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FOLIATION BY CONSTANT MEAN CURVATURE SPHERES

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Let M be a Riemannian manifold of dimension $n+1$ and $p \in M$. Geodesic spheres around p of small radius constitute a smooth foliation. We shall show that this foliation can be perturbed into a foliation whose leaves are spheres of constant mean curvature, provided that p is a nondegenerate critical point of the scalar curvature function of M. The obtained foliation is actually the *unique* foliation by constant mean curvature hypersurfaces which is *regularly centered at* p (Definition 1.1). On the other hand, if p is not a critical point of the scalar curvature function, then there exists no such foliation.

0. Introduction. The perturbation procedure consists not only of normal perturbation of geodesic spheres, but also of perturbation of their center, which is used to deal with the kernel of the linearized operator of the perturbation equation. On the other hand, it is crucial to control the magnitude of the center perturbation in order to retain the foliation property. Note that the above mentioned "regularly centered" condition is a natural geometric one, but it appears fairly restrictive. What weaker geometric conditions imply it? Are all foliations by constant mean curvature spheres (or hypersurfaces) which are centered at p (Definition 1.1) automatically regularly centered at p? We shall treat these problems in a subsequent paper. (The answer to the second question is yes in dimension $n = 1$ and "almost" yes in dimension $n = 2$.)

The techniques in this paper can be applied to produce foliations by constant mean curvature spheres on asymptotically flat manifolds of nonzero mass.

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Notation. $O(r^m) = r^m \cdot$ a smooth function. Throughout the paper we use $\| \ \|$ to denote the norm of a vector and the specific norm used in each case should be clear from the context.

1. Perturbation. For convenience, we assume $M \in C^\infty$. We refer to [2] and [1] for basic terminology in local Riemannian geometry. Let $S_r(q)$, $B_r(q)$ denote respectively the geodesic sphere and open geodesic ball of radius $r > 0$ and center $q \in M$. Consider a (henceforth fixed) point p and choose an orthonormal basis $\{e_j\}$ for T_pM. Put $r_p = \frac{1}{8}\cdot$ the injectivity radius at p, $\mathbb{B}_r = \{x \in \mathbb{R}^{n+1} : \|x\| < r\}$, and $S^n = \partial\mathbb{B}_1$. For $\tau = (\tau^1, \ldots, \tau^{n+1}) \in \mathbb{R}^{n+1}$ with $\|\tau\| < r_p$ we define $\varphi_\tau : \mathbb{B}_{2r_p} \to M$ by

$$\varphi_\tau(x) = \exp_{c(\tau)}(x^i e_i^\tau),$$

where $c(\tau) = \exp_p(\tau^i e_i)$, e_i^τ are the parallel transports of e_i to $c(\tau)$ along the geodesic $c(t\tau)|_{0\le t\le 1}$ and the summation convention is used. For each τ, φ_τ gives rise to a Riemannian normal coordinate system at $c(\tau)$. We put $ds^2 =$ the metric tensor of M, $g_{ij}^\tau(x) =$ the coefficients of ds^2 in the coordinate φ_τ and $g^\tau = \det(g_{ij}^\tau)$. Then by a straightforward computation the inward mean curvature function of $S_\rho(c(\tau))$ (pulled back to $\partial\mathbb{B}_\rho$ by φ_τ) is given by

$$h(x) = \frac{n}{\rho} + \frac{\partial}{\partial r}\log\sqrt{g^\tau(x)}, \qquad x \in \partial\mathbb{B}_\rho, \tag{1.1}$$

where $r = \|x\|$.

This formula together with the following lemma reveals basic relations between the mean curvature of geodesic spheres and the curvature of M.

LEMMA 1.1. *We have*

$$\begin{aligned} g^\tau(x) = 1 &- \frac{1}{3}R_{ij}^\tau(0)x^i x^j - \frac{1}{6}R_{ij;k}^\tau(0)x^i x^j x^k \\ &- \Big(\frac{1}{20}R_{ij;kl}^\tau(0) + \frac{1}{90}R_{pijm}^\tau(0)R_{pklm}^\tau(0) \\ &\qquad - \frac{1}{18}R_{ij}^\tau(0)R_{kl}^\tau(0)\Big)x^i x^j x^k x^l \\ &+ a_{ijklm}(\tau, x)x^i x^j x^k x^l x^m, \end{aligned}$$

where R_{ij}^τ, $R_{ij;k}^\tau$, $R_{ij;kl}^\tau$, R_{pijm}^τ *are respectively the coefficients of* Ric (*Ricci tensor*), ∇Ric, $\nabla\nabla$Ric *and* Riem (*Riemann curvature tensor*) *in the coordinate* φ_τ *and* $a_{ijklm}(\tau, x)$ *denote some smooth functions. We shall omit the superscript* τ *for* $\tau = 0$.

For a proof of this lemma we refer to [2].

Let ν denote the inward unit normal of $S^n := \partial \mathbb{B}_1$ and α_r the dilation $x \mapsto rx$ for $r > 0$. For $\varphi \in C^2(S^n)$ we define $S^n_\varphi = \{x + \varphi(x)\nu(x) : x \in S^n\}$ and $S_{r,\tau,\varphi} = \varphi_\tau(\alpha_r(S^n_\varphi))$. Note that $S_{r,\tau,0} = S_r(c(\tau))$ and S^n_φ is an embedded C^2 surface if only $\|\varphi\|_{C^1} < \delta_0$ for some number $\delta_0 > 0$. Here and in the sequel, unless otherwise stated, we use the standard metric on S^n. For $0 < r < r_p$, $\|\tau\| < r_p$, $\|\varphi\|_{C^1} < \delta_0$ and $x \in S^n$ we put

$$H(r, \tau, \varphi)(x) = \text{the inward mean curvature of the surface } S^n_\varphi \text{ at } x + \varphi(x)\nu(x) \text{ w.r.t. the metric } ds^2_{\tau,r} \text{ on } \mathbb{B}_2,$$

where $ds^2_{\tau,r} = r^{-2}\alpha_r^*(\varphi_\tau^*(ds^2))$. One readily checks that $ds^2_{\tau,r}$ extends smoothly to $r = 0$ with $ds^2_{\tau,0} \equiv$ the euclidean metric. Hence $H(r, \tau, \varphi)$ also extends to $r = 0$. On the other hand, we have

$$H(r, \tau, \varphi)(x) = r \cdot \text{the inward mean curvature of } S_{r,\tau,\varphi} \text{ at } \varphi_\tau(r(x + \varphi(x)\nu(x))). \tag{1.3}$$

In particular, by (1.1)

$$H(r, \tau, 0)(x) = n + r\frac{\partial}{\partial r}\log\sqrt{g^\tau(rx)}, \tag{1.4}$$

which implies on account of (1.2)

$$H(r, \tau, 0)(x) = n - \frac{1}{3}R^\tau_{ij}(0)x^i x^j r^2 - \frac{1}{4}R^\tau_{ij;k}(0)x^i x^j x^k r^3 + a_{ijkl}(\tau)x^i x^j x^k x^l r^4 + O(r^5) \tag{1.5}$$

where $a_{ijkl}(\tau)$ denotes some smooth functions.

We consider $H(r, \tau, \cdot)$ as a mapping from $C^{2,1/2}(S^n)$ into $C^{0,1/2}(S^n)$ and let H_φ denote the differential of H w.r.t. φ. Clearly $H_\varphi(r, \tau, 0)$ is just the Jacobi operator $\Delta + \|B\|^2 + \mathrm{Ric}(\nu)$ on S^n relative to the metric $ds^2_{\tau,r}$, where B denotes the second fundamental form. We indicate the dependence on $ds^2_{\tau,r}$ as follows

$$H_\varphi(r, \tau, 0) = \Delta_{r,\tau} + \|B_{r,\tau}\|^2 + \mathrm{Ric}_{r,\tau}(\nu). \tag{1.6}$$

It follows that $H_\varphi(0, \tau, 0) = L := \Delta_{S^n} + n$, where Δ_{S^n} is the standard Laplace operator on S^n. The Jacobi operator L has an $(n+1)$-dimensional kernel K consisting of first order spherical harmonics. Correspondingly, we obtain L_2-decompositions $C^{2,1/2}(S^n) = K \oplus K^\perp$ and $C^{0,1/2}(S^n) = K \oplus L(K^\perp)$. Let P denote the orthogonal projection from $C^{0,1/2}(S^n)$ onto K and $T: K \to \mathbb{R}^{n+1}$ the isomorphism sending $x^i|_{S_n}$ to $\mathbf{e}_i =$ the ith coordinate basis. Put $\tilde{P} = TP$.

LEMMA 1.2. *We have*

$$\widetilde{P}(H(r,\tau,\varphi)) = -\frac{\omega_{n+1}}{2(n+3)} r^3 R^{\tau}_{,l}(0) e_l + \widetilde{P}\left(\int_0^1 H_{\varphi}(r,\tau,t\varphi)\varphi\, dt\right) + O(r^5), \tag{1.7}$$

where $\omega_{n+1} = \mathrm{Vol}(\mathbb{B}_1)$ *and* $R^{\tau}_{,l}$ *denotes the derivatives of the scalar curvature function* R *of* M *in the coordinate system* φ_τ.

Proof. From (1.5) we deduce

$$\widetilde{P}(H(r,\tau,0)) = -\frac{1}{4} r^3 e_l R^{\tau}_{ij;k}(0) \int_{S^n} x^i x^j x^k x^l\, dA + O(r^5).$$

But ($R^{\tau}_{ij;k}$ stands for $R^{\tau}_{ij;k}(0)$. No summation over l.)

$$\begin{aligned}
&R^{\tau}_{ij;k} \int_{S^n} x^i x^j x^k x^l\, dA \\
&\quad = R^{\tau}_{ll;l} \int_{S^n} (x^l)^4\, dA + \sum_{j,k\neq l} R^{\tau}_{lj;k} \int_{S^n} (x^l)^2 x^j x^k\, dA \\
&\qquad + \sum_{k,i\neq l} R^{\tau}_{il;k} \int_{S^n} (x^l)^2 x^i x^k\, dA + \sum_{i,j\neq l} R^{\tau}_{ij;l} \int_{S^n} (x^l)^2 x^i x^j\, dA \\
&\qquad + \sum_{i,j,k\neq l} R^{\tau}_{ij;k} \int_{S^n} x^i x^j x^k x^l\, dA \\
&\quad = R^{\tau}_{ll;l} \int_{S^n} (x^l)^4\, dA + (R^{\tau}_{lj;j} + R^{\tau}_{il;i} + R^{\tau}_{ii;l}) \int_{S^n} (x^1)^2 (x^2)^2\, dA \\
&\qquad - 3R^{\tau}_{ll;l} \int_{S^n} (x^1)^2 (x^2)^2\, dA = \frac{2\omega_{n+1}}{n+3} R^{\tau}_{,l},
\end{aligned}$$

since $\int_{S^n} (x^l)^4\, dA = 3\int_{S^n} (x^1)^2 (x^2)^2\, dA = 2\omega_{n+1}/(n+3)$ and $R^{\tau}_{lj;j} = R^{\tau}_{il;i} = \frac{1}{2} R^{\tau}_{,l}$ by the second Bianchi identity. Hence (1.7) follows. □

LEMMA 1.3. *Set*

$$H_{\varphi r} = \frac{\partial}{\partial r} H_{\varphi}, \quad H_{\varphi rr} = \frac{\partial^2}{\partial r^2} H_{\varphi}.$$

We have for all τ

(1) $H_{\varphi r}(0,\tau,0) = 0$,

(2) $H_{\varphi rr}(0,\tau,0)$ *is an even operator in the sense that if* φ *is an even function, i.e.* $\varphi(-x) = \varphi(x)$, *then* $H_{\varphi rr}(0,\tau,0)\varphi$ *is also an even function.*

Proof. Consider an arbitrary point $x_0 \in S^n$. We may assume $x_0 = (0, \dots, 0, 1)$. Choose a local coordinate system $x = x(\theta)$ of S^n around x_0 with $x(0) = x_0$, which is normal at x_0 w.r.t. the standard metric. Then

$$\Delta_{r,\tau} = \frac{1}{\sqrt{h}} \frac{\partial \sqrt{h}}{\partial \theta_i} h^{ij} \frac{\partial}{\partial \theta_j} + \frac{\partial h^{ij}}{\partial \theta_i} \frac{\partial}{\theta_j} + h^{ij} \frac{\partial^2}{\partial \theta_i \partial \theta_j} \tag{1.8}$$

where

$$(h^{ij}(\theta)) = (h_{ij}(\theta))^{-1}, \; h(\theta) = \det(h_{ij}(\theta)) \quad \text{and}$$

$$h_{ij}(\theta) = ds^2_{r,\tau}(\partial/\partial\theta_i, \partial/\partial\theta_j)$$

(the dependence on r, τ is suppressed). Let the matrix $a(\theta) = (a_i^j(\theta))$ be defined by the relation

$$\left(\frac{\partial}{\partial \theta_1}, \dots, \frac{\partial}{\partial \theta_n}, \frac{\partial}{\partial r}\right) = \left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n+1}}\right) a(\theta).$$

Then we have

$$a^{n+1}(0) = 0, \qquad 1 \le i \le n; \tag{1.9}$$

$$\frac{\partial}{\partial \theta_k} a_i^j(0) = 0, \qquad 1 \le i, j, k \le n; \tag{1.10}$$

$$h(\theta) = [\det a(\theta)]^2 g^\tau(rx(\theta)). \tag{1.11}$$

Note that (1.10) follows from the assumption that $x(\theta)$ is normal at $x_0 = x(0)$. Since $\det a(\theta) = \| \frac{\partial}{\partial \theta_1} \wedge \cdots \wedge \frac{\partial}{\partial \theta_n} \|$, we also deduce from this assumption that

$$\frac{\partial}{\partial \theta_i} (\det a)(0) = 0, \qquad 1 \le i \le n. \tag{1.12}$$

From (1.10), (1.12) and the fact $(\partial g^\tau / \partial x^k)(0) = 0$, $1 \le k \le n+1$ we immediately obtain

$$\frac{\partial}{\partial r} h(0)\Big|_{r=0} = 0, \qquad \frac{\partial}{\partial \theta_i} h(0)\Big|_{r=0} = 0, \tag{1.13}$$

$$\frac{\partial}{\partial r} \frac{\partial}{\partial \theta_i} h(0)\Big|_{r=0} = 0, \qquad 1 \le i \le n.$$

On the other hand, we have

$$h_{ij}(\theta) = a_i^k(\theta) a_j^l(\theta) g_{kl}^\tau(rx(\theta)); \tag{1.14}$$

hence

$$\frac{\partial}{\partial r}h_{ij}(0)\Big|_{r=0} = a_i^k(0)a_j^l(0)g^\tau_{kl,m}(0)x^m(0) = 0, \tag{1.15}$$

$$\begin{aligned}\frac{\partial}{\partial r}\frac{\partial}{\partial \theta_k}h_{ij}(0)\Big|_{r=0} &= \frac{\partial}{\partial \theta_k}(a_i^m a_j^l)(0)g^\tau_{ml,m'}(0)x^{m'}(0) \\ &\quad + a_i^m(0)a_j^l(0)g^\tau_{ml,m'}(0)\frac{\partial x^{m'}}{\partial \theta_k}(0) = 0.\end{aligned} \tag{1.16}$$

From (1.8), (1.15) and (1.16) one easily derives $(\partial/\partial r)\Delta_{r,\tau}|_{r=0} = 0$.

Next we compute $\|B_{r,\tau}\|^2$. Using the Gram-Schmidt orthogonalization procedure and the fact $g^\tau_{ij}(rx) = \delta_{ij} + O(r^2)$ we find orthonormal bases $e_i(r, \tau)$, $1 \le i \le n$, for $T_{x_0}S^n$ relative to $ds^2_{r,\tau}$ with $e_i(r, \tau) = \partial/\partial x^i + O(r^2)$. We compute the coefficients of the second fundamental form $B_{r,\tau}$

$$b^{r,\tau}_{ij} = \nabla^{r,\tau}_{e_i(r,\tau)}\left(x^k\frac{\partial}{\partial x^k}\right)\cdot e_j(r, \tau),$$

where $\nabla^{r,\tau}$ and the dot denote respectively the covariant differentiation and the inner product relative to the metric $ds^2_{r,\tau}$. Observe

$$\nabla^{r,\tau}_{\partial/\partial x^i}\frac{\partial}{\partial x^j} = r^{(\tau)}\Gamma^k_{ij}(rx)\frac{\partial}{\partial x^k},$$

where ${}^{(\tau)}\Gamma^k_{ij}$ are the Christoffel symbols of the metric ds^2 of M in the coordinates φ_τ. Consequently

$$\begin{aligned} b^{r,\tau}_{ij} &= \nabla^{r,\tau}_{\partial/\partial x^k}\left(x^k\frac{\partial}{\partial x^k}\right)\cdot\frac{\partial}{\partial x^j} + O(r^3) \\ &= g_{ij}(rx_0) + rx_0^k{}^{(\tau)}\Gamma^l_{ki}(rx_0)g_{lj}(rx_0) + O(r^3), \end{aligned} \tag{1.17}$$

$$\begin{aligned}\|B_{r,\tau}\|^2(x_0) = &\sum_{1\le i,j\le n} g^2_{ij}(rx_0) \\ &+ 2rx_0^k\sum_{1\le i,j\le n} g_{ij}(rx_0)^{(\tau)}\Gamma^l_{ki}(rx_0)g_{lj}(rx_0) + O(r^3).\end{aligned}$$

It follows that $(\partial/\partial r)\|B_{r,\tau}\|^2(x_0)|_{r=0} = 0$. On the other hand it is easy to see that

$$\mathrm{Ric}_{r,\tau}(\nu)(x_0) = r^2R_{ij}(rx_0)x_0^ix_0^j;$$

hence $\partial/\partial r\,\mathrm{Ric}_{r,\tau}(\nu)(x_0)|_{r=0} = 0$ and therewith the first part of the lemma is proved.

Notice now $(\partial^2/\partial r^2)\,\mathrm{Ric}_{r,\tau}(\nu)(x_0)|_{r=0} = 2R_{ij}(0)x_0^i x_0^j$. From (1.17) we also get

$$\frac{\partial^2}{\partial r^2}\|B_{r,\tau}\|^2(x_0)\Big|_{r=0} = 2g_{ij}(0)g_{ij,kl}(0)x_0^k x_0^l + 2x_0^k \sum_{i\le i,\, j\le n} g_{ij}(0)\cdot \frac{\partial}{\partial x^m}{}^{(\tau)}\Gamma_{ki}^l(0)x_0^m g_{lj}(0).$$

Using $x = -x(\theta)$ as a coordinate system around $-x_0$ and performing the same computations as above we then deduce

$$\frac{\partial^2}{\partial r^2}\,\mathrm{Ric}_{r,\tau}(\nu)(-x_0)\Big|_{r=0} = \frac{\partial^2}{\partial r^2}\,\mathrm{Ric}_{r,\tau}(\nu)(x_0)\Big|_{r=0} \quad \text{and}$$

$$\frac{\partial^2}{\partial r^2}\|B_{r,\tau}\|^2(-x_0)\Big|_{r=0} = \frac{\partial^2}{\partial r^2}\|B_{r,\tau}\|^2(x_0)\Big|_{r=0}.$$

From (1.8), (1.9), (1.13), (1.14), (1.15) and (1.16) one obtains through elementary computations that at x_0 the operator

$$(\partial^2/\partial r^2)\Delta_{r,\tau}|_{r=0}$$

equals

$$-\frac{1}{2}\frac{\partial^2}{\partial r^2}\frac{\partial}{\partial\theta_i}h(0)\Big|_{r=0}\frac{\partial}{\partial\theta_i} + \left(\frac{\partial^2}{\partial r^2}A_{ij}(0)\Big|_{r=0} - \frac{\partial^2}{\partial r^2}h(0)\Big|_{r=0}\delta_{ij}\right)\cdot\frac{\partial^2}{\partial\theta_l\partial\theta_j},$$

where A_{ij} denotes the cofactors of the matrix (h_{ij}). But

$$\frac{\partial^2}{\partial r^2}\frac{\partial}{\partial\theta_i}h(0)\Big|_{r=0} = \frac{\partial^2}{\partial r^2}\frac{\partial}{\partial\theta_i}g^\tau(rx(\theta))\Big|_{\theta=0,\,r=0} = \frac{\partial^2 g^\tau}{\partial x^k\partial x^l}(0)\frac{\partial x^k}{\partial\theta_i}(0)x^l(0);$$

hence it is quadratic in (x, Dx), where $Dx = (\partial x^i/\partial\theta_j)_{1\le i\le n+1,\,1\le j\le n}$. Easy computations also show that the coefficients of $\partial^2/\partial\theta_l\partial\theta_j$ above are quadratic in (x, Dx). Employing the coordinates $-x(\theta)$ around $-x_0$ we then see that $(\partial^2/\partial r^2)\Delta_{r,\tau}|_{r=0}$ is an even operator. Since

$$\frac{\partial^2}{\partial r^2}\,\mathrm{Ric}_{r,\tau}(\nu)\Big|_{r=0} \quad \text{and} \quad \frac{\partial^2}{\partial r^2}\|B_{r,\tau}\|^2\Big|_{r=0}$$

have been shown to be even, the second part of the lemma is proved. □

DEFINITION 1.1. A smooth codimension 1 foliation $\mathcal{F}$ of $U\backslash\{p\}$ for a neighborhood U of p is called a *foliation centered at* p, provided that its leaves are all closed. If furthermore

$$\sup_{S\in\mathcal{F}}\left(\sup_{S}\|B_S\|\operatorname{diam}S\right)<\infty,$$

where B_S denotes the second fundamental form of S, then $\mathcal{F}$ is called a *foliation regularly centered at* p

THEOREM 1.1. *If p is a nondegenerate critical point of the scalar curvature function, then there exist $\delta>0$ and smooth functions $\tau=\tau(r)$, $\varphi=\varphi(r)$ with $\tau(0)=0$ such that $H(r,\tau(r),r^2\varphi(r))\equiv n$ for $0\le r\le\delta$. Hence the family $\mathcal{F}=\{S_r:=S_{r,\tau(r),r^2\varphi(r)}:0\le r\le\delta\}$ is a smooth family of constant mean curvature spheres with S_r having mean curvature n/r. $\mathcal{F}$ is a foliation regularly centered at p. Moreover, $\mathcal{F}$ can actually be represented as $\{S_{r,0,r\overline{\varphi}(r)}:0<r\le\delta\}$ for some smooth function $\overline{\varphi}(r)$. (In particular, each S_r is a normal graph over $S_r(p)$.)*

Proof. We first consider the equation $\widetilde{P}(H(r,\tau,r^2\varphi))=0$. By Lemma 1.2, Lemma 1.3 and the fact $\widetilde{P}L=0$ we can write this equation as follows (after division by r^3)

$$\begin{aligned}(1.18)\quad &-\frac{\omega_{n+1}}{2(n+3)}R^{\tau}_{,l}(0)\mathbf{e}_l+O(r^2)\\ &+\widetilde{P}\left(r\int_0^1\int_0^1 tH_{\varphi\varphi}(sr,\tau,str^2\varphi)\varphi\varphi\,ds\,dt\right.\\ &\quad+r\int_0^1\int_0^1\int_0^1 sH_{\varphi rr}(usr,\tau,ustr^2\varphi)\varphi\,du\,ds\,dt\\ &\quad+r^2\int_0^1\int_0^1\int_0^1 stH_{\varphi\varphi r}(usr,\tau,ustr^2\varphi)\\ &\qquad\left.\times 2\varphi\varphi\,du\,ds\,dt\right)=0,\end{aligned}$$

where

$$H_{\varphi\varphi}(r,\tau,\psi)\varphi\varphi'=\frac{d}{dt}H_{\varphi}(r,\tau,\psi+t\varphi')\varphi\Big|_{t=0},$$

and $H_{\varphi\varphi r}=(\partial/\partial r)H_{\varphi\varphi}$. By the assumption, $R^{\tau}_{,l}(0)|_{\tau=0}=0$ and the Hessian matrix

$$\left(\frac{\partial}{\partial\tau_k}R^{\tau}_{,l}(0)\Big|_{\tau=0}\right)_{k,l}$$

is nonsingular. Applying the implicit function theorem we obtain a solution $\tau = \tau(r, \varphi)$ of the equation $\tilde{P}(H(r, \tau, r^2\varphi)) = 0$ around $r = 0$, $\varphi = \varphi_0$ with $\tau(0, \varphi_0) = 0$, where $\varphi_0 \in K^\perp$ is determined by the relation $L\varphi_0 = \frac{1}{3}R_{ij}(0)x^i x^j$ ($R_{ij}(0)x^i x^j$ denotes the function defined on S^n whose value at $x \in S^n$ is given by the quadratic form $R_{ij}(0)x^i x^j$).

Now we consider the mapping $(r, \varphi) \mapsto H(r, \tau(t, \varphi), r^2\varphi)$ whose values lie in $K^\perp$ by the construction of $\tau(r, \varphi)$. We restrict φ to be in $K^\perp$. From (1.5) and Lemma 1.3 we obtain (for small r, τ, φ)

$$
\begin{aligned}
(1.19)\quad H(r, \tau, \varphi) = n &- \frac{1}{3}R^\tau_{ij}(0)x^i x^j r^2 \\
&- \frac{1}{4}R^\tau_{ij;k}(0)x^i x^j x^k r^3 + O(r^4) \\
&+ \int_0^1 \int_0^1 tH_{\varphi\varphi}(sr, \tau, st\varphi)\varphi\varphi \, ds\, dt \\
&+ r^2 \int_0^1 \int_0^1 \int_0^1 sH_{\varphi r v}(usr, \tau, ust\varphi)\varphi \, du\, ds\, dt \\
&+ r \int_0^1 \int_0^1 \int_0^1 stH_{\varphi\varphi r}(usr, \tau, ust\varphi)\varphi\varphi \, du\, ds\, dt \\
&+ L\varphi.
\end{aligned}
$$

By the implicit function theorem we can solve $H(t, \tau(r, \varphi), r^2\varphi) = n$ after dividing it out by r^2 to get $\varphi = \varphi(r)$, $0 < r \leq \delta$ for some $\delta > 0$ with $\varphi(0) = \varphi_0$.

Intermediate Remark. Define a map W from a neighborhood of $(0, 0, \varphi_0) \in \mathbb{R} \times \mathbb{R}^{n+1} \times K^\perp$ into $\mathbb{R}^{n+1} \times L(K^\perp)$ as follows

$$
W(r, \tau, \varphi) = \left(\frac{1}{r^3}\tilde{P}(H(r, \tau, r^2, \varphi)), \frac{1}{r^2}P^\perp(H(r, \tau, r^2\varphi) - n) \right),
$$

where $P^\perp$ denotes the L_2 orthogonal projection from $C^{0,1/2}(S^n)$ onto $L(K^\perp)$. Then $W(r, \tau(r), \varphi(r)) \equiv 0$ and the implicit function theorem implies: for $(r, \tau, \varphi) \in \mathbb{R} \times \mathbb{R}^{n+1} \times K^\perp$ with r, τ and $\|\varphi - \varphi_0\|$ sufficiently small we have $\tau = \tau(r)$, $\varphi = \varphi(r)$, provided that $W(r, \tau, \varphi) = 0$.

It is clear that the family $\{S_r = S_{r, \tau(r), r^2\varphi(r)} : 0 < r \leq \delta\}$ with $\tau(r) = \tau(r, \varphi(r))$ is a smooth family of embedded constant mean curvature spheres with S_r having mean curvature n/r. Geometrically, we obtained this family by moving the center p of the geodesic spheres $S_r(p)$ to $\exp_p(\tau^i(r)e_i)$ and then performing the normal perturbation $r^3\varphi(r)$. In order to show that this family constitutes a foliation, we

need to be sure that the new centers are sufficiently close to p. From (1.18) we see that

$$\tau(r) = O(r^2), \tag{1.20}$$

whenever the following is true

$$\tilde{P}(H_{\varphi\varphi}(0, \tau, 0)\varphi_0\varphi_0) = 0, \qquad \tilde{P}(H_{\varphi rr}(0, \tau, 0)\varphi_0) = 0. \tag{1.21}$$

To show (1.21) we first consider the equation

$$\Delta_{S^n}\varphi_0 + n\varphi_0 = \tfrac{1}{3}R_{ij}(0)x^i x^j.$$

By the unique solvability of this equation under the constraint $\varphi_0 \in K^{\perp}$ and the invariance of the Laplacian under isometries we deduce that φ_0 is even. Lemma 1.3 then implies the second equation in (1.21). On the other hand, we conclude that the mean curvature functions $H(0, 0, \tau\varphi_0)$ of the hypersurfaces $S^n_{t\varphi_0}$ (in the euclidean metric) are even. Since $H_{\varphi\varphi}(0, \tau, 0)\varphi_0\varphi_0 = H_{\varphi\varphi}(0, 0, 0)\varphi_0\varphi_0 = (\partial^2/\partial t^2)H(0, 0, t\varphi_0)|_{t=0}$, the first equation in (1.21) also follows and therewith (1.20) is established.

Now we define $\Psi^r = \exp_p^{-1} \exp_{c(\tau(r))}$, $\psi(r, x) = \Psi^r(r(x+r^2\varphi(r)(x)))$ and $\beta(r, x) = \psi/\|\psi\|$ if $\|\psi(r, x)\| \neq 0$, where $x \in S^n$.

Claim. For sufficiently small $r > 0$, $\|\psi(r, x)\|$ is nonzero for all $x \in S^n$ and $\beta(r, \cdot)$ is a diffeomorphism from S^n onto itself. Moreover, $\beta(r, x)$ extends smoothly to $r = 0$ with $\beta(0, \cdot) =$ the identity.

In fact, we have

$$\frac{\partial\psi}{\partial r} = (d_x\Psi^r)(x + r^2\varphi(r)(x) + r(r^2\varphi(r)(x))_r) + \left(\frac{\partial\Psi^r}{\partial r}\right)(r(x + r^2\varphi(r)(x))),$$

$$\left.\frac{\partial\Psi^r}{\partial r}\right|_{r=0} = \left.\frac{\partial}{\partial\tau^i}(\exp_p^{-1}\exp_{c(\tau)})\right|_{\tau=0}\left.\frac{\partial\tau^i}{\partial r}\right|_{r=0} = 0$$

by (1.19), hence $(\partial\psi/\partial r)(0, x) = x$. It follows that $\psi(r, x) = rx + O(r^2)$. Consequently $\psi(r, x) \neq 0$ for sufficiently small $r > 0$ and $\beta(r, x) = (x + O(r))/\|x + O(r)\|$. Clearly β extends smoothly to $r = 0$ with $\beta(0, \cdot) =$ the identity. For r near 0, $\beta(r, \cdot)$ then is a diffeomorphism.

Now set $\varphi(r, x) = \|\psi(r, \beta^{-1}(r, x))\|$, where $\beta^{-1}(r, \cdot)$ denotes the inverse of $\beta(r, \cdot)$. We have

$$\frac{\partial \varphi}{\partial r} = \frac{1}{\|\psi\|}\psi \cdot \left(\frac{\partial \psi}{\partial r} + (d_x \psi)\left(\frac{\partial}{\partial r}\beta^{-1}\right)\right)$$
$$= \frac{1}{\|x + O(r)\|}(x + O(r)) \cdot (x + O(r)),$$

whence $(\partial\varphi/\partial r)|_{r=0} = 1$. We conclude that $\varphi(r, x)$ is strictly increasing w.r.t. r for r small and hence $\tilde{S}_r$, $\tilde{S}_{r'}$ are disjoint for small r, r' with $r \neq r'$. The family $\{S_r\}_{0<r\leq\delta}$ therefore constitutes a foliation if δ is chosen small enough. It is easy to see that this foliation is regularly centered at p. The verification of the last statement of the theorem is also straightforward. □

2. Uniqueness and nonexistence. Let $\mathcal{F}$ be a foliation regularly centered at p, whose leaves have constant mean curvature.

LEMMA 2.1. *The following holds*:

(1) *There is a neighborhood* Ω *of* p *together with a constant* $c > 1$ *such that the absolute value of the mean curvature of* S *lies in the interval* $(1/c \operatorname{diam} S, c/\operatorname{diam} S)$, *provided that* S *is a leaf of* $\mathcal{F}$ *and* $S \subset \Omega$;

(2) $\operatorname{diam} S \to 0$ *as* $\operatorname{dist}(p, s) \to 0$ *for leaves* S *of* $\mathcal{F}$;

(3) *each leaf of* $\mathcal{F}$ *bounds a domain containing* p;

(4) *the leaves of* $\mathcal{F}$ *can be parametrized as a smooth family* S_t, $0 < t \leq 1$ *with* $S_t \neq S_{t'}$ *if* $t \neq t'$ *and* $\lim_{t\to 0} \operatorname{diam} S_t = 0$.

Proof. (1) Choose Ω such that $\operatorname{dist}^2(p, \cdot)$ is smooth and strictly convex on $\overline{\Omega}$. For $S \in \mathcal{F}$, $S \subset \Omega$ let ρ denote the restriction of this function to S. At a maximum point we have $\Delta\rho \leq 0$, which by simple computations implies the desired lower bound. The upper bound follows from Definition 1.1.

(2) This is a simple consequence of the assumption that all leaves of $\mathcal{F}$ are closed.

(3) and (4) Consider a leaf S sufficiently close to p. Clearly S bounds a domain. But a compact manifold with nonempty connected boundary cannot be smoothly foliated by closed leaves, whence this domain contains p. By a continuity argument one then derives (3)

and that the leaf topology of $\mathscr{F}$ is equivalent to $(0, 1]$. The last fact implies (4). □

LEMMA 2.2. *For t sufficiently small, S_t is unstable, i.e. the first eigenvalue of the Jacobi operator $\Delta + \|B\|^2 + \mathrm{Ric}(\nu)$ on S_t is negative. Consequently, $H'(t) \neq 0$, where $H(t)$ denotes the inward mean curvature of S_t.*

Proof. Consider a sequence $t_k \to 0$ and write S_{t_k} as S_k. Define $\widetilde{S}_k = \alpha_{1/d(S_k)}(\varphi_0(S_k))$. Then the diameter of $\widetilde{S}_k$ relative to the metric $ds^2_{0,d(S_k)}$ is 1. Since $ds^2_{0,d(S_k)}$ converges to the euclidean metric as $k \to \infty$ and each $\widetilde{S}_k$ encloses the origin, we may assume $\widetilde{S}_k \subset \mathbb{B}_2$ for all k. Then the second fundamental form bound postulated in Definition 1.1 and the standard elliptic PDE theory imply that local pieces of $\widetilde{S}_k$ subconverge smoothly. Consequently one of the following two cases occurs: (i) (after selection of a subsequence of S_k) there are pieces $\widetilde{S}'_k$, $\widetilde{S}''_k$ of $\widetilde{S}_k$ such that $\widetilde{S}'_k$ and $\widetilde{S}''_k$ converge to the same piece of surface, while the outer unit normal approaches opposite directions. This implies that the mean curvature of $\widetilde{S}_k$ relative to $ds^2_{0,d(S_k)}$ converges to zero. (ii) A subsequence $\widetilde{S}_{k'}$ converges smoothly to an immersed, closed hypersurface S of euclidean constant mean curvature, which is the boundary of a domain. The classical argument of Alexandrov shows that S is a round sphere. By Lemma 2.2, case (1) cannot happen. Since round spheres are unstable in the euclidean metric (the first eigenvalue of the Jacobi operator on a round sphere of radius 1 is $-n$), $\widetilde{S}_{k'}$ is unstable whenever k' is large enough. Since the sequence t_k is arbitrary, we conclude that for t sufficiently small, S_t is unstable.

Now assume $H'(t) = 0$. Representing $S_{t'}$ as normal graphs over S_t for t' near t, we derive that the Jacobi operator on S_t has a positive eigenfunction with eigenvalue 0. This would imply that S_t is stable. □

Since $H'(t) \neq 0$, we can introduce a new parameter $r = n/H(t)$. Then we obtain a parametrization S_r, $0 < r \leq r_0$ (with some $r_0 > 0$) for the leaves of (a restriction of) $\mathscr{F}$ with the property that the inward mean curvature of S_r equals n/r. We put $\widetilde{S}_r = \alpha_{1/r}(\varphi_0^{-1}(S_r))$ and $x(r) =$ the center of mass for $\widetilde{S}_r$.

Let $S^n(a)$ denote the unit sphere of a center a in $\mathbb{R}^{n+1}$. For a smooth function φ on $S^n(a)$, $S^n(a)_\varphi$ is defined in a similar way to S^n_φ.

LEMMA 2.3. *There exists some* $\bar{r}_0 > 0$ *such that for each* $r \in (0, \bar{r}_0)$, $\widetilde{S}_r = S^n(x(r))_{\overline{\varphi}(r)}$ *for a smooth function* $\overline{\varphi}(r)$ *on* $S^n(x(r))$ *with* $\lim_{r\to 0} \|\overline{\varphi}(r)\|_{C^3} = 0$. *(It follows automatically that* $\overline{\varphi}(r)$ *depends smoothly on* r.)

Proof. Assume the contrary. Then there is a sequence $r_k \to 0$ together with a number $\delta > 0$ such that for each k, either $\widetilde{S}_{r_k}$ is not a smooth normal graph over $S^n(x(r_k))$ or $\widetilde{S}_{r_k} = S^n(x(r_k))_{\varphi_k}$ for a function φ_k with $\|\varphi_k\|_{C^3} \geq \delta$. Note that φ_k is uniquely determined and the mean curvature of $\widetilde{S}_r$ in the metric of $ds^2_{0,r}$ equals n. By the arguments in the proof of Lemma 2.2 we can find a subsequence $\widetilde{S}_{r_{k'}}$ which converges smoothly to a round sphere of radius 1. For k' large we then have $\widetilde{S}_{r_{k'}} = S^n(x(r_{k'}))_{\varphi_{k'}}$ with $\|\varphi_{k'}\|_{C^3} < \delta$, a contradiction. □

COROLLARY 2.1. *There are* $r_1 > 0$, $\delta > 0$ *such that for* $r \leq r_1$ *and* $\|a\| \leq \delta$, $\alpha_{1/r}(\varphi^{-1}_{r(x(r)+a)}(S_r))$ *is a smooth normal graph over* S^n, *thus* $S_r = S_{r, r(x(r)+a), \varphi(r,a)}$ *for a smooth function* $\varphi(r, a)$ *on* S^n. *We have* $\|\varphi(r, a)\|_{C^3} \to 0$ *as* $r \to 0$, $a \to 0$.

Proof. Firstly, there are $r_1 > 0$, $\delta > 0$ such that for $r \leq r_1$, $\|a\| \leq \delta$, $\widetilde{S}_r$ is a smooth (euclidean) normal graph over $S^n(a + r(r))$, thus $\widetilde{S}_r = S^n(a+x(r))_{\overline{\varphi}(r,a)}$ for a smooth function $\overline{\varphi}(r, a)$ on $S^n(a+x(r))$. Since the metric $ds^2_{0,r}$ approaches the euclidean metric as $r \to 0$, for r small enough (and $\|a\| \leq \delta$), $\widetilde{S}_r$ will be a smooth (geodesic) normal graph over the geodesic sphere $S^n_*(a + x(r))$ of radius 1 and center $a + x(r)$ relative to $ds^2_{0,r}$. We denote the defining function of this normal graph relative to the inward normal by $\overline{\varphi}(r, a)$. It follows that S_r is a smooth normal graph over $S_r(\varphi_0(r(a + x(r))))$, which in turn implies $S_r = S_{r, r(a+x(r)), \varphi(r,a)}$. We can relate $\varphi(r, a)$ to $\overline{\varphi}(r, a)$ as follows. Denote by $\psi_{r,a}$ the exponential map at $a + x(r)$ relative to the metric $ds^2_{0,r}$. Then $\varphi(r, a)(x) = \overline{\varphi}(r, a)(\psi_{r,a}(x^i e_i^*))$, where $e_i^* = (\varphi_0^{-1})_*(e_i^{r(a+x(r))})$. The last statement of the corollary follows easily. □

Next we want to compute the dependence of $\varphi(r, a)$ on a. Let us consider a hypersurface S which is a smooth normal graph over $S^n(a_0)$ as well as $S^n(a_0 + a)$, thus $S = S^n(a_0)\varphi = S^n(a_0 + a)_{\varphi'}$ for some φ, φ'. Given $x \in S^n(a_0)$, we consider the point $y \in S$ above

x, $y = a_0 + (1 - \varphi(x))(x - a_0)$. For a unique point $x' \in S^n(a_0 + a)$ we also have $y = a_0 + a + (1 - \varphi'(x'))(x' - a_0 - a)$. It follows that

$$x' = F(x) := a + a_0 + \frac{(1 - \varphi(x))(x - a_0) - a}{\|(1 - \varphi(x))(x - a_0) - a\|}, \tag{2.1}$$
$$\varphi'(x') = 1 - \|(1 - \varphi(x))(x - a_0) - a\|.$$

We consider the projection of φ' onto the spherical harmonics on $S^n(a_0 + a)$,

$$\begin{aligned} &\frac{1}{\omega_{n+1}} \int_{S^n(a_0+a)} \varphi'(x')(x' - a_0 - a)\, dA \\ &\quad = \frac{1}{\omega_{n+1}} \int_{S^n(a_0)} (1 - \|(1 - \varphi(x))(x - a_0) - a\|) \\ &\qquad \cdot \frac{(1 - \varphi(x))(x - a_0) - a}{\|(1 - \varphi(x))(x - a_0) - a\|} J_F(x)\, dA, \end{aligned} \tag{2.2}$$

where J_F denotes the Jacobian of the map $F: S^n(a_0) \to S^n(a_0 + a)$. Elementary computations show (for a, $\|\varphi\|_{C^1}$ sufficiently small)

$$\begin{aligned} &\frac{1}{\omega_{n+1}} \int_{S^n(a_0+a)} \varphi'(x')(x' - a_0 - a)\, dA \\ &\quad = \frac{1}{\omega_{n+1}} \int_{S^n(a_0)} \varphi(x)(a - a_0)\, dA + a + B_{a,\varphi}(a, \varphi), \end{aligned} \tag{2.3}$$

where $B_{a,\varphi}: \mathbb{R}^{n+1} \times C^1(S^n(a_0)) \to \mathbb{R}^{n+1}$ stands for a continuous bilinear form depending smoothly on a, φ.

LEMMA 2.3. *There is a smooth function* $a(r)$, $0 < r \le r_1'$ *for some* $r_1' > 0$ *such that* $\lim_{r \to 0} a(r) = 0$ *and* $P(\varphi(r, a(r))) = 0$ *for every* r.

Proof. Define a map $G(r, a) = \widetilde{P}(\varphi(r, a))$. We have

$$\begin{aligned} G(r, a) &= \frac{1}{\omega_{n+1}} \int_{S^n} \varphi(r, a)(x) x\, dA \\ &= \frac{1}{\omega_{n+1}} \int_{S^n} \overline{\varphi}(r, a)(\psi_{r,a}(x^i e_i^*)) x\, dA. \end{aligned}$$

Note the following fact: as $r \to 0$ (with $|a| \le \delta$), the geodesic sphere $S_*^n(a + x(r))$ and the euclidean sphere $S^n(a + x(r))$ approach each other smoothly and all geometric quantities measured in $ds_{0,r}^2$ converge to those measured in the euclidean metric. Hence $\varphi(r, a) \circ Q_r$ and $\overline{\varphi}(r, a)$ approach each other smoothly as $r \to 0$, where Q_r denotes the radial projection from $S^n(a + x(r))$ onto $S_*^n(a + x(r))$. Besides, the mappings $x \mapsto \psi_{r,a}(x^i e_r^*)$ and $x \mapsto a + x(r) + x$ approach

each other as $r \to 0$. We conclude

$$G(r, a) = \frac{1}{\omega_{n+1}} \int_{S^n} \overline{\varphi}(r, a)(a + x(r) + x)x \, dA + u(r)(a)$$
$$= \frac{1}{\omega_{n+1}} \int_{S^n(a+x(r))} \overline{\varphi}(r, a)(x)(x - a - x(r)) \, dA + u(r)(a),$$

where $u(r)$ denotes a smooth map depending smoothly on $r > 0$ with $\|u(r)\|_{C^1(\mathbb{B}_\delta)} \to 0$ as $r \to 0$. Applying (2.3) we then get ($\overline{\varphi}(r)$ was defined in Lemma 2.3)

$$G(r, a) = \frac{1}{\omega_{n+1}} \int_{S^n(x(r))} \overline{\varphi}(r)(x)(x - x(r)) \, dA + a + B_{a,\varphi(r)}(a, \overline{\varphi}(r)) + u(r)(a). \tag{2.4}$$

By the inverse mapping theorem and the properties of $B_{a,\varphi}$, $u(r)$ and $\overline{\varphi}(r)$, there are $r_1' > 0$, $\delta' > 0$ and a natural number m such that for $r < r_1'$, $G(r, \cdot)$ is a diffeomorphism from $\mathbb{B}_{\delta'}$ onto a domain containing $\mathbb{B}_{\delta'/m}$. Hence we get a unique solution $a(r)$ of the equation $G(r, a) = 0$. Clearly $a(r) \to 0$ as $r \to 0$. □

Henceforth we put $\tau_1(r) = r(x(r) + a(r))$, $\varphi^*(r) = \varphi(r, a(r))$. We have obtained a nice parametrization $S_r = S_{r,\tau(r),\varphi^*(r)}$ for the leaves of (a restriction of) $\mathcal{F}$. The crucial property is $P(\varphi(r)) = 0$, $\lim_{r\to 0} \tau_1(r) = 0$ and $\lim_{r\to 0} \|\varphi^*(r)\|_{C^3} = 0$.

We shall call a smooth codimension 1 foliation with constant mean curvature leaves a constant mean curvature foliation.

THEOREM 2.1. (1) *If p is not a critical point of the scalar curvature function, then there exists no constant mean curvature foliation regularly centered at p.*

(2) *Assume that p is a nondegenerate critical point of the scalar curvature function and let $\mathcal{F}_0$ be the constant mean curvature foliation obtained in Theorem* 1.1. *If $\mathcal{F}$ is a constant mean curvature foliation regularly centered at p, then either $\mathcal{F}_0$ is a restriction of $\mathcal{F}$ or $\mathcal{F}$ is a restriction of $\mathcal{F}_0$.*

Proof. Let $\mathcal{F}$ be a constant mean curvature foliation regularly centered at p and $S_r = S_{r,\tau_1(r),\varphi^*(r)}$ the parametrization obtained above. Since the mean curvature of S_r is n/r, we have $H(r, \tau_1(r), \varphi^*(r)) = n$ and hence

$$P^{\perp}(H(r, \tau_1(r), \varphi^*(r)) - n) = 0, \tag{2.5}$$

$$\widetilde{P}(H(r, \tau_1(r), \varphi^*(r))) = 0. \tag{2.6}$$

From (2.5) and (1.19) we derive $(L+A(r))\varphi^*(r) = b(r, \tau_1)r^2$, where $b(r, \tau) \in K^\perp$ denotes a smooth function of r and τ, $A(r)\colon K^\perp \to L(K^\perp)$ denotes a linear operator with $\lim_{r\to 0} \|A(r)\| = 0$ and $K^\perp$, $L(K^\perp)$ are endowed with $C^{2,1/2}$ and $C^{0,1/2}$ norms respectively. We immediately deduce $\|\varphi^*(r)\|_{C^{2,1/2}} \leq cr^2$ for a constant c. Now define $\varphi_1(r) = \varphi^*(r)/r^2$. Inserting $\tau = \tau_1(r)$ and $\varphi = r^2\varphi_1(r)$ into (1.19), dividing the resulting equation by r^2 we easily deduce that $\varphi^*(r)$ converges to φ_0 as r goes to zero. By the Intermediate Remark in the proof of Theorem 1.1 we conclude that $\tau_1(r) = \tau(r)$ and $\varphi_1(r) = \varphi(r)$ for r small enough, provided that p is a nondegenerate critical point of the scalar curvature function. This finishes the proof of the theorem. □

References

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