.5. *If R is a type* III *algebra then $I(\varphi) = 0$ and $J(\varphi) = \infty$.*

Proof. By slightly changing the proof of Proposition 3.3, given $\varepsilon > 0$, we can find two projections p and q in R with $p \vee q = 1$ and $\varphi(p) + \varphi(q) < \varepsilon$. This proves that $I(\varphi) = 0$.

We can now see that $J(\varphi) = \infty$ since the supremum of $\varphi(p \vee q)/\varphi(p+q)$ over the set of projections p and q with p and q in M_φ and $p \vee q = 1$ is already ∞. □

Actually, we can see that $J(\varphi) = \infty$ as soon as R has a nonzero type III direct summand. Also $I(\varphi) = 0$ if φ is finite and R is properly infinite.

4. The semifinite case. Comparison of M_φ and M_τ. We begin with a discussion of "central intervals" which we need throughout the rest of the present work. Let Z be the center of the von Neumann algebra R and $\widetilde{Z}$ be the set of all densely defined selfadjoint elements affiliated with Z. For each $z \in \widetilde{Z}$, there is a sequence $\{e_n\}$ of mutually orthogonal projections in Z of sum 1 such that ze_n is in Z for every n. If x and y are in $\widetilde{Z}$, then we write $x < y$ (respectively, $x \leq y$) if there is a sequence $\{e_n\}$ of mutually orthogonal projections in Z of sum 1 such that xe_n and ye_n are bounded and $xe_n < ye_n$ (respectively, $xe_n \leq ye_n$) for every n.

Now let h be a selfadjoint element affiliated with R and let χ be the spectral resolution of h. Let $\{e_n\}$ and $\{f_n\}$ be sequences of mutually orthogonal projections in Z of sum 1 and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in R. If $\sum \alpha_n e_n \leq \sum \beta_n f_n$, we have that

$$\sum \chi(-\infty, \alpha_n)e_n \leq \sum \chi(-\infty, \beta_n)f_n.$$

Now let $z \in \widetilde{Z}$. Then the family of all sums of the form $\sum \alpha_n e_n$ with $\sum \alpha_n e_n < z$ is upward directed. Thus, the following definition is possible.

DEFINITION 4.1. Let h be a selfadjoint element affiliated with R with spectral resolution χ and let $z \in \widetilde{Z}$. Then let $\chi(-\infty, z)(h) = \chi(-\infty, z)$ be the least upper bound of the increasing family of projections of the form $\sum \chi(-\infty, \alpha_n)e_n$ for all sequences $\{e_n\}$ of central projections of sum 1 and sequences $\{\alpha_n\}$ of real numbers with $\sum \alpha_n e_n < z$.

We similarly define the other central spectral projection $\chi(z, \infty)$ as the least upper bound of the increasing family of projections of the form $\sum \chi(\alpha_n, \infty)e_n$ for all sequences $\{e_n\}$ of central projections of sum 1 and sequences $\{\alpha_n\}$ of real numbers with $\sum \alpha_n e_n > z$. We note that both $\chi(-\infty, z)$ and $\chi(z, \infty)$ are in the von Neumann algebra generated by Z and h.

Now let $z \in \widetilde{Z}$ and let h be a selfadjoint element affiliated with R. Then the operator $z - h$ is a densely defined operator. In fact, there is a sequence of mutually orthogonal projections $\{p_n\}$ in R of sum 1 commuting with z and h such that $p_n H$ is contained in the domain $D(z) \cap D(h)$ of $z - h$. The closure of this operator is a selfadjoint operator affiliated with R. We again denote this selfadjoint operator by $z - h$. The operator $z - h$ is affiliated with the von Neumann algebra generated by Z and h as are the positive selfadjoint operators

$$(z-h)^+ = (z-h)\chi(0, \infty)(z-h)$$

and

$$(z-h)^- = -(z-h)\chi(-\infty, 0)(z-h)$$

(cf. [**8**, 2.7.10]).

We have a different characterization of the central spectral projections.

Proposition 4.2. *Let R be a von Neumann algebra and let h be a selfadjoint element affiliated with R. Then, for every $z \in \widetilde{Z}$, $\chi(-\infty, z)(h) = R((z-h)^+)$ and $\chi(z, \infty)(h) = R((z-h)^-)$.*

Proof. We verify only the first relationship. Since $z - h$ is a selfadjoint operator affiliated with the von Neumann algebra generated by Z and h, it has a spectral resolution χ' and $R((z-h)^+) = \chi'(0, \infty)$. In particular, the spectral resolutions χ and χ' commute. Now let $\{e_n\}$ be a sequence of central projections of sum 1 and let $\{\alpha_n\}$ be a sequence of real numbers such that $\sum \alpha_n e_n < z$. There is no loss of generality in the assumption that $ze_n - \alpha_n e_n$ is a bounded invertible positive operator since we may decompose the e_n further. Then we have, for $-m < \alpha_n$, that

$$z\chi(-m, \alpha_n)e_n - \chi(-m, \alpha_n)he_n \geq z\chi(-m, \alpha_n)e_n - \alpha_n\chi(-m, \alpha_n)e_n$$

shows that $z\chi(-m, \alpha_n)e_n - \chi(-m, \alpha_n)he_n$ is an invertible positive operator on $\chi(-m, \alpha_n)e_nH$. Since $\chi(-m, \alpha_n)e_n$ commutes with $z - h$, we have that

$$R((z-h)^+) \geq \chi(-m, \alpha_n)e_n.$$

By taking the least upper bounds, we have that

$$R((z-h)^+) \geq \chi(-\infty, z).$$

Conversely, let k be given. We can find a sequence of central projections $\{e_n\}$ of sum 1 and a sequence $\{\alpha_n\}$ of real numbers such that

$$(\alpha_n - (1/3k))e_n < ze_n < (\alpha_n + (1/3k))e_n.$$

For $p = 1, 2, \ldots$ we have that

$$\begin{aligned} \frac{1}{k} e_n \chi'\left(\frac{1}{k}, p\right) \chi\left(\alpha_n - \frac{1}{3k}, m\right) &\leq (z-h)\chi'\left(\frac{1}{k}, p\right) \chi\left(\alpha_n - \frac{1}{3k}, m\right) \\ &\leq \frac{2}{3k} e_n \chi'\left(\frac{1}{k}, p\right) \chi\left(\alpha_n - \frac{1}{3k}, m\right). \end{aligned}$$

This shows that

$$e_n \chi'(1/k, p)\chi(\alpha_n - (1/3k), m) = 0$$

for all m and p. By taking least upper bounds, we get

$$e_n \chi'(1/k, \infty)\chi(\alpha_n - (1/3k), \infty) = 0,$$

and consequently, that

$$e_n \chi'(1/k, \infty) \leq e_n \chi(-\infty, \alpha_n - (1/3k)] \leq e_n \chi(-\infty, z).$$

Summing over n, we get the inequality

$$\chi'(0, -\infty) \leq \chi(-\infty, z),$$

which is the reverse of the inequality found in the last paragraph. □

From the preceding proposition we see that the usual properties of spectral resolutions are true for the central spectral resolutions $\chi(-\infty, z)(h) = \chi(-\infty, z)$ on $\widetilde{Z}$. In particular, we note the properties:

(1) $\chi(-\infty, z) + \chi[z] + \chi(z, \infty) = 1$;
(2) $h\chi(-\infty, z) \leq z\chi(-\infty, z)$;
(3) $h\chi(z, \infty) \geq z\chi(z, \infty)$; and
(4) $h\chi[z] = z\chi[z]$.

Here we let equation (1) define $\chi[z]$. From (4) we see that all subprojections of $\chi[z]$ are in $\{h\}'' \vee Z$. In addition $\chi(-\infty, z)$ has the usual continuity properties

(5) $\text{lub}\{\chi(-\infty, z) \mid z < x\} = \chi(-\infty, x)$

as well as translation properties

(6) $\chi(-\infty, wz)(wh) = \chi(-\infty, z)(h)$, for every $w > 0$ affiliated with Z and

(7) $\chi(-\infty, z)(x+h) = \chi(-\infty, z-x)(h)$ for x in $\widetilde{Z}$. Here (6) arises from the relation

$$\begin{aligned}\chi(-\infty, wz)(wh) &= R((wz - wh)^+) = R(w(z-h)^+) \\ &= R((z-h)^+) = \chi(-\infty, z)(h)\end{aligned}$$

while (7) follows from the relation

$$R((z-(x+h))^+) = R(((z-x)-h)^+).$$

We now recall some facts on the essential central spectrum from [4] and [14]. Let R be a properly infinite semifinite algebra with center Z and let J be the ideal in R generated by the finite projections of R. Let Ω be the maximal ideal space of Z. For $\omega \in \Omega$, let $J(\omega)$ be the ideal of R generated by J and ω. Then for every selfadjoint element h in R, the set $Z-\sigma^e(h)$ of all $z \in Z$ such that $z^\frown(\omega)$ is in the spectrum of h modulo $J(\omega)$ for every $\omega \in \Omega$ is called the *essential central spectrum* of h. Here $z^\frown$ is the Gelfand transform of z. The essential central spectrum is nonempty.

PROPOSITION 4.3. *Let z be in the essential central spectrum of the selfadjoint operator h in the properly infinite semifinite von Neumann algebra R. Let χ be the spectral resolution of h. Then*

$$\chi(z-w, z+w) = \chi(-\infty, z+w)\chi(z-w, \infty) \sim 1$$

for any w in Z with $w > 0$.

Proof. There is a sequence of orthogonal central projections $\{e_n\}$ of sum 1 and sequence $\{\varepsilon_n\}$ of strictly positive numbers such that $\varepsilon_n e_n < w e_n$ for every $n = 1, 2, \ldots$. Let p_n be the spectral projection of $(z-h)e_n$ corresponding to the interval $[-\varepsilon_n, \varepsilon_n]$. Then $p_n \sim e_n$ [4, Proposition 3.13]. But we have that

$$p_n \leq \chi(z-w, z+w)e_n$$

by the property (7) listed for the spectral resolutions. So we have that $\chi(z-w, z+w) \sim 1$. □

We start by comparing $P(M_\varphi)$ to $P(M_\tau)$. We need the following lemma.

LEMMA 4.4. *Let* φ *be a f.s.n. weight on a semifinite von Neumann algebra* R. *Let* τ *be a f.s.n. trace on* R *and let* χ *be the spectral resolution of the Radon-Nikodym derivative of* φ *with respect to* τ. *Then*

(i) *if* $\tau(\chi[\beta, \infty)) + \varphi(\chi[\beta, \infty)) = \infty$ *for every* $\beta > 0$, *then there is an element* x *in* R *such that* $\varphi(x) = \infty$ *and* $\tau(x) < \infty$.

Furthermore, if R *has no type* I *factor direct summands, then the following are true*:

(ii) *if* $\tau(\chi[\beta, \infty)) + \varphi(\chi[\beta, \infty)) = \infty$ *for every* $\beta > 0$, *then, for every* $\gamma > 0$, *there is a* φ*-semifinite projection* p *in* R *majorized by* $\chi[\gamma, \infty)$ *such that* $\varphi(p) = \infty$ *and* $\tau(p) < \infty$; *and*

(iii) *if* $\tau(\chi[\beta, \infty)e) + \varphi(\chi[\beta, \infty)e) = \infty$ *for every* $\beta > 0$ *and every nonzero central projection* e, *then there are orthogonal equivalent* φ*-semifinite projections* p *and* q *in* R *such that* $\varphi(p) < \infty$, $\varphi(q) = \infty$, *and* $\tau(p) = \tau(q) < \infty$.

Proof (i) *and* (ii). There is no loss of generality in the assumption that $\tau(\chi[\beta, \infty)) = \infty$ for every $\beta > 0$. Indeed, if there is a $\beta > 0$ with $\tau(\chi[\beta, \infty)) < \infty$, then there is no loss of generality in the assumption that $\beta > \gamma$. The projection $p = \chi[\beta, \infty)$ then satisfies the requirements of (i) and (ii) since it is φ-semifinite such that $\varphi(p) = \infty$ and $\tau(p) < \infty$. So we assume that $\tau(\chi[\beta, \infty)) = \infty$ for every $\beta > 0$. By induction, we can find a monotonely increasing sequence $\{\beta_n\}$ of real numbers with $\beta_n \geq 2^n$ and projections $p_n \leq \chi[\beta_n, \beta_{n+1})$ with $2^{-n} \leq \tau(p_n) < \infty$ for every $n = 1, 2, \ldots$. We start the induction with $\beta_1 > \gamma$. Suppose we have constructed $p_1, \ldots, p_n$ and $\beta_1, \ldots, \beta_{n+1}$. Because

$$\mathrm{lub}_\gamma\, \tau(\chi[\beta_{n+1}, \gamma)) = \infty$$

by hypothesis, there is a $\beta_{n+2} > \max\{\beta_{n+1}, 2^{n+1}\}$ such that $\tau(\chi[\beta_{n+1}, \beta_{n+2})) \geq 2^{n+1}$. There is a projection p_{n+1} with $p_{n+1} \leq \chi[\beta_{n+1}, \beta_{n+2})$ and $2^{-n-1} \leq \tau(p_{n+1}) < \infty$. This completes the induction step. We note we may assume that $\tau(p_n) = 2^{-n}$ for every n provided that R has no type I factor direct summands. Setting $x = \sum 2^{-n}\tau(p_n)^{-1}p_n$ in the general case (respectively, $p = \sum p_n$ in the case that R has no type I factor direct summands), we get a positive element x (respectively, a φ-semifinite projection p) in R such that

$$\tau(x) = \sum 2^{-n}\tau(p_n)^{-1}\tau(p_n) = \sum 2^{-n} < \infty$$

and

$$\varphi(p) = \sum 2^{-n}\tau(p_n)^{-1}\varphi(p_n) \geq \sum \beta_n 2^{-n}\tau(p_n)^{-1}\tau(p_n) = \infty$$

(respectively,

$$\tau(p) = \sum \tau(p_n) = \sum 2^{-n} < \infty$$

and

$$\varphi(p) = \sum \varphi(p_n) \geq \sum \beta_n \tau(p_n) = \infty).$$

Proof (iii) *Case* I. First assume that $\tau(\chi[\beta, \infty)e) = \infty$ for every $\beta > 0$ and every nonzero central projection e. Choose β_0 so that $\chi(-\infty, \beta_0) \neq 0$ and let p be a nonzero projection of finite trace majorized by $\chi(-\infty, \beta_0)$. Then $\varphi(p) \leq \beta_0\tau(p) < \infty$. Using the fact that R has no type I factor direct summands, we write p as an infinite sum of mutually orthogonal nonzero projections $p = \sum p_n$. There is an increasing sequence $\beta_n \geq \beta_0$ such that $\sum \beta_n\tau(p_n) = \infty$. We now construct by induction a sequence of mutually orthogonal φ-s. projections $q_n \leq \chi[\beta_n, \infty)$ such that $p_n \sim q_n$. Suppose that we have constructed mutually orthogonal projections $q_1, q_2, \dots, q_n$ and a sequence $\{e_{ni}\}_i$ of mutually orthogonal central projections of sum 1 and a corresponding sequence $\{\gamma_{ni}\}_i$ of positive numbers such that (1) $q_j \leq \chi[\beta_j, \infty)$ for $1 \leq j \leq n$; (2) $p_j \sim q_j$ for $1 \leq j \leq n$; and (3) $(q_1 + \cdots + q_n)e_{ni} \leq \chi[0, \gamma_{ni})e_{ni}$ for all i. We construct q_{n+1}, $\{e_{n+1i}\}_i$, and $\{\gamma_{n+1i}\}_i$ satisfying (1), (2) and (3). There is no loss of generality in the assumption that $e_{ni} = 1$. Then let $\beta = \gamma_{ni}$. Then it is sufficient to find a single nonzero central projection e, a $\gamma > 0$, and a projection q_{n+1} with q_{n+1} orthogonal to $q_1, q_2, \dots, q_n$ and

(1) $q_{n+1} \leq \chi[\beta_{n+1}, \infty)$;

(2) $ep_{n+1} \sim q_{n+1}$; and

(3) $(q_1 + \cdots + q_n + q_{n+1})e \leq \chi[0, \gamma)e$.

Indeed, a maximal set of nonzero mutually orthogonal central projections e satisfying the foregoing properties will have sum 1. To find q_{n+1} let Φ be an operator valued trace on R. We have that

$$q_1 + \cdots + q_n \leq \chi[0, \beta),$$

for some $\beta > 0$ due to (3) and the assumption that $e_{ni} = 1$. Let $\delta = \beta_{n+1} \vee \beta$. We have that $\chi[\delta, \infty) \sim 1$. Since $\mathrm{lub}\{\chi[\delta, \gamma) \mid \delta < \gamma\} = \chi[\delta, \infty)$, there is a nonzero central projection e and a $\gamma > \delta$ with $e\Phi(\chi[\delta, \gamma)) \geq e\Phi(p_{n+1})$. This means that ep_{n+1} is equivalent to a subprojection q_{n+1} of the projection $e\chi[\delta, \gamma)$. Then we have that

(1) $q_{n+1} \leq e\chi[\delta, \gamma) \leq e\chi[\beta_{n+1}, \infty)$;

(2) $ep_{n+1} \sim q_{n+1}$; and

(3) $(q_1 + \cdots + q_n + q_{n+1})e \leq \chi[0, \beta)e + e\chi[\delta, \gamma) \leq e\chi[0, \gamma)$.

This completes the induction step.

We notice that each projection $q_n e_{ni}$ is in M_φ since

$$\varphi(q_n e_{ni}) \leq \varphi(\chi[0, \beta_{ni}]q_n) \leq \beta_{ni}\tau(q_n) = \beta_{ni}\tau(p_n) < \infty.$$

Thus, the projections $q_n = \sum_i q_n e_{ni}$ and $q = \sum_n q_n$ are φ-semifinite due to Proposition 2.4(i). We also have that

$$p = \sum p_n \sim \sum q_n = q$$

and

$$\varphi(q) = \sum \varphi(q_n) \geq \sum \beta_n \tau(p_n) = \infty.$$

This completes the proof of the first case.

Proof (iii) *Case* II. Now assume that $\tau(\chi[\beta_0, \infty)e) < \infty$ for some $\beta_0 > 0$ and some nonzero central projection e. Then we must have that $\varphi(\chi[\beta_0, \infty)f) = \infty$ for every nonzero central projection f majorized by e. Since φ is a semifinite faithful normal weight on the semifinite algebra R_e without type I factor direct summands, there is no loss of generality in the assumption that $\tau(\chi[\beta_0, \infty)) < \infty$ and that $\varphi(\chi[\beta, \infty)e) = \infty$ for every nonzero central projection e and every $\beta \geq \beta_0$. By using the normality of φ and τ, we can find two monotonely increasing sequences $\beta_0 < \beta_n < \gamma_n \leq \beta_{n+1}$ such that $\tau(\chi[\beta_n, \infty)) \leq 2^{-n}$ and $\varphi(\chi[\beta_n, \gamma_n)) \geq 1$. Let $q_n = \chi[\beta_n, \gamma_n)$ and let $q = \sum q_n$. We also note that q is a φ-s. projection (Proposition 2.4(ii)). By construction, we have that $q \notin M_\varphi$ but $q \in M_\tau$. Since $\chi[\beta_0, \infty)$ is a finite projection, we have that $\chi(-\infty, \beta_0) \sim 1$. Because $\chi(-\infty, \beta_0)$ is a φ-s. projection, we can find a φ-s. projection $p \leq \chi(-\infty, \beta_0)$ with $p \sim q$ (Lemma 2.5). Then $\varphi(p) \leq \beta_0\tau(p) < \infty$. □

The next lemma treats the lower part of the spectral resolution of the Radon-Nikodym derivative in a manner similar to Lemma 4.4.

LEMMA 4.5. *Let φ be a f.s.n. weight on a semifinite von Neumann algebra R. Let τ be a f.s.n. trace on R and let χ be the spectral resolution of the Radon-Nikodym derivative of φ with respect to τ. Then*

(i) *if $\tau(\chi(-\infty, \alpha)) = \infty$ for every $\alpha > 0$, then, for $\varepsilon > 0$, there is a positive element x in R such that $\varphi(x) < \varepsilon$ and $\tau(x) = \infty$; and*

(ii) *if* $\tau(\chi(-\infty, \alpha)e) = \infty$ *for every* $\alpha > 0$ *and every nonzero central projection* e, *then, given* $\varepsilon > 0$, *there is a projection* p *in* R *such that* $\varphi(p) < \varepsilon$ *and* $p \sim 1 - p \sim 1$.

Furthermore,

(iii) *if* R *has no type* I *factor direct summands, and if* $\tau(\chi(-\infty, \alpha)) = \infty$ *for every* $\alpha > 0$, *then, for* $\varepsilon > 0$ *and every* $\gamma > 0$, *there is a projection* p *in* R *majorized by* $\chi(-\infty, \gamma)$ *such that* $\varphi(p) < \varepsilon$ *and* $\tau(p) = \infty$.

Proof (i) *and* (iii). Let $\{\alpha_n\}$ be a strictly decreasing sequence of positive real numbers such that $\alpha_1 = \gamma$ and such that $\sum \alpha_n < \infty$. By induction we can find a decreasing sequence $\{\beta_n\}$ of positive numbers such that $\beta_n \leq \alpha_n$ and a sequence of projections p_n with $p_n \leq \chi(\beta_{n+1}, \beta_n]$ such that $\alpha_n \leq \tau(p_n) < \infty$. As in Lemma 4.4, we can find a projection p_n with $p_n \leq \chi(\beta_{n+1}, \beta_n]$ such that $\alpha_n = \tau(p_n)$ provided R has no type I factor direct summands. Setting $x = \sum \tau(p_n)^{-1} p_n$ in the general case (respectively, $p = \sum p_n$ in the case that R has no type I factor direct summands), we get a positive element x (respectively, a φ-semifinite projection p by Proposition 2.4(ii)) in R such that

$$\tau(x) = \sum \tau(p_n)^{-1}\tau(p_n) = \infty$$

and

$$\varphi(x) = \sum \beta_n \tau(p_n)^{-1}\tau(p_n) \leq \sum \alpha_n < \infty$$

(respectively,

$$\tau(p) = \sum \tau(p_n) = \infty$$

and

$$\varphi(p) \leq \sum \beta_n \tau(p_n) < \infty).$$

Proof (ii). The hypothesis is equivalent to the statement that $\chi(-\infty, \alpha) \sim 1$ for every $\alpha > 0$. This means in particular that R is properly infinite. Let r_0 be a finite projection in R such that $\tau(r_0) = \delta$ for some $\delta > 0$. Let $\{\alpha_n\}$ be a monotonely decreasing sequence of strictly positive real numbers such that $\sum \delta\alpha_n < \varepsilon$. We construct a sequence $\{r_n\}$ of mutually orthogonal projections such that

$$r_0 \sim r_n \leq \chi(-\infty, \alpha_n)$$

for every $n = 1, 2, \ldots$. Suppose that we have constructed the finite set $r_1, \ldots, r_n$; we construct r_{n+1}. Let $r = r_1 + \cdots + r_n$. We see that

$$\chi(-\infty, \alpha_{n+1}) - R(\chi[(-\infty, \alpha_{n+1})r)) \sim 1$$

because $\chi(-\infty, \alpha_{n+1}) \sim 1$ by hypothesis and because the range projection $R(\chi(-\infty, \alpha_{n+1})r)$ of $\chi(-\infty, \alpha_{n+1})r$ is a finite projection since the range projection is equivalent to a subprojection of r. We can therefore find a projection r_{n+1} with

$$r_0 \sim r_{n+1} \leq \chi(-\infty, \alpha_{n+1}) - R(\chi(-\infty, \alpha_{n+1})r).$$

Then we have that

$$r_{n+1}r = r_{n+1}\chi(-\infty, \alpha_{n+1})r = r_{n+1}(R(\chi(-\infty, \alpha_{n+1})r))r = 0.$$

Thus, we have completed the induction step. We now have a sequence of mutually orthogonal projections $\{r_n\}$ with $r_0 \sim r_n$ and with $r_n \leq \chi(-\infty, \alpha_n)$ for every $n = 1, 2, \ldots$. Setting $p = \sum r_n$, we get that

$$\varphi(p) = \sum \varphi(r_n) = \sum \tau(hr_n) \leq \sum \alpha_n \tau(r_n) = \sum \alpha_n \tau(r_0) < \infty.$$

We also have that $p \sim c(r_0)$.

Now let $\{p_n\}$ be a maximal set of nonzero projections in R with orthogonal central supports such that $\varphi(p_n) < \varepsilon$ and $p_n \sim c(p_n)$. The material in the previous paragraph shows that $\sum p_n = 1$. Since Rp_n is properly infinite, we can decompose each p_n into 2^n mutually equivalent orthogonal projections $\{p_{nk}\}_k$ of sum p_n. One of the projections p_{nk} satisfies $\varphi(p_{nk}) < \varepsilon/2^n$ and $p_{nk} \sim c(p_{nk}) - p_{nk} \sim c(p_{nk})$. So we may assume that $\varphi(p_n) < \varepsilon/2^n$ and $p_n \sim c(p_n) - p_n \sim c(p_n)$. Therefore, the projection $p = \sum p_n$ satisfies $\varphi(p) < \varepsilon$ and $p \sim 1 - p \sim 1$. □

If the ideals of definition of a f.s.n. weight and a f.s.n. trace are related by inclusion, then the weight and the trace bound one another up to a finite functional.

PROPOSITION 4.6. *Let R be a semifinite von Neumann algebra. Let φ be a f.s.n. weight on R and let τ be a f.s.n. trace on R. Then* a(i) *$M_\varphi \subset M_\tau$ if and only if* a(ii) *there is an $\alpha > 0$ and a positive normal functional ω_1 such that $\alpha\tau - \omega_1 \leq \varphi$; and* b(i) *$M_\tau \subset M_\varphi$ if and only if* b(ii) *there is a $\beta > 0$ and a positive normal functional ω_2 such that $\varphi \leq \beta\tau + \omega_2$.*

Proof. We only need to show that (i) implies (ii). We use Lemmas 4.4(i) and 4.5(i). First we show that a(i) implies a(ii). Let χ be the spectral resolution of the Radon-Nikodym derivative of φ with respect to τ. By Lemma 4.5(i), there is an $\alpha > 0$ such that

$\tau(\chi(-\infty, \alpha)) < \infty$ for some $\alpha > 0$; otherwise, there would be a positive element x in R with $\varphi(x) < \infty$ and $\tau(x) = \infty$ contrary to a(i). But then we have that

$$\alpha\tau \leq \varphi + \alpha\tau(\chi(-\infty, \alpha)).$$

Letting $\omega_1 = \alpha\tau(\chi(-\infty, \alpha)\cdot)$, we see that ω_1 is finite and $\alpha\tau - \omega_1 \leq \varphi$.

The implication b(i) implies b(ii) follows in a similar manner from Lemma 4.4(i). □

When the algebra R has no type I factor direct summands, we can rephrase Proposition 4.6 in terms of the projection lattices.

PROPOSITION 4.7. *Let R be a semifinite von Neumann algebra with no type* I *factor direct summands. Let φ be a f.s.n. on R and let τ be a f.s.n. trace on R. Then $M_\varphi \subset M_\tau$ if and only if $P(M_\varphi) \subset P(M_\tau)$ and $M_\tau \subset M_\varphi$ if and only if $P(M_\tau) \subset P(M_\varphi)$.*

Proof. First let $P(M_\varphi) \subset P(M_\tau)$. Let χ be the spectral resolution of the Radon-Nikodym derivative of φ with respect to τ. By Lemma 4.5(ii), there is an $\alpha > 0$ such that $\tau(\chi(-\infty, \alpha)) < \infty$ for some $\alpha > 0$; otherwise, there would be a projection p with $\varphi(p) < \infty$ and $\tau(p) = \infty$ contrary to the assumption that $P(M_\varphi) \subset P(M_\tau)$. As in Proposition 4.5(a), there is a normal functional ω on R such that

$$\alpha\tau - \omega \leq \varphi.$$

This shows that $M_\varphi \subset M_\tau$.

The proof of the second part of Proposition 4.6 is similar. Here Lemma 4.4(ii) is used instead of Lemma 4.5(ii). □

In the remainder of this section we consider f.s.n. weights φ such that $P(M_\varphi)$ is a lattice. Here we need to separate two cases: with type I factor summands and without such summands. We first prove a lemma that is used in both cases.

LEMMA 4.8. *Let φ be a f.s.n. weight on a properly infinite semifinite von Neumann algebra R such that $P(M_\varphi)$ is a lattice. If $\{e_n\}$ is a sequence of mutually orthogonal central projections such that $\varphi(e_n) < \infty$ for every n, then $\sum \varphi(e_n) < \infty$.*

Proof. Let χ be the spectral resolution of the Radon-Nikodym derivative of φ with respect to a f.s.n. trace τ on R. Then the projection $\chi[\beta, \infty)e_n$ has finite trace for every $\beta > 0$ because

$$\beta\tau(\chi[\beta, \infty)e_n) \leq \varphi(\chi[\beta, \infty)e_n) \leq \varphi(e_n) < \infty.$$

Because R is properly infinite $\tau(\chi(-\infty, \alpha)e) = \infty$ for every $\alpha > 0$ and every nonzero central projection e majorized by e_n. Lemma 4.5(ii) applied to Re_n implies the existence of a projection $p_n \leq e_n$ with $\varphi(p_n) \leq 2^{-n}$ such that $p_n \sim e_n - p_n \sim e_n$. However, the projection $p = \sum p_n$ now satisfies $\varphi(p) \leq 1$ and

$$p \sim \sum(e_n - p_n) \sim \sum e_n - p \sim \sum e_n.$$

So we have obtained two orthogonal equivalent φ-semifinite projections p and $\sum e_n - p$ (cf. Proposition 2.4(i) and (ii)). Now the characterization of the lattice property of $P(M_\varphi)$ in Proposition 3.2 forces

$$\varphi\left(\sum e_n - p\right) < \infty,$$

which taken together with $\varphi(p) < \infty$ forces

$$\sum \varphi(e_n) < \infty.$$

□

Now we consider the first of the two cases.

THEOREM 4.9. *Let R be a properly infinite semifinite von Neumann algebra without type* I *factor direct summands. Let φ be a f.s.n. weight on R. Then $P(M_\varphi)$ is a lattice if and only if there is a central projection e and a f.s.n. trace τ on R such that*

(i) *φ restricted to R_e is a functional; and*

(ii) *$P(M_\tau) = P(M_\varphi)$ on $R_{(1-e)}$.*

Proof. Let φ be a f.s.n. weight on R such that $P(M_\varphi)$ is a lattice. First let $\{e_n\}$ be a maximal set of mutually orthogonal nonzero central projections such that $\varphi(e_n) < \infty$. Setting $e = \sum e_n$, we get a central projection e such that φ is a functional on R_e (Lemma 4.8) and such that $\varphi(f) = \infty$ for every nonzero central projection f in $R_{(1-e)}$. We note that the finite projections of $\varphi|R_{(1-e)}$ is still a lattice. So by reducing to the f.s.n. weight on the properly infinite semifinite algebra $R_{(1-e)}$ with no type I factor direct summands, there is no loss of generality in the assumption $\varphi(f) = \infty$ for every central projection f in R.

We construct a f.s.n. trace τ with $P(M_\tau) = P(M_\varphi)$. Let τ be any f.s.n. trace on R and let h be the Radon-Nikodym derivative of φ

with respect to τ. Let χ be the spectral resolution of h. We modify τ by multiplying it by an element affiliated with the center constructed from h in order to get the desired trace.

First, let $\{e_n\}$ be a maximal set of nonzero mutually orthogonal central projections in R such that for each e_n there is a $\beta_n > 0$ with

$$\tau(\chi[\beta_n, \infty)e_n) + \varphi(\chi[\beta_n, \infty)e_n) < \infty.$$

We must have that $\sum e_n = 1$; otherwise, we must have that every nonzero projection e majorized by the nonzero projectional $1 - \sum e_n$ satisfies

$$\tau(\chi[\beta, \infty)e) + \varphi(\chi[\beta, \infty)e) = \infty$$

for every $\beta > 0$. Then there would be two orthogonal equivalent projections p and q in R such that $\varphi(p) < \infty$, $\varphi(q) = \infty$ (Lemma 4.4(iii)) and $P(M_\varphi)$ would not be a lattice (Proposition 3.2). So we must have that $\sum e_n = 1$.

Second, let $\{f_n\}$ be a maximal set of nonzero orthogonal central projections in R such that for each f_n there is an $\alpha_n > 0$ with

$$\tau(\chi(-\infty, \alpha_n)f_n) < \infty.$$

By the same reasoning as the preceding paragraph we have that $\sum f_n = 1$. Here we use Lemma 4.5(ii).

Now by combining the sets $\{e_n\}$ and $\{f_n\}$ into a single set, we may assume that there is a sequence of mutually orthogonal central projections $\{e_n\}$ of sum 1 and two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ of real numbers with $0 < \alpha_n < \beta_n$ such that

$$\tau(\chi(-\infty, \alpha_n)e_n) + \tau(\chi[\beta_n, \infty)e_n) + \varphi(\chi[\beta_n, \infty)e_n) < \infty$$

for every $n = 1, 2, \ldots$. We can also write

$$\tau(\chi(-\infty, \alpha_n)e_n) + \varphi(\chi(-\infty, \alpha_n)e_n) \\ + \tau(\chi[\beta_n, \infty)e_n) + \varphi(\chi[\beta_n, \infty)e_n) < \infty.$$

We have that $\tau(p) < \infty$ if and only if $\varphi(p) < \infty$ for every projection $p \leq e_n$. In fact, we have that $\tau(p) < \infty$ (respectively, $\varphi(p) < \infty$) if and only if $\tau(p\chi[\alpha_n, \beta_n)e_n) < \infty$ (respectively, $\varphi(p\chi[\alpha_n, \beta_n)e_n) < \infty$) so that the relation

$$\alpha_n\tau(\chi[\alpha_n, \beta_n)pe_n) \leq \varphi(\chi[\alpha_n, \beta_n)pe_n) \leq \beta_n\tau((\chi[\alpha_n, \beta_n)pe_n))$$

shows that τ and φ mutually bound each other on projections majorized by e_n.

Now we have that each h_n given by

$$h_n = h(1 - (\chi(-\infty, \alpha_n) + \chi[\beta_n, \infty)))e_n$$

is a bounded positive operator on the properly infinite von Neumann algebra R_{e_n}. Let z_n be an element in the essential central spectrum of h_n. We have that

$$\alpha_n e_n \le z_n \le \beta_n e_n$$

since

$$\alpha_n e_n \le h_n \le \beta_n e_n$$

modulo the ideal generated by finite projections in R_{e_n}. Thus, for any projection p, the number $\tau(z_n p)$ is finite if and only if $\varphi(z_n p)$ is finite.

Now we show that the trace

$$\tau_0(x) = \sum \tau(z_n x)$$

is the desired trace. From this point to the end of the proof we do not use the fact that R has no type I factor direct summands. We present an argument based entirely on the fact that $\alpha_n e_n \le z_n \le \beta_n e_n$. So we must show that $P(M_{\tau_0}) = P(M_\varphi)$. First let p be an arbitrary projection in R with $\tau(p_n) + \varphi(p_n) < \infty$ for every $n = 1, 2, \ldots$. Here $p_n = pe_n$. Let $\{\varepsilon_n\}$ be a sequence of positive real numbers such that

$$\sum \varepsilon_n \tau(p_n) < \infty.$$

Then we have that

$$\chi(z_n - \varepsilon_n e_n, z_n + \varepsilon_n e_n) \sim e_n$$

by Proposition 4.3. The range projection

$$r_n = R(\chi(z_n - \varepsilon_n e_n, z_n + \varepsilon_n e_n)p_n)$$

is a finite projection and satisfies

$$\varphi(r_n) \le \beta_n \tau(p_n) < \infty.$$

So we have that

$$r_n' = \chi(z_n - \varepsilon_n e_n, z_n + \varepsilon_n e_n) - r_n$$

is a φ-s. projection (Proposition 2.4(iii)) equivalent to e_n. Then we can find a φ-s. projection q_n with

$$p_n \sim q_n \le r_n'$$

due to Lemma 2.5. Actually, the projection q_n is in M_φ. We see that p_n is orthogonal to q_n since

$$q_n p_n = q_n \chi(z_n - \varepsilon_n e_n, z_n + \varepsilon_n e_n) p_n = q_n r_n \chi(z_n - \varepsilon_n e_n, z_n + \varepsilon_n e_n) p = 0.$$

We also see that

$$\tau((z_n + \varepsilon_n) q_n) - \varepsilon_n \tau(q_n) \le \tau_0(p_n) = \tau_0(q_n) \le \tau((z_n - \varepsilon_n) q_n) + \varepsilon_n \tau(q_n)$$

since

$$\varphi(q_n) = \tau(h_n \chi(z_n - \varepsilon_n e_n, z_n + \varepsilon_n e_n) q_n) \le \tau((z_n + e_n) q_n)$$

and likewise that

$$\tau((z_n - \varepsilon) q_n) \le \varphi(q_n).$$

Setting $\sum q_n = q$, we get a φ-s. projection q with $pq = 0$ and $q \sim p$ such that

$$\varphi(q) - \sum \varepsilon_n \tau(q_n) \le \tau_0(p) = \tau_0(q) \le \varphi(q) + \sum \varepsilon_n \tau(q_n).$$

Now we use the material in the last paragraph to complete the proof. Suppose that $p \in P(M_\varphi)$. Let $p_n = pe_n$. Then we have $\tau(p_n) + \varphi(p_n) < \infty$ for every $n = 1, 2, \dots$ due to the first part of the proof. This means that the projection q constructed in the previous paragraph is in $P(M_\varphi)$; otherwise, the set $P(M_\varphi)$ would not be a lattice by Proposition 3.2. Since $\sum \varepsilon_n \tau(q_n) < \infty$, we have that p is in $P(M_{\tau_0})$. So we have that $P(M_\varphi) \subset P(M_{\tau_0})$.

Conversely, suppose that $p \in P(M_{\tau_0})$. Again we have $\tau_0(p_n) < \infty$ for every $n = 1, 2, \dots$. This means that $\tau(p_n) < \infty$ for every n and thus that $\varphi(p_n) < \infty$ for every n. Now by the previous part of the proof we find a φ-s. projection q with $pq = 0$, $p \sim q$, and $\sum \varepsilon_n \tau(q_n) < \infty$ such that

$$\varphi(q) - \sum \varepsilon_n \tau(q_n) \le \tau_0(p) = \tau_0(q) \le \varphi(q) + \sum \varepsilon_n \tau(q_n).$$

So we get that q is in $P(M_\varphi)$, and consequently, p is in $P(M_\varphi)$ by Proposition 3.2. This means that $P(M_{\tau_0}) \subset P(M_\varphi)$. □

Now we complete the analysis of the properly infinite semifinite case.

THEOREM 4.10. *Let $R = \sum \bigoplus R_n$ where R_n are type I_∞ factors and let φ be f.s.n. weight on R. Then $P(M_\varphi)$ is a lattice if and only if the identity is the sum of three orthogonal central projections e, f and g such*

(i) *φ is a finite functional on R_e;*

(ii) $P(M_\varphi) \subset P(M_{\mathrm{Tr}})$ *on* R_f; *and*
(iii) $P(M_\varphi) = P(M_\tau)$ *on* R_g.
Here $\mathrm{Tr} = \sum \bigoplus \mathrm{tr}$ *where* tr *is the canonical trace on a type* I_∞ *factor and* τ *is f.s.n. trace on* R_g.

Proof. We show that $P(M_\varphi) \subset P(M_{\mathrm{Tr}})$ on R implies $P(M_\varphi)$ is a lattice. Let e_n be the central projection of R which is the identity on R_n. Let $p, q \in P(M_\varphi)$; then $p \vee q \in P(M_{\mathrm{Tr}})$. Since $p \vee q$ is a sum of minimal projections, each with trace one, we see that $(p \vee q)e_n = 0$ for all but a finite number of indices. For these indices

$$(p \vee q)e_n \leq \gamma_n(p + q)e_n$$

for some $\gamma_n > 0$ because pe_n and qe_n are finite dimensional projections. But then

$$\varphi(p \vee q) \leq \max\{\gamma_n\}\varphi(p + q) < \infty,$$

whence $p \vee q \in P(M_\varphi)$. Thus $P(M_\varphi)$ is a lattice.

Now assume that $P(M_\varphi)$ is a lattice. Let χ be the Radon-Nikodym derivative h of φ with respect to Tr and as before let e_n be the central projection of R such that $Re_n = R_n$. First let S_1 be the set

$$S_1 = \{n \mid \varphi(e_n) < \infty\}.$$

Then we must have that

$$\sum\{\varphi(e_n) \mid n \in S_1\} < \infty$$

by Lemma 4.8, i.e., φ is a finite functional on R_e where $e = \sum\{e_n \mid n \in S_1\}$.

Now we consider φ on the direct sum $\sum \bigoplus\{R_n \mid n \notin S_1\}$. Again there is no loss of generality in the assumption that

$$\sum\{e_n \mid n \notin S_1\} = 1.$$

First we show that, for every n, there is $\alpha_n > 0$ such that $\chi(0, \alpha_n)e_n = 0$. We have that $\varphi(e_n) = \infty$. If $\mathrm{tr}(\chi(-\infty, \alpha)e_n) = \infty$ for all $\alpha > 0$, then we could again find a projection $p \leq e_n$ with $\varphi(p) < \infty$ and $p \sim e_n - p \sim e_n$ by Lemma 4.5(ii). This also contradicts Proposition 3.2. So we must have that $\mathrm{tr}(\chi(-\infty, \alpha)e_n) < \infty$ for some $\alpha > 0$. We have that

$$\mathrm{glb}\{\mathrm{tr}(\chi(-\infty, \alpha)e_n) \mid \alpha > 0\} = 0$$

since φ is a f.s.n. when restricted to Re_n. Because $\mathrm{tr}(\chi(-\infty, \alpha)e_n)$ is integer valued, we must have that $\mathrm{tr}(\chi(-\infty, \alpha)e_n) = 0$ for some $\alpha > 0$. Let

$$\alpha_n = \mathrm{lub}\{\alpha \mid \mathrm{tr}(\chi(-\infty, \alpha)e_n) = 0\}.$$

We have that $\chi(-\infty, \alpha_n)e_n = 0$ while $\chi(-\infty, \alpha_n]e_n \neq 0$. The latter is due to the fact that $\alpha \to \mathrm{tr}(\chi(-\infty, \alpha)e_n)$ is integer valued and left continuous.

Now let

$$S = \{n \mid \chi[1, \infty)e_n \sim e_n\}.$$

Suppose that $n \in S$. We show that

$$\gamma = \mathrm{glb}\{\alpha_n \mid n \in S\} > 0.$$

We obtain a contradiction if $\gamma = 0$. By passing to a subset of S, we may assume that $\sum \alpha_n < \infty$. Then there are two infinite orthogonal sequences of one dimensional projections $\{p_n\}$ and $\{q_n\}$ such that

$$p_n \leq \chi(-\infty, \alpha_n]e_n$$

and

$$q_n \leq \chi(1, \beta_n]e_n$$

for every n. Here β_n is some number $\beta_n \geq 1$. However, this would give two equivalent φ-s. projections $p = \sum p_m$ and $q = \sum q_m$ with

$$\varphi(p) \leq \sum \alpha_n \,\mathrm{tr}(p_n) = \sum \alpha_n < \infty$$

and

$$\varphi(q) > \sum \beta_n \,\mathrm{tr}(q_n) = \sum \beta_n = \infty.$$

Again this would contradict Proposition 3.2. So we must have $\gamma > 0$. Now let $f = \sum\{e_n \mid n \in S\}$. Then the weight φ restricted to R_f is a f.s.n. weight with

$$\varphi(p) \geq \gamma \,\mathrm{Tr}(p\chi[\gamma, \infty)) = \gamma \,\mathrm{Tr}(pf(\chi(-\infty \cdot \gamma) + \chi[\gamma, \infty))) = \gamma \,\mathrm{Tr}(pf)$$

for every projection p majorized by f. This means that $P(M_\varphi)f \subset P(M_{\mathrm{Tr}})$.

Now we consider the final set of indices, the complement of S and S_1. We have that $\chi[1, \infty)e_n$ is a finite projection in Re_n; otherwise, the projection $\chi[1, \infty)e_n$ would be infinite and n would be in S. Since $\mathrm{lub}_\beta \,\mathrm{tr}(\chi[\beta_n, \infty)e_n) = 0$, there is a $\beta_n > 0$ with $\chi[\beta_n, \infty)e_n = 0$. Thus, there are numbers $0 < \alpha_n < \beta_n$ such that $\chi(-\infty, \alpha_n)e_n = \chi[\beta, \infty)e_n = 0$. Now we can finish the proof in the same way we finished the proof of Theorem 4.9. Let δ_n be in the essential spectrum of the bounded operator he_n. Let τ be the trace $\sum \bigoplus \delta_n \,\mathrm{tr}$. Then we have that $P(M_\varphi) = P(M_\tau)$. □

REMARK 4.11. In the case of type I_∞ factors the inclusion $P(M_\varphi) \subset P(M_{\mathrm{tr}})$ does not in general imply that $P(M_\varphi) = P(M_{\mathrm{tr}})$. In fact, if

h is the Radon-Nikodym derivative of φ with respect to tr, then $P(M_\varphi) = P(M_{\rm tr})$ implies that h is bounded. Indeed, if there were a vector $\xi \notin D(h^{1/2})$, then the rank 1 projection on the subspace generated by ζ would be in $P(M_{\rm tr})$ but not in $P(M_\varphi)$. So $P(M_\varphi) = P(M_{\rm tr})$ implies that h is defined on the whole Hilbert space and thus that h is bounded.

On the other hand $P(M_\varphi) = P(M_{\rm tr})$ does not in general imply that $M_\varphi = M_{\rm Tr}$. For example, the weight $\varphi = \sum \bigoplus n\,{\rm tr}$ satisfies $P(M_\varphi) = P(M_{\rm Tr})$ but $M_\varphi \neq M_{\rm Tr}$.

5. Finite algebras. To treat finite algebras we need to develop additional functional calculus for central intervals.

PROPOSITION 5.1. *Let R be a finite von Neumann algebra with center Z, let Φ be the canonical center valued trace on R, let $h\eta R$, $h = h^* \geq 0$ with null space $N(h) = 0$ and let*

$$z = \mathrm{lub}\{x\,\eta\,Z \mid x > 0,\, \Phi(\chi(-\infty, x)) \leq (1/2)1\}.$$

Then $z\eta Z$, $z > 0$ and

$$\Phi(\chi(-\infty, z)) \leq (1/2)1 \quad \textit{and} \quad \Phi(\chi(z, \infty)) \leq (1/2)1.$$

Proof. Let $\{e_n\}$ (respectively, $\{f_n\}$) be a maximal family of nonzero mutually orthogonal central projections such that there are strictly positive numbers $\{\alpha_n\}$ (respectively, $\{\beta_n\}$) with $\Phi(\chi(-\infty, \alpha_n)e_n) \leq 2^{-1}e_n$ (respectively, $\Phi(\chi(-\infty, \beta_n)f_n) \geq (2/3)f_n$). Then we must have that $\sum e_n = 1$ (respectively, $\sum f_n = 1$) since the limit in the strong operator topology of $\{\Phi(\chi(-\infty, \alpha))\}$ as α goes to 0 (respectively, ∞) is 0 (respectively, 1). We have that

$$\sum \alpha_n e_n \leq \sum \beta_n f_n.$$

Then the set of all sums $\sum \alpha_n e_n$ where $\{e_n\}$ is a sequence of mutually orthogonal projections of sum 1 and $\{\alpha_n\}$ is a sequence of strictly positive numbers such that $\sum \Phi(\chi(-\infty, \alpha_n)e_n) \leq 2^{-1}1$ is upward by the ordering described in the introduction to §4. In addition the set of all sums is bounded above by the sums $\sum \beta_n f_n$. Thus, we have that

$$z = \mathrm{lub}\left\{\sum \alpha_n e_n \mid \sum e_n = 1 \text{ and } \sum \Phi(\chi(-\infty, \alpha_n)e_n) \leq 2^{-1}\right\}$$

is a positive selfadjoint element affiliated with Z such that

$$\Phi(\chi(-\infty, z)) \leq 2^{-1}1.$$

Now let x and y be positive selfadjoint elements affiliated with Z such that $z < x < y$. Then we have that

$$\Phi(\chi(-\infty, x)) \geq 2^{-1}1$$

and

$$\Phi(\chi(y, \infty)) \leq 1 - \Phi(\chi(-\infty, x)) \leq 2^{-1}1.$$

Taking the least upper bound of $\Phi(\chi(y, \infty))$ for all $y > z$, we get

$$\Phi(\chi(z, \infty)) \leq 2^{-1}1. \qquad \square$$

Now let φ be a f.s.n. weight on the finite von Neumann algebra R. Let τ_1 and τ_2 be two f.n.s. traces on R and let h_1 and h_2 be the Radon-Nikodym derivatives of φ with respect to τ_1 and τ_2 respectively. Let χ_1 and χ_2 be the spectral resolutions of h_1 and h_2 respectively. Let z_1 and z_2 be the operators associated to h_1 and h_2 respectively by Proposition 5.1, viz.,

$$z_i = \mathrm{lub}\{x\eta Z \mid x > 0, \Phi(\chi_i(-\infty, x)) \leq (1/2)1\}.$$

There is a $w\eta Z^+$ with $w > 0$ such that

$$\tau_1(wx) = \tau_2(x)$$

for all $x \in R^+$. So we have that $wh_2 = h_1$. We have already seen in Property 6 of the central spectral resolution given in §4 that

$$\begin{aligned}\chi_1(-\infty, z) &= \chi(-\infty, z)(h_1) \\ &= \chi(-\infty, wz)(wh_1) = \chi(-\infty, wz)(h_2) = \chi_2(-\infty, wz).\end{aligned}$$

Taking into account the definition of z_1 and z_2 as least upper bounds, we get that $z_1 = wz_2$. This means that

$$\tau_1(z_1x) = \tau_1(wz_2x) = \tau_2(z_2x)$$

for all x in R^+.

Now the following definition makes sense.

DEFINITION 5.2. Let R be a finite von Neumann algebra and φ be a f.s.n. weight. Let τ be a f.s.n. trace on R and let

$$z = \mathrm{lub}\{x\eta Z \mid x > 0, \Phi(\chi(-\infty, x)) \leq (1/2)1\}$$

where Φ is the canonical operator valued trace on R and χ is the spectral resolution of the Radon-Nikodym derivative h of φ with respect to τ. Then the trace $\tau_\varphi = \tau \cdot z$ is called *canonical trace* associated with the weight φ.

The canonical trace balances at 1 the spectral resolution of the Radon-Nikodym derivative.

PROPOSITION 5.3. *Let φ be a f.s.n. weight on the finite von Neumann algebra R with center Z, let τ_φ be the canonical trace associated with φ, and let h be the Radon-Nikodym derivative of φ with respect to τ_φ. Then*

$$1 = \operatorname{lub}\{x\eta Z \mid x > 0, \Phi(\chi(-\infty, x)(h)) \leq (1/2)1\},$$

where Φ is the canonical operator valued trace on R.

Proof. Setting

$$z = \sup\{x\eta Z \mid x > 0, \Phi(\chi(-\infty, x)(h)) \leq (1/2)1\},$$

we have that $\tau_\varphi = \tau_\varphi \cdot z$, and consequently, we have that $z = 1$. □

PROPOSITION 5.4. *Let R be a finite von Neumann algebra, let φ be a f.s.n. weight on R, let τ_φ be the canonical trace associated with φ, and h be the Radon-Nikodym derivative of φ with respect to τ_φ. Then there are equivalent projections r_- and r_+ in R^φ such that* (i) $hr_- \leq r_-$, (ii) $r_+ \leq hr_+$; *and* (ii) $h(1-(r_- + r_+)) = 1 - (r_- + r_+)$.

Proof. By taking a central decomposition, we may assume that R is either a type II_1 or a type I_n algebra. First assume that R is a type II_1 algebra. Then we have that $h\chi(-\infty, 1) \leq \chi(-\infty, 1)$, $\chi(1, \infty) \leq h\chi(1, \infty)$, and $h\chi[1] = \chi[1]$. Now we have that

$$\Phi(\chi(-\infty, 1)) \leq (1/2)1 \leq 1 - \Phi(\chi(1, \infty)) = \Phi(\chi(-\infty, 1)) + \Phi(\chi[1]).$$

Since R is a continuous algebra, there is a subprojection p of $\chi[1]$ such that $\Phi(\chi[0, 1) + p) = 1$. Setting $r_- = \chi(-\infty, 1) + p$ and $r_+ = 1 - r_-$, we get two projections r_- and r_+ in R^φ due to the fact that every subprojection of $\chi[1]$ is in R^φ. We also have that

$$hr_- \leq h(\chi(-\infty, 1) + \chi[1])r_- \leq r_-$$

and

$$hr_+ = h(\chi(1, \infty) + (\chi[1] - p))r_+ \leq r_+.$$

Since the condition $\Phi(r_-) = \Phi(r_+)$ implies that $r_- \sim r_+$, we have constructed the desired projections in the type II_1 case. Note that $r_+ + r_- = 1$ in this case.

If R is a type I_n algebra, we may assume that by passing to a central summand that $\chi(-\infty, 1)$, $\chi[1]$, and $\chi(1, \infty)$ are equal respectively to the sum of n_1, n_2 and n_3 mutually orthogonal maximal abelian projections. Then $n_1/(n_1+n_2+n_3) \leq 1/2$ while $n_3/(n_1+n_2+n_3) \leq 1/2$. Now it is clear that one can find r_- and r_+ using the fact that two maximal abelian projections are equivalent. □

It is instructive to consider a type I_n factor algebra M_n. Let tr be the trace $\mathrm{tr}((a_{ij})) = \sum a_{ii}$. We may assume that the Radon-Nikodym derivative h of the weight φ with respect to tr is the diagonal matrix

$$h = \mathrm{diag}(a_1, \ldots, a_n)$$

with $0 < a_1 \leq a_2 \leq \cdots \leq a_n$. Then a decomposition of the identity satisfying the requirements of Proposition 5.4 is

$$r_- = \mathrm{diag}(1, \ldots, 1, 0, \ldots, 0),$$

and

$$r_+ = \mathrm{diag}(0, \ldots, 0, 1, \ldots, 1),$$

where there are $[n/2]$ ones in both r_- and r_+. Here $[n/2]$ is the integer part of $n/2$. We note that the decomposition of Proposition 5.4 is not unique. However, the canonical trace is unique and is given by

$$\tau_\varphi = a_{[(n+1)/2]}\,\mathrm{tr}.$$

We can now discuss the lattice properties of $P(M_\varphi)$ for a finite algebra.

THEOREM 5.5. *Let R be a finite algebra, let φ be a f.s.n. weight and let τ_φ be the associated canonical trace. Then $P(M_\varphi)$ is a lattice if and only if $P(M_\varphi) = P(M_{\tau_\varphi})$.*

Proof. We can prove the necessity in the separate cases (i) R is a direct sum of type I factors and (ii) R has no type I factor direct summands. We have already presented a proof for the direct sum of type I factors in [7]. We sketch the proof again for the sake of completeness.

First let R be the direct sum of matrix algebras

$$R = \sum_n \bigoplus M_{k(n)}.$$

We have already seen that the canonical trace for φ is

$$\tau_\varphi = \sum_n \bigoplus a_{n,m(n)}\,\mathrm{tr},$$

where $m(n) = [(k(n)+1)/2]$. Given any projection $p = \sum_n \bigoplus p_n$ in R, we define

$$r_n = \mathrm{diag}(1, \dots, 1, 0, \dots, 0) \quad \text{and} \quad s_n = \mathrm{diag}(0, \dots, 0, 1, \dots, 1),$$

both with $\min\{\mathrm{tr}(p_n), [k(n)/2]\}$ ones, and we set

$$r = \sum_n \bigoplus r_n \quad \text{and} \quad s = \sum_n \bigoplus s_n.$$

Here we shall again use the assumption that h_n are diagonal matrices. The projections r and s are orthogonal equivalent projections. By definition of r_n and s_n and by the monotonicity of $\{a_{n,j}\}$, we have

$$\mathrm{tr}(r_n) \le \mathrm{tr}(p_n) \le 2\,\mathrm{tr}(r_n),$$

and

$$\mathrm{tr}(h_n r_n) \le a_{n,m(n)}\,\mathrm{tr}(r_n) \le \mathrm{tr}(h_n s_n).$$

We have that

$$\mathrm{tr}(h_n r_n) \le \sum_{j=1}^{\mathrm{tr}(p_n)} a_{n,j} \le \sum_{j=1}^{k(n)} a_{n,m}(p_n)_{jj} = \mathrm{tr}(h_n p_n)$$

and likewise that $\mathrm{tr}(h_n p_n) \le 3\,\mathrm{tr}(h_n s_n)$. Therefore, we obtain the inequalities

$$\varphi(r) \le \tau_\varphi(r) \le \varphi(s),$$
$$\tau_\varphi(r) \le \tau_\varphi(p) \le 3\tau_\varphi(r),$$
$$\varphi(r) \le \varphi(p) \le 3\varphi(s).$$

Now assume that $P(M_\varphi) \ne P(M_{\tau_\varphi})$. We shall obtain a contradiction from the preceding inequalities by showing the existence of two orthogonal φ-semifinite projections, one of which is in M_φ and one of which is not. This is impossible on account of Proposition 3.2. On the one hand, if $p \in M_\varphi$ but $p \notin M_{\tau_\varphi}$, then $r \in M_\varphi$, $r \notin M_{\tau_\varphi}$ and hence $s \notin M_\varphi$. On the other hand, if $p \in M_{\tau_\varphi}$ but $p \notin M_\varphi$, then $s \notin M_\varphi$ and hence $r \in M_\varphi$. In either case, we have two equivalent orthogonal projections r and s, one of which is in M_φ and one of which is not. But every projection in a finite von Neumann algebra is φ-semifinite (Proposition 2.3). Thus, we have obtained a contradiction. Hence, we conclude that $P(M_\varphi) = P(M_{\tau_\varphi})$ whenever $P(M_\varphi)$ is a lattice.

Now assume that R has no type I factor summands and let r_- and r_+ be the equivalent, orthogonal projections in R^φ given by Proposition 5.4 applied to the canonical trace τ_φ with respect to φ. First we show that $P(M_\varphi) \not\subset P(M_{\tau_\varphi})$ leads to a contradiction. Indeed, let χ be the spectral resolution of the Radon-Nikodym derivative of φ with respect to τ_φ. We have already seen that $\tau_\varphi(\chi(-\infty, \alpha)) < \infty$ for some $\alpha > 0$ implies that $P(M_\varphi) \subset P(M_{\tau_\varphi})$ since

$$\tau_\varphi(x) \leq \alpha^{-1}\varphi(x) + \tau_\varphi(\chi(-\infty, \alpha)x)$$

whenever $x \geq 0$. So if $P(M_\varphi) \not\subset P(M_{\tau_\varphi})$ were true, we must have that $\tau_\varphi(\chi(-\infty, \alpha)) = \infty$ for all $\alpha > 0$. Thus, there would be a projection $p \leq \chi(-\infty, 1) = r_-$ such that $\varphi(p) < \infty$ and $\tau_\varphi(p) = \infty$ (Lemma 4.5(iii)). Since $r_- \sim r_+$, we could find a φ-semifinite projection $q \leq r_+ \leq \chi[1, \infty)$ which is equivalent to p due to Lemma 2.5. However, we would then have that

$$\varphi(q) = \tau_\varphi(hq) \geq \tau_\varphi(q) = \tau_\varphi(p) = \infty.$$

This would now mean that $P(M_\varphi)$ is not a lattice due to Proposition 3.2. On the other hand, if $P(M_{\tau_\varphi}) \not\subset P(M_\varphi)$, then we also get a contradiction. We would have that

$$\tau_\varphi(\chi[\beta, \infty)) + \varphi(\chi[\beta, \infty)) = \infty$$

for all $\beta > 0$ and so we could find a φ-s. projection $p \leq \chi(1, \infty) \leq r_+$, such that $\tau_\varphi(p) < \infty$ and $\varphi(p) = \infty$ (Lemma 4.4(ii)). But then there is a projection $q \leq r_-$ with $q \sim p$ because $r_- \sim r_+$ (Proposition 5.4). Since

$$\varphi(q) = \tau_\varphi(hq) \leq \tau_\varphi(q) = \tau_\varphi(p) < \infty,$$

we would conclude, again by Proposition 3.2, that $P(M_\varphi)$ is not a lattice. Thus, we must have that $P(M_{\tau_\varphi}) \subset P(M_\varphi)$. Combining this with the previous inclusion found in the first part of the proof, we get that $P(M_{\tau_\varphi}) = P(M_\varphi)$ whenever $P(M_\varphi)$ is a lattice. □

Now we can compute $I(\varphi)$ for finite algebras.

THEOREM 5.6. *Let φ be a f.s.n. weight on a finite algebra R and let h be the Radon-Nikodym derivative of φ with respect to the canonical trace τ_φ of φ. Let r_- and r_+ be any projections in R^φ with $r_- \sim r_+$ in R such that* (i) $hr_- \leq r_-$, (ii) $r_+ \leq hr_+$, *and* (iii) $h(1 - r_- - r_+) = 1 - r_- - r_+ = r_0$. *Then* $I(\varphi) = 2\varphi(r_-) + \varphi(r_0)$.

Proof. We note that projections r_- and r_+ exist by Proposition 5.4 but are not necessarily unique since the piece of $\chi[1]$ is not determined. Then there is a sequence $\{e_n\}$ of mutually orthogonal central projections of sum 1 such that $\tau_\varphi(e_n) < \infty$ for every $n = 1, 2, \ldots$. We have that $\tau_\varphi \cdot e_n$ is the canonical trace associated with the weight $\varphi \cdot e_n$ on R_{e_n}. We also have that

$$\sum I(\varphi \cdot e_n) = I(\varphi)$$

and

$$\sum (2\varphi(e_n r_-) + \varphi(e_n r_0)) = 2\varphi(r_-) + \varphi(r_0).$$

Thus, there is no loss of generality in the assumption that τ_φ is a finite trace. Then we have that

$$\varphi(r_-) = \tau_\varphi(hr_-) \leq \tau_\varphi(r_-) < \infty$$

so that $\varphi(r_- \cdot)$ is a finite functional. Therefore, for every projection p, we can decompose $\varphi(p)$ as

$$\begin{aligned}\varphi(p) &= \varphi(r_-) - \varphi(r_-(1-p)) + \varphi((1-r_-)p) \\ &\geq \varphi(r_-) - \tau_\varphi(r_-(1-p)) + \tau_\varphi((1-r_-)p) \\ &= \varphi(r_-) + \tau_\varphi(p) - \tau_\varphi(r_-).\end{aligned}$$

Now we take now a second projection q such that $p \vee q = 1$. We have that $1 - p \sim q - p \wedge q$ by the Parallelogram Law. In defining $I(\varphi)$, we have already remarked that there is no loss of generality in the assumption that $p \wedge q = 0$. Using this assumption, the Parallelogram Law for p and q becomes $1 - p \sim q$. Applying the inequality in the preceding paragraph to both p and q and adding the results, we get

$$\begin{aligned}I(\varphi) \geq \varphi(p+q) &\geq 2\varphi(r_-) + \tau_\varphi(p+q) - 2\tau_\varphi(r_-) \\ &= 2\varphi(r_-) + \tau_\varphi(1) - \tau_\varphi(r_- + r_+) \\ &\geq 2\varphi(r_-) + \tau_\varphi(r_0) \\ &\geq 2\varphi(r_-) + \varphi(r_0).\end{aligned}$$

To prove the reverse inequality, we apply Lemma 3.1 to the pair of orthogonal equivalent projections r_- and r_+. Notice that r_+ is φ-semifinite since it is in R^φ (Proposition 2.4(ii)). Thus, for every $\varepsilon > 0$, there is a projection q such that $r_- \vee q = r_- + r_+$ and $\varphi(q) < \varphi(r_-) + \varepsilon$. Because $(r_- + r_0) \vee q = 1$, we get

$$I(\varphi) \leq \varphi(r_- + r_0 + q) \leq 2\varphi(r_-) + \varphi(r_0) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have that

$$I(\varphi) \leq 2\varphi(r_-) + \varphi(r_0). \qquad \square$$

In particular we see that $2\varphi(r_-) + \varphi(r_0)$ does not depend on the choice of r_- and r_+ with the properties (i)–(iii) of Proposition 5.4.

6. Calculation of $I(\varphi)$ and $J(\varphi)$ for semifinite algebras. We have already calculated $I(\varphi)$ for finite algebras. We also know that $I(\varphi) = \infty$ if the identity is not the supremum of two projections in M_φ. We complete the remaining case in the next theorem.

First we need to extend the notion of essential central spectrum to an unbounded selfadjoint operator h affiliated with a von Neumann algebra R with center Z. The selfadjoint operator z affiliated with z will be said to be in the *essential central spectrum* of h if

$$\chi(z - w\,,\, z + w)(h) \sim 1$$

for every $w > 0$ affiliated with z. This corresponds to the behavior of the essential central spectrum for bounded operators (cf. Proposition 4.3).

THEOREM 6.1. *Let φ be a f.s.n. weight on a properly infinite semifinite von Neumann algebra R and let h be the Radon-Nikodym derivative of φ with respect to a f.s.n. trace τ. Then the following are equivalent:*

(i) $0 \in Z - \sigma^e(h)$;
(ii) $I(\varphi) = 0$; *and*
(iii) $I(\varphi) < \infty$, *i.e., the identity is the supremum of two projections in M_φ.*

Proof (i) *implies* (ii). Let $0 \in Z - \sigma^e(h)$; then $\chi(0, \alpha) \sim 1$ for all $\alpha > 0$. This means that, given $\varepsilon > 0$, there is a projection p in R with $\varphi(p) < \varepsilon$ and $p \sim 1 - p \sim 1$ (Lemma 4.5(ii)). We can find a projection q with $\varphi(q) < \varepsilon$ and $p \vee q = p + (1-p) = 1$ (Lemma 3.1) because $1 - p$ is φ-semifinite (Proposition 2.4(iii)). This means that $I(\varphi) < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have that $I(\varphi) = 0$.

Proof (iii) *implies* (i). Suppose that there are two projections p and q in M_φ with $p \vee q = 1$. We get a contradiction if $\chi(-\infty, \alpha)e$ is a finite projection for some nonzero central projection e and some $\alpha > 0$. In fact, if $\chi(-\infty, \alpha)e$ is a finite projection, we have that

$$\alpha\tau(\chi[\alpha, \infty)pe) \leq \tau(h\chi[\alpha, \infty)pe) \leq \varphi(pe) < \infty.$$

This means that $\chi[\alpha, \infty)pe$ and consequently

$$pe = \chi(-\infty, \alpha)pe + \chi(\alpha, \infty)pe$$

are finite projections. Likewise, the projection qe is finite so that $pe \vee qe = e$ is finite contrary to our assumption that R is properly infinite. Thus $\chi(-\infty, \alpha) \sim 1$ for all $\alpha > 0$. □

Let φ be a n.s.f. weight on the von Neumann algebra R. Using Theorem 6.1, we compute the parameter $J(\varphi)$. Recall that we have already shown in Proposition 3.5 that $J(\varphi) = \infty$ whenever R has a nonzero type III direct summand. So we can restrict our attention to semifinite algebras.

Let h be a positive faithful selfadjoint element affiliated with the von Neumann algebra R with center Z. Let χ be the spectral resolution of h. On the one hand, there is a set $\{e_n\}$ of central projections of sum 1 and a sequence $\{\alpha_n\}$ of positive numbers such that $\sum \chi(-\infty, \alpha_n)e_n = 0$. Thus, the set of elements in Z given by

$$\{z \in Z \mid \chi(-\infty, z) = 0\}$$

is upward directed and

$$v_h = \text{lub}\{z \in Z \mid \chi(-\infty, z) = 0\}$$

is a positive selfadjoint element affiliated with Z.

On the other hand, there is a maximal set $\{f_n\}$ of nonzero orthogonal central projections and a set $\{\beta_n\}$ of numbers such that $\sum \chi(\beta_n, \infty)f_n = 0$. Let $\sum f_n = f_h$ and let

$$w_h = \text{glb}\{z \ \eta \ Zf_h \mid \chi(z, \infty) = 0\} + \infty(1 - f_h).$$

Since $wf_h = \text{glb}\{\chi(z, \infty) \mid z\eta Zf_h\}$ is a positive selfadjoint element affiliated with Zf_h due to the fact that h is faithful and positive selfadjoint, the element w is in $\widehat{Z}^+$ (cf. [**2**]).

PROPOSITION 6.2. *Let h be a faithful positive selfadjoint element affiliated with the von Neumann algebra R. Let χ be the spectral resolution of h. Let*

$$v_h = \text{lub}\{z \in Z \mid \chi(-\infty, z) = 0\}$$

and

$$w_h = \text{glb}\{z \ \eta \ Zf_h \mid \chi(z, \infty) = 0\} + \infty(1 - f_h)$$

where $(1 - f_h)$ is the largest projection f in Z such that $\chi(\alpha, \infty)$ has central support f for every α. Then

(i) *$\chi(-\infty, z)$ has central support* 1 *for every z affiliated with Z such that $z > v_h$;*

(ii) $\chi(z, \infty)$ *has central support* 1 *for every* z *affiliated with* Z *such that* $z < w_h$; *and*

(iii) $v_h \leq h \leq w_h$.

Proof (i). Suppose that z is affiliated with Z with $v_h < z$. Then there is a sequence $\{e_n\}$ of central projections of sum 1 and sequences of numbers $\{\alpha_n\}$ and $\{\beta_n\}$ such that

$$v_h e_n < \alpha_n e_n < z e_n < \beta_n e_n.$$

We must have that $\chi(-\infty, \alpha_n)e_n$ has central support e_n by the definition of v_h. Thus, the projection $\sum \chi(-\infty, \alpha_n)e_n$ and consequently the projection $\chi(-\infty, z)$ has central support 1.

Proof (ii). Same as (i).

Proof (iii). Let $\{e_n\}$ be an orthogonal sequence of central projections of sum 1 and let $\{\alpha_n\}$ be a sequence of numbers with

$$\sum \chi(-\infty, \alpha_n)e_n = 0.$$

Then we have that, for all $m_n > \alpha_n$,

$$\alpha_n \chi(-\infty, m_n)e_n \leq h\chi(-\infty, m_n)e_n.$$

Then for all finite sums we have that

$$\sum \alpha_n \chi(-\infty, m_n)e_n = \sum h\chi(-\infty, m_n)e_n.$$

Hence, we have that

$$\sum \alpha_n e_n \leq h,$$

and finally, that

$$v_h \leq h$$

(cf. [**2**, §1]).

Now let $\{e_n\}$ be an orthogonal sequence of central projections of sum f_h and let $\{\beta_n\}$ be a sequence of numbers with $\sum \chi(\beta_n, \infty)e_n = 0$. Then we have that

$$h\chi(j_n, k_n)e_n \leq \beta_n \chi(j_n, k_n)e_n$$

for all $j_n < \beta_n < k_n$. So we have that

$$h\chi(j_n, k_n)e_n \leq w_h e_n$$

as a relation for bounded operators and

$$hf_h \leq w_h f_h$$

by taking least upper bounds. Since $h(1-f_h) \leq \infty(1-f_h)$, we get

$$h \leq w_h. \qquad \square$$

DEFINITION 6.3. Let φ be a f.s.n. weight on the semifinite von Neumann algebra R with center Z, let τ be a f.s.n. trace on R, and let h be the Radon-Nikodym derivative of φ with respect to τ. Let χ be the spectral resolution of h and let

$$v = v_h = \text{lub}\{z \in Z \mid \chi(-\infty, z) = 0\}$$

and

$$w = w_h = \text{glb}\{z \; \eta \; Zf_h \mid \chi(z, \infty) = 0\} + \infty(1-f_h)$$

where $(1-f_h)$ is the largest projection f in Z such that $\chi(\alpha, \infty)$ has central support f for every α. Then the *central size* of φ is the number

$$\gamma = \gamma_\varphi = \begin{cases} \|v^{-1}w\| & \text{if } v^{-1}w \text{ is bounded}, \\ \infty & \text{otherwise.} \end{cases}$$

Notice that v and w do not depend on the choice of the trace. In fact, if τ' is a second n.s.f. trace on R, then there is an $x > 0$ affiliated with Z such that

$$\tau = \tau' \cdot x.$$

The Radon-Nikodym derivative of φ with respect to τ' is xh. Then we have that $\chi(-\infty, z)(h) = 0$ if and only if $\chi(-\infty, \chi z)(xh) = 0$ (cf. §4, Property 6) so that $\chi v_h = v_{xh}$. Similar reasoning gives $\chi w_h = w_{xh}$.

THEOREM 6.4. *If R is a semifinite algebra, let φ be a f.s.n. weight on R and let $\gamma = \gamma_\varphi$ be the central size of φ; then*

$$J(\varphi) - \frac{1}{2}(1+\gamma).$$

Proof. Let us first prove that $J(\varphi) \leq \frac{1}{2}(1+\gamma)$. Clearly we need to consider only the case that $\gamma \neq \infty$. Since

$$\tau' \leq \varphi \leq \gamma\tau',$$

where $\tau' = \tau(v\cdot)$, we have $M_\varphi = M_{\tau'}$. Let p and q be nonzero projections in M_φ; then $r = p \vee q$ is in $M_{\tau'}$. Hence R_r is a finite algebra and so the restriction φ' of φ to R_r is finite and normal and faithful. Thus, by Proposition 5.4 applied to R_r, we can find a decomposition of $r = r_- + r_0 + r_+$ into three mutually orthogonal projections such that $r_- \sim r_+$ and

$$\varphi'(p+q) = \varphi(p+q) \geq I(\varphi') \geq 2\varphi'(r_-) + \varphi'(r_0) \geq 2\varphi(r_-) + \varphi(r_0).$$

Then we get

$$\begin{aligned}\frac{\varphi(p \vee q)}{\varphi(p+q)} &\leq \frac{\varphi(r_-) + \varphi(r_0) + \varphi(r_+)}{2\varphi(r_-) + \varphi(r_0)} \\ &\leq \frac{1}{2} + \frac{2\tau(wr_+) + \varphi(r_0)}{2\tau(vr_-) + \varphi(r_0)} \leq \frac{1}{2} + \frac{2\gamma\tau(vr_+) + \varphi(r_0)}{2\tau(vr_+) + \varphi(r_0)}\end{aligned}$$

so that

$$\frac{\varphi(p \vee q)}{\varphi(p+q)} \leq \frac{1}{2}(1+\gamma).$$

since $\gamma \geq 1$. Since p and q are arbitrary, we obtain

$$J(\varphi) \leq \frac{1}{2}(1+\gamma).$$

Now we prove the reverse inequality. First suppose that χ is the spectral resolution of h. Suppose that $\chi(\alpha, \alpha')$ and $\chi(\beta, \beta')$ have the same nonzero central support e for some $0 \leq \alpha < \alpha' < \beta < \beta'$. Then there are finite equivalent p and q with $p \leq \chi(\alpha, \alpha')$ and $q \leq \chi(\beta, \beta')$. For every $\eta > 0$, there is a projection q_η such that $\varphi(q_\eta) < \varphi(p) + \eta$ and $q_\eta \vee p = p + q$. Then we have

$$J(\varphi) \geq \frac{\varphi(p \vee q_\eta)}{\varphi(p + q_\eta)} \geq \frac{\varphi(p+q)}{2\varphi(p) + \eta} \geq \frac{(\alpha+\beta)\tau(p)}{2\alpha'\tau(p) + \eta}.$$

Since $\eta > 0$ is arbitrary, we have that

$$J(\varphi) \geq \frac{\alpha+\beta}{2\alpha'}.$$

Now we consider two cases: (i) the null projection e of v is nonzero and (ii) $e = 0$. In case (i) we have that $\gamma = \infty$. Let $\beta > 0$ be any number such that $\chi(\beta, \infty)e \neq 0$. By replacing e by a smaller nonzero projection if necessary, there is no loss of generality in the assumption that e is a nonzero central projection such that $ve = 0$ and $\chi(\beta, \infty)e$ has central support e. Now by the definition v we have that $\chi(0, \alpha')e$ has central support e for every $\alpha' > 0$. By the previous paragraph there are projections p and q with

$$\frac{\varphi(p \vee q)}{\varphi(p+q)} \geq \frac{\beta}{2\alpha'}.$$

Since $\beta > 0$ is fixed and α' can be arbitrarily small, we get the desired relation $J(\varphi) = \infty$.

Assume now that $e = 0$. For every integer n define

$$\delta_n = \begin{cases} \gamma - \frac{1}{n} & \text{if } \gamma \neq \infty, \\ n & \text{if } \gamma = \infty. \end{cases}$$

If $\gamma = 1$, then $v = w = h$; hence, we see that φ is a trace and therefore $J(\varphi) = 1 = (1+\gamma)/2$.

Assume, therefore, that $\gamma > 1$, and choose n so that also $\delta_n < \gamma$. Then, by the definition of γ, we can find a nonzero central projection f such that $\delta_n f < wv^{-1}f$. For every $\varepsilon > 0$, there are mutually orthogonal projections $\chi(\alpha, \alpha')$ and $\chi(\beta, \infty)$ with $\beta/\alpha' > \delta$ such that

$$c(\chi(\alpha, \alpha'))c(\chi(\beta, \infty)) \neq 0.$$

As before, we have that

$$J(\varphi) \geq 2^{-1}(1 + (\beta/\alpha')) \geq 2^{-1}(1+\delta).$$

This means that $J(\varphi) \geq (1+\gamma)/2$ as desired. □

COROLLARY 6.5. *The weight φ is a trace if and only if $J(\varphi) = 1$.*

Proof. On the one hand, if $J(\varphi) = 1$, then R can have no type III direct summand (Proposition 3.5). Thus, the algebra R is semifinite. Then we have that $\gamma_\varphi = 1$ and $\varphi = \tau \cdot h$ for some injective positive selfadjoint operator with $v \leq h \leq v$ with $v \in \widehat{Z}^+$. This means that $h = v$ and thus that φ is a trace. On the other hand, if φ is a trace, then $\gamma_\varphi = 1$ and $J(\varphi) = (1+\gamma_\varphi)/2 = 1$. □

7. The main result. We can now restate our main theorem.

THEOREM 1. *Let R be a σ-finite von Neumann algebra and let φ be a f.s.n. weight on R. Then $P(M_\varphi)$ is a lattice if and only if there is a decomposition of the identity into mutually orthogonal central projections $e + f + g = 1$ such that R_f is a semifinite algebra and R_g is a direct sum of type I_∞ factors equipped with the f.s.n. trace* Tr *(the direct sum of the canonical traces on the factors) so that*

(a) *φ restricted to R_e is a finite functional,*
(b) *$P(M_{\varphi(f\cdot)}) = P(M_\tau)$ for some f.s.n. trace τ on R_f, and*
(c) *$P(M_{\varphi(g\cdot)}) \subset P(M_{\mathrm{Tr}})$.*

Proof. The proof follows from combining the statements of Proposition 3.3, Theorem 4.9, Theorem 4.10, and Theorem 5.5. Note that some of the type I_∞ factors have been included in (ii) due to Theorem 4.10. □

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