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NICE DIMENSIONS FOR THE I_0 EQUIVALENCE RELATION OF DIAGRAMS OF MAP GERMS

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In this paper we give geometric characterisations for finite determinacy with respect to the various equivalence relations for convergent diagrams of smooth map germs. Using those results and known results on nice as well as semi-nice dimensions for smooth mappings due to Mather and Wall, we determine nice dimensions for some diagrams of smooth map germs. For manifolds with nice dimension finite I_0 determinacy holds in general for diagrams of map germs and topological stability holds in general in the space of diagrams of proper smooth mappings.

0.1. The main theorems, and some relations with ordinary singularity theory, Mather's nice and semi-nice ranges. Let $G = (V, L, \Lambda)$ be a convergent finite tree with a root $v_0 \in V$: $\substack{\searrow \\ \rightrightarrows \\ \nearrow} \!\!\! \succ \to v_0$, where V is the set of vertices, L the set of edges and $\Lambda = (\alpha, \beta): L \to V \times V$ is the orientation: $\alpha(l) \xrightarrow{l} \beta(l)$, $l \in L$. Let $P = (p_v)$ be a tuple of positive integers. In the paper **[N1]**, the author introduced some new equivalence relations, called I equivalence, for diagrams

$$f = (f_l) \in \mathscr{E}(G, P) = \bigoplus_{l \in L} m(P_{\alpha(l)})\mathscr{E}(P_{\alpha(l)}, P_{\beta(l)})$$

of map germs $f_l: (\mathbf{R}^{P_{\alpha(l)}}, 0) \to (\mathbf{R}^{P_{\beta(l)}}, 0)$ along G, for tuples $I = (a_v)$, where each a_v is either a non-negative integer or one of the symbols ∞, $*$. These generalize the idea of contact equivalence due to Mather. Our Thom-Mather theory for diagrams **[N1]** works effectively for generic diagrams of smooth mappings

$$f = (f_l) \in C^\infty(G, M) = \prod_{l \in L} C^\infty(C_{\alpha(l)}, M_{\beta(l)}),$$

provided that finite I_0 determinacy holds in general for multi germs of f of any combinatorial types where $I_0 = (a_v)$, $a_{v_0} = 0$ and $a_v = *$ otherwise.

In this paper we will determine a range of dimensions P for which this determinacy holds in general, and also obtain some properties of various critical point sets of finitely I_0 determined diagrams. For manifolds M_v with dimensions in this nice range, the topological stability theorem is proved in the paper **[N2]**.

In the first chapter, we give a geometric characterization for finite I_0 determinacy in terms of critical point sets. The finite I_0 determinacy of diagrams of map germs $f = (f_l)$, in other words $C_{I_0}(f) \leq r < \infty$, is an algebraic condition on their jets of the order $e(P+r)+1$ (denoted $e(r)$), depending on P and r and increasing with r. This condition defines algebraic sets $\Sigma^{e(r)}$ in the jet spaces $J^{e(r)}(G, P)$. So the set $\Sigma \subset \mathscr{E}(G, P)$ of non-finitely I_0 determined diagrams is the pro algebraic set defined by $\Sigma^{e(r)}$, $r = 0, 1, \dots$ (Proposition 2.2.2). We say that finite I_0-determinacy holds in general if $\operatorname{codim}\Sigma = \lim_{r\to\infty} \operatorname{codim}\Sigma^{e(r)} = \infty$. This condition is independent of whether we are in the real or complex case.

In §1.3, we define various critical point sets for diagrams, which are, in the complex case, rephrased in terms of the coherent sheaves $\theta(U_v)$, $\theta(f_l)$ of holomorphic vector fields on U_v, sections of the bundles $f_l^* TU_{\beta(l)} \to U_{\alpha(l)}$ ($f_l\colon U_{\alpha(l)} \to U_{\beta(l)}$), and the morphisms tf_l, ωf_l. Then finite I_0 determinacy is equivalent to the finiteness of stalks of certain coherent sheaves $N_{\alpha(l)}$, $\beta(l) = v_0$ considered as modules over $\mathscr{O}(U_{v_0})$ via f_l; in other words, the $f_l \mid \operatorname{sup}\mathscr{N}_{\alpha(l)}$ are finite to one ($\operatorname{supp} N_{\alpha(l)}$ is the non-trivial locus of f) (Proposition 1.5.2).

Using this geometric characterization, our problem is reduced to one of estimation of the set of non-trivial as well as non-stable jets. In particular for the graph of height 2 (unions of two compositions) our nice range is completely determined using the function ${}^2\sigma(n, p)$ due to Mather.

Now we recall some known results in the singularity theory of single mappings. The function ${}^m\sigma(n, p)$ is roughly defined to be the codimension of the union of $\mathscr{K}$ orbits of modality $\geq m$ in the jet space $J^\infty(n, p)$, and Mather showed that the set $\Sigma_{\text{uns}} \subset J(n, p)$ of unstable jets has codimension $> n$ (resp. $\geq n$) if and only if ${}^1\sigma(n, p) > n$: nice (resp. ${}^2\sigma(n, p) \geq n$: semi-nice). He also determined the range of such nice pairs (n, p) for which C^∞ stable mappings are dense in the proper mapping space $C^\infty_{\text{pr}}(N^n, P^p)$ **[Ml-2]**. The range of semi-nice pairs was recently determined by Wall **[W3]**, and in these dimensions the spaces $C^\infty_{\text{pr}}(N, P)$, $J(N, P)$ possess some remarkable properties, which we now briefly describe.

Gaffney proved in his thesis that a holomorphic map germ $f: \mathbf{C}^n, 0 \to \mathbf{C}^p, 0$ is finitely $\mathscr{A}$ $(= RL)$ determined (given that f is finitely $\mathscr{K}$ determined, i.e., $f \mid \Sigma(f)$ is finite to one) if and only if f is infinitesimally (C^∞) multi-stable on a deleted neighbourhood of $0 \in \mathbf{C}$. Later du Plessis proved that finite $\mathscr{A}$ determinacy holds in general if and only if $\operatorname{codim} \Sigma_{\mathrm{uns}} \geq n \Leftrightarrow^2 \sigma(n, p) \geq n$ **[P]**.

From our point of view, a composition $f, g: \mathbf{C}^n, 0 \to \mathbf{C}^p, 0 \to \mathbf{C}^r, 0$ is regarded as a family of mappings of varieties:

$$f_u: (g \circ f)^{-1}(u) \to g^{-1}(u)$$

parametrized by $u \in \mathbf{C}^r$. By definition, two compositions (f, g), (f', g') are I_0 equivalent if and only if f_0, f_0' are RL equivalent in a certain algebraic sense (see §0.2), and then the compositions are C^∞ equivalent if each is infinitesimally stable **[N1]**. This suggests that I_0 determinacy holds in general if and only if the pair $(n-r, p-r)$ of dimensions of the varieties is semi-nice. In fact, this is a part of Theorem 1.

Let $G = (V, L, \Lambda)$ be a tree with orientation $\Lambda = (\alpha, \beta): L \to V \times V$. The relation $<$ given by $\alpha(l) < \beta(l)$, $l \in L$, generates a partial order relation on the set of vertices V, i.e., $v < v'$ if and only if there is an oriented path $v \to \cdots \to v'$ by edges joining v to v'. We say a finite connected tree G is *convergent* if V has only the maximal element in the order $<$ and call the maximal element the *sink* or *root* of the tree G, denoted by $v_0 \in V$. It is then easily seen that any vertex $v \neq v_0$ has just one edge $l_v \in L$ with $\alpha(l_v) = v$. We denote $\beta(l_v) = \beta(v)$ and define $\beta^n(v) \in V$ and the function $h(v)$ inductively by $\beta^{n+1}(v) = \beta(\beta^n(v))$ and $h(\beta^{n+1}(v)) = h(\beta^n(v)) - 1$, $h(v_0) = 0$. We call $\max\{h(v) \mid v \in V\}$ the height of G. Henceforward the graph G is always connected and convergent.

The *branch* $G_v = (V_v, L_v, \Lambda)$ of G *on* $v \in V$ is a subgraph of G consisting of vertices $v' \leq v$ and edges $l \in L$, $\beta(l) \leq v$. The *prolongation* G_v^- of G_v is the union of G_v and the edge $l_v: v \to \beta(v)$.

Let $P = (p_v)_{v \in V}$ be a tuple of positive integers and let $\mathscr{E}(G, P) = \bigoplus_{l \in L} m(p_{\alpha(l)}) \mathscr{E}(p_{\alpha(l)}, p_{\beta(l)})$ denote the set of diagrams of smooth map germs $f = (f_l)_{l \in L}$, $f_l: (\mathbf{R}^{p_{\alpha(l)}}, 0) \to (\mathbf{R}^{p_{\beta(l)}}, 0)$. Here $m(p)$ denotes the maximal ideal of function germs on $(\mathbf{R}^p, 0)$ vanishing at 0 in the local ring $\mathscr{E}(p)$ of smooth function germs, and $\mathscr{E}(n, p) = \bigoplus^p \mathscr{E}(n)$. The C^r equivalence relation of these diagrams is defined in the same way as for global diagrams in $C^\infty(G, M) = \prod_{l \in L} C^\infty(M_{\alpha(l)}, M_{\beta(l)})$.

THEOREM 1. *Let $G = (V, L, \Lambda)$ be a convergent diagram of height 2 with root v_0, and let $P = (p_v)_{v\in V}$ be a tuple of positive integers. Then finite I_0 determinacy holds in general if and only if, for any triple $v_2 < v_1 < v_0$, one of the following conditions holds*:

(1) $p_{v_1} \le p_{v_0}$,
(2) $p_{v_2} \le p_{v_0}$,
(3) $p_{v_1}, p_{v_2} > p_{v_0}$ *and* ${}^2\sigma(p_{v_2} - p_{v_0}, p_{v_1} - p_{v_0}) \ge p_{v_2} - p_{v_0}$.

THEOREM 2. *For a convergent diagram G, finite I_0 determinacy holds in general if one of the following conditions holds for each vertex $v \in V$*:

(1) $p_v \le p_{v'}$ *for* $v \le v'$,
(2) $p_{\beta(v)} \le p_{v'}$ *for* $\beta(v) \le v'$,
(3) $p_{\beta^2(v)} \le p_{v'}$ *for* $\beta^2(v) \le v'$ *and*

$$ {}^2\sigma(p_v - p_{\beta^2(v)}, p_{\beta(v)} - p_{\beta^2(v)}) \ge p_v - p_{\beta^2(v)}. $$

THEOREM 3. *Let $G = (V, L, \Lambda)$ be the union of k edges $l_1, \dots, l_k$ with a common root v_1 and let $p = (p_v)$ be a tuple of positive integers. Then for any infinitesimally stable diagram $f = (f_{l_i}) \in \mathscr{E}(G, P)$ (resp. $\mathscr{O}_{\mathbf{R}}(G, P)$, $\mathscr{O}_{\mathbf{C}}(G, P)$) and any generic smooth germ $g: (\mathbf{R}^{p_{v_1}}, 0) \to (\mathbf{R}^{p_{v_0}}, 0)$ (resp. real or complex analytic germ), the composition (f, g) along the prolongation $G^- = G \cup v_1 \to v_0$ (of height 2) is finitely I_0 determined if and only if one of the following conditions holds for each $i = 1, \dots, k$*:

(1) $p_{v_0} \le p_{\alpha(l_i)}, p_{v_1}$ *and*

$$ p_{\alpha(l_i)} - p_{v_0} \le {}^2\sigma(P_{\alpha(l_i)} - p_{v_0}, p_{v_1} - p_{v_0}). $$

(2) $p_{v_0} > \min\{p_{v_1}, p_{\alpha(l_i)}\}$.

Here we recall a result on the function ${}^2\sigma(n, p)$ from the paper **[W3]**: $(4 \le n)$

$$ {}^2\sigma(n, p) = \begin{cases} -7(n-p)+7 & \text{if } n-p \le -4, \\ -4(n-p)+16 & \text{if } -3 \le n-p \le -1, \\ 13 & \text{if } 0, \\ 11 & \text{if } 1, \\ 13 & \text{if } 2, \\ 2(n-p)+4 & \text{if } 3 \le n-p \le 7, \\ (n-p)+11 & \text{if } 7 \le n-p. \end{cases} $$

0.2. Terminology and preliminaries. The k jet spaces of $C^\infty(G, M)$ and $\mathscr{E}(G, P)$ are respectively

$$J^k(G, M) = \prod_{l\in L} J^k(M_{\alpha(l)}, M_{\beta(l)}),$$
$$J^k(G, \mathbf{P}) = \prod_{l\in L} J^k(p_{\alpha(l)}, p_{\beta(l)}),$$

and the projection ev: $J^k(G, M) \to \prod_{l\in L}(M_{\alpha(l)} \times M_{\beta(l)})$ is given by $\mathrm{ev}((J^k f_l(x_l))_{l\in L}) = ((x_l, f_l(x_l))_{l\in L})$. Note that ev has the canonical fibre $J^k(G, P)$. Sometimes $J^k(G, P)$ is identified with

$$\mathscr{E}(G, P)/m^k\mathscr{E}(G, P) \\ = \bigoplus_{l\in L} m(p_{\alpha(l)})\mathscr{E}(p_{\alpha(l)}, p_{\beta(l)})/m(p_{\alpha(l)})^{k+1}\mathscr{E}(p_{\alpha(l)}, p_{\beta(l)}).$$

We denote by $\pi^k: (G, P) \to J^k(G, P)$, $\pi_{kl}: J^k(G, P) \to J^l(G, P)$ $(k > l)$ the natural projections.

The *unfoldings* of $f \in \mathscr{E}(G, P)$ used in this paper are diagrams $F = (F_l) \in \mathscr{E}(G, P + r)$, $P + r = (p_v + r)_{v\in V}$, such that $\imath_{\beta(l)} \circ f_l = F_l \circ \imath_{\alpha(l)}$ holds for $l \in L$ and $\imath_{v_0}: \mathbf{R}^{p_{v_0}} \to \mathbf{R}^{p_{v_0}+r}$ is transversal to all compositions $F_{vv_0}: \mathbf{R}^{p_v} \to \mathbf{R}^{p_{v_0}}$, where $\imath_v: \mathbf{R}^{p_v} \to \mathbf{R}^{p_v+r}$, $v \in V$, are the natural inclusions. It is easily seen that, after a suitable coordinate transformation, we may assume F is of the following normal form:

$$F_l(x, u) = (f_{lu}(x), u), \qquad f_{l0} = f_l, \ x \in \mathbf{R}^{p_{\alpha(l)}}, \quad u \in \mathbf{R}^r, \ l \in L.$$

For unfoldings of the normal form above, the jet sections

$$\overline{J}^k F: \prod_{l\in L} \mathbf{R}^{p_{\alpha(l)}} \times \mathbf{R}^r \to J^k(G, \mathbf{R}^p), \qquad \mathbf{R}^p = (\mathbf{R}^{p_v})_{v\in V},$$

are defined by

$$\overline{J}^k F((x_l), u) = J^k f_u((x_l)) \quad \text{for } (x_l) \in \prod_{l\in L} \mathbf{R}^{p_{\alpha(l)}}, \ u \in \mathbf{R}^r,$$

where $f_u = (f_{lu})_{l\in L}$.

We now introduce the new I equivalence relations for tuples of "integers" $I = (a_v)_{v\in V}$, $0 \le a_v \le \infty$, $*$, in terms of $C^\infty\mathbf{R}$ algebra.

A $C^\infty\mathbf{R}$ *algebra* is an $\mathbf{R}$ algebra R with the following properties:

(1) for any smooth function germ $f \in \mathscr{E}(n)$, $n = 0, 1, \dots,$ and elements $x_1, \dots, x_n \in R$, the composite $f(x_1, \dots, x_n) \in R$ is given,

(2) the morphism $\mathrm{ev}_x: \mathscr{E}(n) \to R$ given by $\mathrm{ev}_x(f) = f(x_1, \dots, x_n)$ is an $\mathbf{R}$ algebra homomorphism for any $x = (x_1, \dots, x_n)$,

(3) for any smooth functions $g_1, \dots, g_n \in \mathscr{E}(p)$, $f \in \mathscr{E}(n)$ and $x_{ij} \in R$, $i = 1, \dots, n$, $j = 1, \dots, p$, the equality

$$f(g_1, \dots, g_n)(x_{ij}) = f(g_1(x_{11}, \dots, x_{1p}), \dots, g_n(x_{nl}, \dots, x_{np}))$$

holds, i.e., $\mathrm{ev}_x \circ \mathrm{ev}_g = \mathrm{ev}_{\mathrm{ev}_x(g)}$.

A *morphism of* $C^\infty\mathbf{R}$ *algebras of* R *to* R' is an $\mathbf{R}$ algebra homomorphism $h: R \to R'$ with the property:

(4) $h(f(x_1, \dots, x_n)) = f(h(x_1), \dots, h(x_n))$ for any $f \in \mathscr{E}(n)$ and $x_1, \dots, x_n \in R$.

It is an easy exercise to see that any quotient $\mathscr{E}(p)/J$ of the $\mathbf{R}$ algebra $\mathscr{E}(p)$ by an ideal J is a $C^\infty\mathbf{R}$ algebra, and the pullback $f^*: \mathscr{E}(p)/J \to \mathscr{E}(n)/f^*J$ induced from a smooth map germ $f: (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ is a $C^\infty\mathbf{R}$ algebra homomorphism. Moreover all $C^\infty\mathbf{R}$ algebra isomorphisms of $\mathscr{E}(n)$ are given by pullbacks ϕ^* of germs of diffeomorphisms of $(\mathbf{R}^n, 0)$.

As a convention we introduce a symbol $*$ with the properties, $*+i = *$, $* + \infty = *$ and $i, \infty < *$ for $i \in N$. We write $N^* = N \cup \{\infty, *\}$, and define $m(n)^* = 0$. Also, we denote by $f^*m^{I+1}\mathscr{E}(p_v)$ the ideal of $\mathscr{E}(p_v)$ generated by $f^*_{vv'}m(p_{v'})^{a_{v'}+1}$, $v \leq v'$. The pullbacks $f_l^*: \mathscr{E}(p_{\beta(l)}) \to \mathscr{E}(p_{\alpha(l)})$, $l \in L$, induce $C^\infty\mathbf{R}$ algebra homomorphisms

$$\overline{f}_l^*: \mathscr{E}(p_{\beta(l)})/f^*m^{I+1}\mathscr{E}(p_{\beta(l)}) \to \mathscr{E}(p_{\alpha(l)})/f^*m^{I+1}\mathscr{E}(p_{\alpha(l)}).$$

We denote the tuple of $\overline{f}_l^*$ by $Q_I^*(f)$. Then two diagrams $f, g \in \mathscr{E}(G, P)$ are defined to be I *equivalent* if the tuples $Q_I^*(f)$, $Q_I^*(g)$ are equivalent as diagrams of $C^\infty\mathbf{R}$ algebra homomorphisms on the dual diagram G^* of G with the reversed orientation, in other words, there are $C^\infty\mathbf{R}$ algebra isomorphisms

$$\phi_v: \mathscr{E}(p_v)/g^*m^{I+1}\mathscr{E}(p_v) \to \mathscr{E}(p_v)/f^*m^{I+1}\mathscr{E}(p_v)$$

such that $\phi_{\alpha(l)} \circ f_l^* = g_l^* \circ \phi_{\beta(l)}$ for $l \in L$. Clearly the I equivalence class of f depends only on $\max\{a_v\}$ jets of f_l, $l \in L$, when the a_v are all finite. Note that the I equivalence relation for $I = (*)_{v \in V}$ coincides with the ordinary C^∞ equivalence relation.

Let $f \in \mathscr{E}(G, P)$ $(C^\infty(G, M))$ be a diagram of smooth map germs (global mappings). Let $\theta(n)$ $(\theta(N))$ denote the $\mathscr{E}(n)$ $(C^\infty(N))$ module of smooth vector fields on $(\mathbf{R}^n, 0)$ (N) and $\theta(f_l)$ the $\mathscr{E}(p_{\alpha(l)})$ $(C^\infty(M_{\alpha(l)}))$ module of sections of the pullback bundle $f_l^*T\mathbf{R}^{p_{\beta(l)}} \to \mathbf{R}^{p_{\alpha(l)}}$ $(f_l^*TM_{\beta(l)} \to M_{\alpha(l)})$. The morphisms $tf_l: \theta(p_{\alpha(l)}) \to \theta(f_l)$, $\omega f_l: \theta(p_{\beta(l)}) \to \theta(f_l)$ are defined by the differential and the pullback of f_l and similarly for the global case.

For simplicity we write

$$\theta(P) = \bigoplus_{v \in V} \theta(p_v), \quad \theta(M) = \bigoplus_{v \in V} \theta(M_v), \quad \theta(f) = \bigoplus_{l \in L} \theta(f_l)$$

and define the morphism

$$T(f) \colon \theta(P) \to \theta(f) \quad (\colon \theta(M) \to \theta(f))$$

by

$$T(f)\left(\bigoplus_{v \in V} \chi_v\right) = \bigoplus_{l \in L} t f_l(\chi_{\alpha(l)}) - \omega f_l(\chi_{\beta(l)}).$$

We say f is *infinitesimally stable* if $T(f)$ is surjective.

The I equivalence class of f is denoted by $\mathscr{O}^I(f)$ and $\pi^k(\mathscr{O}^I(f)) = \mathscr{O}^{Ik}(f)$. By a result in the paper **[N1]**, the $\mathscr{O}^{Ik}(f)$ are all semi algebraic submanifolds of $J^k(G, P)$ and locally C^∞ trivial if $a_v \neq \infty$ for $v \in V$ (Theorem 2.4.1–2 **[N1]**). For infinitesimally stable diagrams, the I_0 equivalence relation, $I_0 = (a_v)$, $a_{v_0} = 0$, $a_v = *$ otherwise, and the ordinary C^∞ equivalence relations are the same (Theorem 4.2.1 **[N1]**). By this fact together with Proposition 0.3.1 and Theorem 0.3.2, we see that finitely I_0 determined diagrams f, g with $C_{I_0}(f)$, $C_{I_0}(g) < \infty$ are I_0 equivalent if and only if they admit unfoldings equivalent with each other.

By definition we have

Proposition 0.3.1. *Let $a_{v_0} = 0$. Then unfoldings $F, G \in \mathscr{E}(G, P + r)$ of diagrams $f, g \in \mathscr{E}(G, P)$ are I equivalent if and only if f and g are so.*

Now we recall a criterion for infinitesimal stability.

Theorem 0.3.2 (*Theorem* 5.1.1 **[N1]**). *Let $F \in \mathscr{E}(G, P + r)$ be an unfolding of $f \in \mathscr{E}(G, P)$ of the above normal form: $F(x, u) = (f_u(x), u)$. Then the following conditions are equivalent.*

(1) *F is infinitesimally stable.*

(2) *The jet section $\overline{J}^k F \colon \prod_{l \in L} \mathbf{R}^{p_{\alpha(l)}} \times \mathbf{R}^r \to J^k(G, \mathbf{R}^{p_1})$ is transversal to $\mathscr{O}^{Ik}(f) \times \Delta$ for a tuple $I = (a_v)$ and a k such that $e_v(F) \leq a_v$, k, where*

$$\Delta = \left\{(x_l, y_l) \in \prod (\mathbf{R}^{p_{\alpha(l)}} \times \mathbf{R}^{p_{\beta(l)}}) \mid y_l = x_{l'}, \text{ if } \beta(l) = \alpha(l')\right\}$$

and the $e(F) = (e_v(F))$ is a tuple of integers such that $e_{v_0}(F) = 0$ and all entries are bounded by the function $e(G, P + r)$ depending on P and r and increasing with r.

(3) $(\partial f_{lu}/\partial u_i\ (u=0))_{l\in L}$, $i=1,\ldots,r$, *span* $\theta(f)/T(f)(\theta(P)) + f^* m^{e(F)+1}\theta(f)$ *over* $\mathbf{R}$.

With the above condition (3) in mind, we define the I codimension $C_I(f)$ by

$$C_I(f) = \dim_{\mathbf{R}} \theta(f)/T(f)(\theta(P)) + \bigoplus_{l\in L} f^* m^{I+1}\theta(f_l),$$

for a tuple $I=(a_v)$ of integers $a_v = 0, 1, \ldots, \infty, *$.

The following is a consequence of Theorem 0.3.2.

THEOREM 0.3.3 (*Proposition* 2.1.1, *Theorems* 3.1.1, 3.2.1 [**N1**]). *Let* $f\in\mathscr{E}(G, P)$ *and* $C_{I_0}(f)\le r<\infty$. *Then* f *is finitely* I_0 *determined: there is an integer* $e(G, P+r)$ *depending only on* P, r *with the property that if* $g\in\mathscr{E}(G, P)$ *has the same* $e(G, P+r)$ *jet as* f *then* g *is* I_0 *equivalent to* f *and* $C_{I_0}(g)\le r$.

More generally we have

THEOREM 0.3.4 (*Theorem* 3.2.1 [**N1**]). *A diagram f is finitely* (*resp. finitely* I_0) *determined if* $C_*(F)<\infty$ (*resp.* $C_{I_0}(f)<\infty$), *where* $*=(*)_{v\in V}$.

1. Trees and branches of convergent diagrams of smooth mappings.

1.1. *Triviality and irreducibility of convergent diagrams of map germs.* Let $G=(V, L, \Lambda)$ be a (finite, connected) convergent diagram, with root v_0. Let $P=(p_v)_{v\in V}$ be a tuple of positive integers and $f=(f_l)_{l\in L}\in\mathscr{E}(G, P)$. For any subgraph $G'=(V', L', \Lambda')\subset G$, $\Lambda'=\Lambda|L'$, we denote by $f_{G'}=(f_l)_{l\in L'}\in\mathscr{E}(G', P')$, $P'=P_{G'}=(P_v)_{v\in V'}$, the restriction of f to G'. We say f is trivial if

$$T(f)\left(\bigoplus_{v\neq v_0}\theta(p_v)\right)=\theta(f) \tag{$*$}$$

holds. Clearly trivial diagrams are infinitesimally stable. By Lemma 1.2.2 [**N1**], f is trivial if and only if

$$T(f)\left(\bigoplus_{v\neq v_0}\theta(p_v)\oplus m(p_{v_0})\theta(p_{v_0})\right)=\theta(f). \tag{$**$}$$

We say f is *irreducible* if, for any vertex $v\in V-v_0$, the restriction $f_{G_v^-}$ of f to the prolongation G_v^- $(=G_v\cup l_v\colon v\to\beta(l))$ of the branch G_v is not trivial, otherwise, we say f is *reducible* or *decomposable*.

PROPOSITION 1.1.1 (*Lemma* 4.2.2 [**N1**]). *A convergent diagram* $f \in \mathscr{E}(G, P)$ *is trivial if and only if the following conditions hold*:

(1) *The compositions* $f_{vv_0}: (\mathbf{R}^{p_v}, 0) \to (\mathbf{R}^{p_{v_0}}, 0)$, $v < v_0$, *along the paths from* v *to* v_0 *are all submersive.*

(2) *The restriction to fibres* $f_0 = (f_{0l})_{l \in L, \beta(l) \neq v_0}$,

$$f_{0l} = f_l \mid f^{-1}_{\alpha(l)v_0}(0): (f^{-1}_{\alpha(l)v_0}(0), 0) \to (f^{-1}_{\beta(l)v_0}(0), 0)$$

is infinitesimally stable.

PROPOSITION 1.1.2. *Assume that* G *has more than* p_{v_0} *stalks branching off the root* v_0. *Then any infinitesimally stable diagram* $f \in \mathscr{E}(G, P)$ *is decomposable.*

Proof. Let

$$\begin{aligned}\delta(f): \mathbf{R}^{p_{v_0}} = \theta(p_{v_0})/m(p_{v_0})\theta(p_{v_0}) \\ \to \theta(f)/T(f)\left(\bigoplus_{v \neq v_0} \theta(p_v) \oplus m(p_{v_0})\theta(p_{v_0})\right)\end{aligned}$$

be the morphism induced from the restriction $T(f): 0 \oplus \theta(p_{v_0}) \subset \theta(P) \to \theta(f)$. If f is infinitesimally stable ($T(f)$ is surjective), $\delta(f)$ is also surjective. Let $v_1, \dots, v_k \in V$ be all vertices of G of height 1. Then the morphism induced from $\delta(f)$

$$\overline{\delta}(f) = \bigoplus_{i=1}^{k} \delta(f_{G^-_{v_i}}): \mathbf{R}^{p_{v_0}} \to \bigoplus_{i=1}^{k} \theta(f_{G^-_{v_i}})/T(f)\left(\bigoplus_{v \leq v_i} \theta(p_v) \oplus m(p_{v_0})\theta(p_{v_0})\right)$$

is surjecture. If $k > p_{v_0}$ then

$$\theta(f_{G^-_{v_1}}) = T(f)\left(\bigoplus_{v \leq v_i} \theta(p_v) \oplus m(p_{v_0})\theta(p_{v_0})\right)$$

holds for some i, and the prolongation $f_{G^-_{v_i}}$ is trivial by the second criterion (**) for triviality.

By the proposition above we have immediately the following corollaries.

COROLLARY 1.1.3. *If a pair* (G, P) *admits an infinitesimally stable and irreducible diagram* $f \in \mathscr{E}(G, P)$, *then* G *is a* P *tree, i.e., any* *vertex $v \in V$ has less than $p_v + 1$ edges $l \in L$ with $\beta(l) = v$, and consequently G has at most $\prod_{h(v)\leq h-1} p_v$ vertices of height h.*

COROLLARY 1.1.4. *If a pair (G, P) admits an irreducible diagram $f \in \mathscr{E}(G, P)$ with $C_I(f) \leq r$, then G is a $P+r$ tree, i.e., any vertex $v \in V$ has less than $p_v + r + 1$ edges $l \in L$ with $\beta(l) = v$.*

Proof. By Theorem 0.3.3 f admits an infinitesimally stable unfolding $F \in \mathscr{E}(G, P+r)$ of dimension r. If G is not a $P+r$ tree, by Corollary 1.1.3, some prolongation $F_{G^-_{v_l}}$ in F is trivial, from which it follows that the prolongation $f_{G^-_{v_l}}$ is trivial by Proposition 1.1.1. This is a contradiction to the assumption that f is irredicible.

1.2. *Maximal trees and branches of convergent diagrams of smooth mappings.* Let $G = (V, L, \Lambda)$ be an oriented graph, $M = (M_v)_{v\in V}$ a collection of smooth manifolds, and $f = (f_l)_{l\in L} \in C^\infty(G, M)$ a smooth mapping on (G, M). Let $X = \bigcup_{l\in L} X_l$, $X_l \subset M_{\alpha(l)}$, be a disjoint union of finite sets, and write $f(X) = \bigcup_{l\in L} f_l(X_l)$. The *graph* $G_{fX} = (V_X, L_X, \Lambda)$ *of* f is the oriented graph consisting of the set of vertices $V_X = V \cup f(X)$ *and* the set of edges $L_X = f_X = \{f_{lx} \mid x \in X_l, l \in L\}$, and the orientation $\Lambda(f_{lx}) = (x, f_l(x)) \in V_X \times V_X$, where f_{lx} denotes the germ of f_l at $x \in M_{\alpha(l)}$. The multigerm f_X is naturally regarded as a diagram of map germs along the graph G_{fX}. Clearly if G is a convergent tree then all graphs G_{fX} are also unions of convergent trees: forests. (We call a forest sometimes a tree.)

From now on, we assume G is a finite convergent tree (connected), and we call f_X simply a tree of f (possibly disconnected). The *prolongation of a connected tree f_X with root x* is the tree f_{X^-}, where $X^- = X \cup f(X) = X \cup x$, i.e., the union of f_X and the edge $f_{lx}: x \to f_l(x)$, $x \in M_{\alpha(l)}$. A connected tree f_X of f is *maximal* if f_X is irreducible and any connected tree $f_{X'}$ with $X \subsetneqq X'$ is reducible. A maximal tree f_X is called the *maximal tree of* x if $x \in X \cup f(X)$, and denoted by f_{X_x}. The maximal branch $f_{X^{\mathrm{br}}_x}$ on $x \in M_v$ is the branch of the maximal tree f_{X_x} of x on x: $X^{\mathrm{br}}_x = X_x \cap (\bigcup_{v'<v} M_{v'})$. By definition the maximal tree of x always exists (possibly empty), and its uniqueness is easily seen.

Let $X^{\mathrm{ch}}_x = X_x \cup \{x, x^1, \dots, x^h\} \cup X_{x^h}$, where $x^i = f^i(x)$ $f_{v\beta'(v)}(x) \in M_{\beta'(v)}$ and $h = h(v)$, for any $x \in M_v$, $v \in V$ (possibly $X_x = \varnothing$, or $X_{x^h} = \varnothing$ for generic x). The tree $f_{X^{\mathrm{ch}}_x}$ is called the *characteristic tree* of x. If $X_x, X_{x^h} \neq \varnothing$, then $f_{X^{\mathrm{ch}}_x}$ is the union of the maximal trees f_{X_x}, $f_{X_{x^h}}$ of x, x^h and the path $x^a \to x^{a+1} \to \cdots \to x^b$ from the root x^a of f_{X_x} to a vertex x^b of $f_{X_{x^h}}$.

1.3. *Critical sets of convergent diagrams.* Let $f = (f_l)_{l\in L}$, $f_l: M_{\alpha(l)} \to M_{\beta(l)}$ be a convergent diagram of smooth (complex analytic) mappings of manifolds M_v, $v \in V$. Let $Q = (q_v)_{v\in V}$ be a tuple of positive integers. We call a tree f_X of f a Q *tree* if its underlying tree G_{fX} is a $Q_{X\cup f(X)}$ tree, where $Q_{X\cup f(X)} = (q_x)_{x\in X\cup f(X)}$ and $q_x = q_v$ for $x \in (X\cup f(X))\cap M_v$, i.e., for any vertex $f_l(x_{\alpha(l)}) \in f(X)$ of f_X there are less than $q_{\beta(l)}+1$ edges $f_{l'x'_{\alpha(l')}}: x'_{\alpha(l')} \to f_{l'}(x'_{\alpha(l')})$ with $\beta(l') = \beta(l)$, $f_{l'}(x'_{\alpha(l')}) = f_l(x_{\alpha(l)})$.

Now we define the *critical set* $C_{vQ}(f) \subset M_v$ for $v \neq v_0$ to be the set of roots of connected Q trees f_X such that the prolongation f_{X^-} is non-trivial.

If $Q = \infty$ we simply write $C_v(f)$. We define the *discriminant sets*

$$D_{vQ}(f) = \bigcup_{\beta(l)=v} f_l(C_{\alpha(l)Q}(f)), \qquad D_v(f) = \bigcup_{\beta(l)=v} f_l(C_{\alpha(l)}(f))$$

and conventionally we define $C_{v_0Q}(f) = D_{v_0Q}(f)$ and $C_{v_0}(f) = D_{v_0}(f)$. By definition, a Q tree f_X, $X = \bigcup_{l\in L} X_l$, $X_l \subset M_{\alpha(l)}$, is irreducible if and only if $X_l \subset C_{\alpha(l)Q}(f)$ for all $l \in L$. It is worth noting that the maximal branch on $x \in M_v$ is the tree $f_{X_x^{\mathrm{br}}}$, $X_x^{\mathrm{br}} = \bigcup_{v'<v} f_{v'v}^{-1}(x)\cap C_{v'}(f)$. We define the unstable set $S_v(F) \subset M_v$ to be the set of roots of noninfinitesimally stable trees of f in M_v.

Let f_X be a connected tree of f. We say f is a *good representative of* f_X if f satisfies the following conditions (1)–(4):

(1) f_X is a maximal tree, if f_X is irreducible.

(2) If $C_{I_0}(f_X) \leq r$ then for any tree $f_{X'}$ of f, $C_{I_0}(f_{X'}) \leq r$ (for the definition of the I_0 codimension $C_{I_0}(f_X)$, see §0.2).

(3) $C_v(f)$, $D_v(f) \subset M_v$ are closed, and the restrictions $f_l|C_{\alpha(l)}(f)$: $C_{\alpha(l)}(f) \to D_{\beta(l)}(f)$, $l \in L$, are proper and uniformly finite to one.

(4) For any subgraph $G' = (V', L', \Lambda')$ of G, the restriction $f_{G'} = (f_l)_{l\in L'}$ is a good representataive of the tree $f_{X'}$, $X' = \bigcup_{l\in L'} X_l$.

By condition (3) all maximal trees of f are Q trees for a common tuple of positive integers Q.

In the next section, we show that any finitely I_0 determined ($C_{I_0}(f) < \infty$) convergent diagram of (either C^∞ or complex analytic) map germs admits a good representative in which it is embedded as a tree.

1.4. *Other definitions of the critical sets by fitting ideals for the complex case.* First we remark that the results stated in §0.2 and also in the paper **[N1]** are all valid for the real and complex analytic cases as well as the smooth case. In this section all concepts and symbols are defined in the complex case. For example a diagram of map germs on a pair (G, P) of an oriented graph $G = (V, L, \Lambda)$ and a tuple of positive integers $P = (p_v)_{v \in V}$ means a collection of holomorphic map germs

$$f = (f_l)_{l \in L}, \qquad f_l \colon (\mathbf{C}^{p_{\alpha(l)}}, 0) \to (\mathbf{C}^{p_{\beta(l)}}, 0).$$

We denote the set of diagrams f by $\mathscr{O}_{\mathbf{C}}(G, P)$, the local ring of holomorphic function germs on $(\mathbf{C}^p, 0)$ by $\mathscr{O}(p)$, and its maximal ideal by $m(p)$. Note that $m(p)^\infty = m(p)^* = 0$.

Let $f \in \mathscr{O}(G, P)$ and let $\tilde{f} = (\tilde{f}_l)_{l \in L}$ be a representative defined on open neighbourhoods U_v of 0 in $\mathbf{C}^{p_v}$. Let $\mathscr{O}(U_v)$ denote the sheaf of germs of holomorphic functions on U_v, $\theta(U_v)$ the sheaf of germs of holomorphic vector fields on U_v and $\theta(\tilde{f}_l)$ the sheaf of germs of holomorphic sections of the bundle $\tilde{f}_l^* TU_{\beta(l)} \to U_{\alpha(l)}$. Clearly $\theta(U_v)$, $\theta(\tilde{f}_l)$ $(\alpha(l) = v)$ are finitely generated sheaves of $\mathscr{O}(U_v)$-modules.

Now we recall the notion of fitting ideals. Let R be a commutative ring with 1 and M an R-module presented by the exact sequence

$$R^p \xrightarrow{\mu} R^q \to M \to 0.$$

The i-th *fitting ideal* $\sigma^i(M)$ of M is the ideal of R generated by all $(q-i) \times (q-i)$ minors of the matrix μ. The i-th fitting ideal $\sigma^i(M)$ is independent of the choice of free resolution, $[T_0, T_e]$.

The i-th fitting ideal of a coherent sheaf of $\mathscr{O}(U)$-modules M is defined locally by the i-th fitting ideals of free resolutions, and globally by patching them up to a sheaf of ideals on U by the uniqueness of the i-th fitting ideals (for details, see **[Ti]**). Since $\sigma^i(M)$ is finitely generated over $\mathscr{O}(U)$, $\sigma^i(M)$ is a coherent sheaf of ideals. We remark that $V(\sigma^0(M)) = \operatorname{supp} M$ holds as topological spaces.

In the following we construct coherent shaves $N_{\alpha(l)}(\tilde{f})$ $(M_{\beta(l)}(\tilde{f}))$ of $\mathscr{O}(U_{\alpha(l)})$ $(\mathscr{O}(U_{\beta(l)}))$-modules, by shrinking the open neighbourhoods U_v if necessary, for any finitely I_0 determined $f \in \mathscr{O}(G, P)$. We define $N_{\alpha(l)}(\tilde{f})$ by the exactness of the following sequence

$$\theta(U_{\alpha(l)}) \xrightarrow{t\tilde{f}_l} \theta(\tilde{f}_l) \to N_{\alpha(l)}(\tilde{f}) \to 0$$

for $l \in L$ such that $\alpha(l)$ is a minimum with respect to the order $<$

on V, and by the exactness of the following sequence

$$\theta(U_{\alpha(l)}) \xrightarrow{\oplus \overline{\omega}\tilde{f}_l, \oplus t\tilde{f}_l} \bigoplus_{\beta(l')=\alpha(l)} \tilde{f}_{l'*}N_{\alpha(l')}(\tilde{f}) \oplus \theta(\tilde{f}_l) \to N_{\alpha(l)}(\tilde{f}) \to 0$$

for other $l \in L$, where $\overline{\omega}\tilde{f}_{l'}$ are the $\theta(U_{\beta(l')})$-homomorphisms induced from $\omega\tilde{f}_{l'}: \theta(U_{\beta(l')}) \to \theta(\tilde{f}_{l'})$. And we define $M_{\beta(l)}(\tilde{f})$ by the exactness of the following sequence

$$\theta(U_{\beta(l')}) \xrightarrow{\oplus \overline{\omega}\tilde{f}_{l'}} \bigoplus_{\beta(l')=\beta(l)} \tilde{f}_{l'*}N_{\alpha(l')} \to M_{\beta(l)}(\tilde{f}) \to 0$$

for $l \in L$.

PROPOSITION 1.4.1. *Let $f \in \mathscr{O}(G, P)$ be finitely I_0 determined, i.e., $C_{I_0}(f) < \infty$. Then f admits a good representative $\tilde{f} = (\tilde{f}_l)_{l \in L'}$, $\tilde{f}_l: U_{\alpha(l)} \to U_{\beta(l)}$, defined on open neighbourhoods U_v of $0 \in \mathbf{C}^{p_v}$, and the sheaves $N_{\alpha(l)}(\tilde{f})$, $f_{l*}N_{\alpha(l)}(\tilde{f})$, $M_{\beta(l)}(\tilde{f})$ are coherent sheaves with stalks*

$$\begin{aligned} N_{\alpha(l)} &= f_{l*}N_{\alpha(l)}(\tilde{f})_{\tilde{f}_l(x)} \\ &= \theta\left(\tilde{f}_{X_x^{\text{br-}}}\right) / T\left(\tilde{f}_{X_x^{\text{br-}}}\right) \left(\bigoplus_{x' \in X_x^{\text{br-}} \cap U_v} \theta(U_v)_{x'} \right), \qquad x \in U_{\alpha(l)}, \\ M_{\beta(l)}(\tilde{f})_x &= \theta\left(\tilde{f}_{X_x^{\text{br}}}\right) / \operatorname{Im} T\left(\tilde{f}_{X_x^{\text{br}}}\right), \qquad x \in U_{\beta(l)}. \end{aligned}$$

Further,

$$\begin{aligned} C_{\alpha(l)}(\tilde{f}) &= \operatorname{supp} N_{\alpha(l)}(\tilde{f}), \\ D_{\beta(l)}(\tilde{f}) &= \operatorname{supp} \bigoplus_{\beta(l')=\beta(l)} f_{l'*}N_{\alpha(l')}(\tilde{f}), \\ S_{\beta(l)}(\tilde{f}) &= \operatorname{supp} M_{\beta(l)}(\tilde{f}). \end{aligned}$$

Proof. We argue by induction on the height of G. So we assume the statement for any convergent diagram lower than G. Let $l_1, \dots, l_k \in L$ be all edges with $\beta(l_i) = v_0$, $v_i = \alpha(l_i)$ and G_{v_i} the branch on v_i. (By the assumption that $C_{I_0}(f) < \infty$, we see that $C_{I_0}(f_{G_{v_i}}) < \infty$ for all i.) So by the induction hypothesis, we may assume the branches $\tilde{f}_{G_{v_i}}$ are good representatives of $f_{G_{v_i}}$. The summand $\bigoplus_{\beta(l)=v_i} \tilde{f}_{l*}N_{\alpha(l)}$

is coherent so $N_{v_l}(\tilde{f})$ is also a coherent sheaf of $\mathscr{O}(U_{v_l})$-modules by Oka's theorem and with stalk

$$\theta(f_{G^-_{v_l}})/T(f_{G^-_{v_l}})\left(\bigoplus_{v'\leq v_l}\theta(p_{v'})\right)$$

at $0\in V_{v_l}$. Again by the assumption that $C_{I_0}(f)<\infty$, we have

$$\dim_{\mathbf{C}}\theta(f_{G^-_{v_l}})/T(f_{G^-_{v_l}})\left(\bigoplus_{v'\leq v_l}\theta(p_{v_l})\right)+f^*m(p_{v_0})\theta(f_{G^-_{v_l}})\leq C_{I_0}(f)<\infty.$$

This shows that $N_{v_l}(\tilde{f})_0$ is finite over $\mathscr{O}(U_{v_0})_0$ via f_{l_l} and also $0\in U_{v_l}$ is an isolated point in $\operatorname{Supp}(N_{v_l}(\tilde{f}))\cap\tilde{f}_l^{-1}(0)$. So by shrinking U_{v_l} and U_{v_0} if necessary, the restriction $\tilde{f}_{v_l}\colon\operatorname{Supp}N_{v_l}(\tilde{f})\to U_{v_0}$ is proper and uniformly finite-to-one and $\operatorname{Supp}N_{v_l}(\tilde{f})\cap\tilde{f}_{l_l}^{-1}(0)=0$. Then by Grauert's coherence theorem, $\tilde{f}_{l_l*}N_{v_l}(\tilde{f})$ is coherent so $M_{v_0}(\tilde{f})$ is also coherent by definition.

Next we check the properties of supports of those sheaves involving trees of $\tilde{f}$. By the induction hypothesis, the maximal branch $\tilde{f}_{X^{\mathrm{br}}_x}$ of f on $x\in U_{v_l}$ is given by $X^{\mathrm{br}}_x=\bigcup_{v<v_l}f_{v_l}^{-1}(x)\cap\operatorname{supp}N_v(\tilde{f})$. So we see that $x\in C_{v_l}(\tilde{f})$ if and only if the prolongation $\tilde{f}_{X^{\mathrm{br}}_x}$, $X^{\mathrm{br}\text{-}}_x=X^{\mathrm{br}}_x\cup x$ is not trivial. Again by the induction hypothesis,

$$\bigoplus_{\beta(l)=v_l}f_{l*}N_{\alpha(l)}(\tilde{f})_x\oplus\theta(\tilde{f}_{l_l})_x=\theta\left(\tilde{f}_{X^{\mathrm{br}\text{-}}_x}\right)/T\left(\tilde{f}_{X^{\mathrm{br}\text{-}}_x}\right)\left(\bigoplus_{\substack{x'\in X^{\mathrm{br}}_x\cap U_v\\ v\in V}}\theta(U_v)\right),$$

so we have

$$N_{v_l}(\tilde{f})_x=\theta\left(\tilde{f}_{X^{\mathrm{br}\text{-}}_x}\right)/T\left(\tilde{f}_{X^{\mathrm{br}\text{-}}_x}\right)\left(\bigoplus_{\substack{x'\in X^{\mathrm{br}\text{-}}_x\cap U_v\\ v\in V}}\theta(U_v)_{x'}\right),$$

by definition.

Therefore we have

$$\operatorname{Supp}N_{v_l}(\tilde{f})=C_{v_l}(\tilde{f})$$

and

$$\operatorname{Supp}\left(\bigoplus\tilde{f}_{l_l*}N_{v_l}(\tilde{f})\right)=\bigcup\tilde{f}_{l_l}(\operatorname{Supp}N_{v_l}(\tilde{f}))=D_{v_0}(\tilde{f}).$$

Similarly, we see that $M_{v_0}(\tilde{f})$ has the stalk

$$M_{v_0}(\tilde{f})_x = \theta(\tilde{f}_{X_x})/\operatorname{Im} T(\tilde{f}_{X_x}),$$

on $x \in U_{v_0}$, where X_x is the maximal tree of x. Therefore we have

$$\operatorname{Supp}(M_{v_0}(\tilde{f})) = S_{v_0}(\tilde{f}).$$

Next we show the condition (2) for good representatives. It suffices to show that if $C_{I_0}(f) = r < \infty$, then $C_{I_0}(\tilde{f}_{X_x}) \leq r$ for any maximal tree $\tilde{f}_{X_x}$ of $x \in U_{v_0}$, by the induction hypothesis. As we have seen above,

$$C_{I_0}(\tilde{f}_{X_x}) = \dim_{\mathbf{C}} M_{v_0}(\tilde{f})_x / m(U_{v_0})_x M_{v_0}(\tilde{f})_x$$

which is the rank of $M_{v_0}(\tilde{f})_x$ over $\mathscr{O}(U_{v_0})_x$. Since $M_{v_0}(\tilde{f})$ is a coherent sheaf of $\mathscr{O}(U_{v_0})$-modules, the rank of stalks is upper semicontinuous. This shows condition (2).

The conditions (1), (4) for good representatives are clear by the argument above.

This completes the proof of Proposition 1.4.1.

The germs of $C_v(\tilde{f})$, $D_v(\tilde{f})$, $S_v(\tilde{f})$ at $0 \in U_v$ are independent of the choice of representative $\tilde{f}$, so we denote them by $C_v(f)$, $D_v(f)$, $S_v(f)$ respectively.

1.5. *Geometric characterization of finite I_0 determinacy.* In this section we prove

LEMMA 1.5.1. *Let $f = (f_l) \in \mathscr{O}_{\mathbf{C}}(G, P)$ be irreducible and assume that all branches $f_{G_{v_i}}$ on vertices $v_i \in V$ of height 1 are finitely I_0 determined. Then f is finitely I_0 determined (i.e. $C_{I_0}(f) < \infty$) if and only if $C_{\alpha(l)}(f) \cap f_l^{-1}(0) = 0$ for all $l \in L$ with $\beta(l) = v_0$, i.e., there is a representative $\tilde{f}$, $\tilde{f}_l\colon U_{\alpha(l)} \to U_{\beta(l)}$ of f such that any connected tree $\tilde{f}_X$ with the root $0 \in U_{v_0}$ and the vertices of height 1 in $\tilde{f}_l^{-1}(0) - 0$, $\beta(l) = v_0$ is trivial.*

From this we have the following propositions.

PROPOSITION 1.5.2. *A convergent diagram $f \in \mathscr{O}_{\mathbf{C}}(G, P)$ is finitely I_0 determined if and only if $C_{\alpha(l)}(f) \cap f_l^{-1}(0) = 0$, for any $l \in L$, i.e., f admits a representative $\tilde{f}$, $\tilde{f}_l\colon U_{\alpha(l)} \to U_{\beta(l)}$, $0 \in U_v \subset \mathbf{C}^{p_v}$, such that the tree $f = \tilde{f}_0$, $0 = (0)_{l \in L}$ is maximal.*

PROPOSITION 1.5.3. *A convergent finitely I_0 determined diagram $f \in \mathscr{O}_{\mathbf{C}}(G, P)$ is finitely determined if and only if $S_{v_0}(\tilde{f}) = 0$, i.e., f admits a representative $\tilde{f}$, $\tilde{f}_l\colon U_{\alpha(l)} \to U_{\beta(l)}$, $0 \in U_v \subset \mathbf{C}^{p_v}$, such that any connected tree $\tilde{f}_X$ with root x in $U_{v_0} - 0$ is infinitesimally stable.*

Proof of Lemma 1.5.1. If f is finitely I_0 determined, f admits a good representative $\tilde{f}$, $\tilde{f}_l\colon U_{\alpha(l)} \to U_{\beta(l)}$ defined on open neighbourhoods U_v of 0 in $\mathbf{C}^{p_v}$. This has the required property. Conversely we assume that f admits such a representative. By Proposition 1.4.1, we may assume that the branches $\tilde{f}_{G_{v_i}}$ are good representatives of $f_{G_{v_i}}$. Then, by definition the sheaves $N_{v_i}(\tilde{f})$ are coherent. By the property in the lemma, we see

$$\operatorname{supp} N_{v_i}(\tilde{f}) \cap \tilde{f}_{l_i}^{-1}(0) = C_{v_i}(\tilde{f}) \cap \tilde{f}_{l_i}^{-1}(0) = 0, \qquad \alpha(l_i) = v_i$$

and by the Nullstellensatz (NST) for coherent sheaves, the stalks $N_{v_i}(\tilde{f})_0$ are finite over $\theta(p_{v_0})$ via $\tilde{f}_{l_i}$. By the equality

$$\begin{aligned}\theta(f)/T(f)&\left(\bigoplus_{v\neq v_0}\theta(p_v)\right) + \bigoplus_{l\in L} f^* m(p_{v_0})\theta(f_l)\\ &= \bigoplus_{i=1}^{k} N_{v_i}(\tilde{f})_0/\tilde{f}_{l_i}^* m(p_{v_0}) N_{v_i}(\tilde{f})_0,\end{aligned}$$

we see that

$$C_{I_0}(f) = \dim_{\mathbf{C}} \theta(f)/T(f)\left(\bigoplus_{v\in V}\theta(p_v)\right) + \bigoplus_{l\in L} f^* m(p_{v_0})\theta(f_l) < \infty;$$

hence f is finitely I_0 determined.

Proof of Proposition 1.5.2. The "only if" part follows from the existence of a good representative (Proposition 1.4.1), and the "if" part is given by applying Lemma 1.5.1 to branches G_v on vertices $v \in V$ inductively descending along G to the root v_0.

Proof of Proposition 1.5.3. By Proposition 1.4.1, f admits a good representative $\tilde{f}$, $\tilde{f}_l\colon U_{\alpha(l)} \to U_{\beta(l)}$. Then the sheaf $M_{v_0}(\tilde{f})$ is a coherent sheaf of $\theta(U_{v_0})$-modules with stalk $M_{v_0}(\tilde{f})_0 = \theta(f)/\operatorname{Im} T(f)$. By NST, $\operatorname{Supp} M_{v_0}(\tilde{f}) \subset \{0\}$ if and only if $\dim_{\mathbf{C}} \theta(f)/\operatorname{Im} T(f) < \infty$ if and only if f is finitely determined, by Theorem 0.3.4.

1.6. *Existence of good representatives of finitely I_0 determined smooth diagrams.*

PROPOSITION 1.6.1. *Finitely I_0 determined convergent diagrams of smooth map germs $f \in \mathscr{E}(G, P)$ admit good representatives.*

Proof. Let $F \in \mathscr{E}(G, P+r)$, $F_l : (\mathbf{R}^{p_{\alpha(l)}+r}, 0) \to (\mathbf{R}^{p_{\beta(l)}+r}, 0)$ be an infinitesimally stable unfolding of f. If F admits a good representative $\widetilde{F}_l$: $U_{\alpha(l)} \to U_{\beta(l)}$, defined on open neighbourhoods U_v of 0 in $\mathbf{R}^{p_v+r}$, then the restriction $\tilde{f} = (\tilde{f}_l)$, $\tilde{f}_l = \widetilde{F}_l \mid U_{\alpha(l)} \cap \mathbf{R}^{p_{\alpha(l)}} \times 0$ is automatically a good representative of f.

By Theorem 0.3.2, F is in particular finitely determined so we may assume that F is a diagram of polynomial mappings. Therefore its complexification $\widetilde{F} \in \mathscr{O}_{\mathbf{C}}(G, P+r)$ is also infinitesimally stable, since infinitesimal stability is an algebraic condition on finite jets. By Proposition 1.4.1, we can take open neighbourhoods $\widetilde{U}_v$ of 0 in $\mathbf{C}^{p_v+r}$ such that $\widetilde{F}_l(\widetilde{U}_{\alpha(l)}) \subset \widetilde{U}_{\beta(l)}$ and the restriction $\widetilde{F}_{\widetilde{U}} = (\widetilde{F}_l : \widetilde{U}_{\alpha(l)} \to \widetilde{U}_{\beta(l)})$ is a good representative of $\widetilde{F} \in \mathscr{O}_{\mathbf{C}}(G, P+r)$.

Now we claim that the restriction

$$F_{\widetilde{U}} = (\widetilde{F}_l : \widetilde{U}_{\alpha(l)} \cap \mathbf{R}^{p_{\alpha(l)}+r} \to \widetilde{U}_{\beta(l)} \cap \mathbf{R}^{p_{\beta(l)}+r})$$

is a good representative of $F \in \mathscr{E}(G, P+r)$. The properties (1), (2) for good representatives are obvious (since the condition $C_{I_0}(f_X) \leq r$ is an algebraic condition on a finite jet of f_X by Theorem 0.3.2 and 0.3.4). Clearly $C_v(F_{\widetilde{U}}) \subset C_v(\widetilde{F}_{\widetilde{U}}) \cap \mathbf{R}^{p_v+r}$ by definition. Let $F_{\widetilde{U}X_x^{\mathrm{br}}}$, $\widetilde{F}_{\widetilde{U}_{\beta(l)}X_x^{\mathrm{br}}}$ denote the maximal branches of $F_{\widetilde{U}}$, $\widetilde{F}_{\widetilde{U}}$ on $x \in \widetilde{U}_v \cap \mathbf{R}^{p_v+r}$. Note that $(X_x^{\mathrm{br}} \subset X_x^{\mathrm{br}})$.

By descending induction on the height of $v \in V$, we prove $C_v(F_{\widetilde{U}}) \subset \widetilde{U}_v \cap \mathbf{R}^{p_v+r}$ is a closed subset, so we assume $C_{v'}(F_{\widetilde{U}}) \subset \widetilde{U}_{v'} \cap \mathbf{R}^{p_{v'}+r}$ are closed for $v' < v$. Since $\widetilde{F}_l : C_{\alpha(l)}(\widetilde{F}_{\widetilde{U}}) \to D_{\beta(l)}(\widetilde{F}_{\widetilde{U}})$ are proper and finite-to-one the restrictions $F_l \mid C_{\alpha(l)}(F_{\widetilde{U}})$, $\alpha(l) < v$ are also proper and finite-to-one. Let $x_i \in C_v(F_{\widetilde{U}})$ be a sequence convergent to x. Then the set $X_{x_i}^{\mathrm{br}}$ converges to X_x^{br}. But if the prolongation $F_{\widetilde{U}X_x^{\mathrm{br}\text{-}}}$ is trivial the prolongation $F_{\widetilde{U}X_{x_i}^{\mathrm{br}\text{-}}}$ must also be trivial for any sufficiently large i, by the geometric characterization of triviality (Proposition 1.1.1). Therefore $C_v(F_{\widetilde{U}}) \subset \widetilde{U}_v \cap \mathbf{R}^{p_v+r}$ is closed. This completes the proof of the induction.

The unions of proper images of closed subsets

$$D_{\beta(l)}(F_{\widetilde{U}}) = \bigcup_{\beta(l')=\beta(l)} F_{l'}(C_{\alpha(l)}(F_{\widetilde{U}}))$$

are closed and the restrictions $F_l\colon C_{\alpha(l)}(F_{\widetilde{U}}) \to D_{\beta(l)}(F_{\widetilde{U}})$ are proper and uniformly finite-to-one. This completes the proof of the property (3) of $F_{\widetilde{U}}$. Condition (4) is clear.

We remark that the argument in the proof above goes the same way for the subsets $C_{vQ}(f) \subset C_v(f)$, $D_{vQ}(f) \subset D_v(f)$, $S_{vQ}(f) \subset S_v(f)$, so we have

PROPOSITION 1.6.2. *Let $f \in \mathscr{E}(G, P)$ be a finitely I_0 determined convergent diagram and let $Q = (q_v)_{v\in V}$ be a tuple of positive integers. Then f admits a representative $\tilde{f}$, $\tilde{f}_l\colon U_{\alpha(l)} \to U_{\beta(l)}$ defined on open neighbourhoods U_v of $0 \in \mathbf{R}^{p_v}$ such that the subsets $C_{vQ}(\tilde{f})$, $D_{vQ}(\tilde{f})$, $S_{vQ}(\tilde{f}) \subset U_v$ are closed and the restrictions $\tilde{f}_l\colon C_{\alpha(l)Q}(\tilde{f}) \to D_{\beta(l)Q}(\tilde{f})$ are proper and uniformly finite-to-one.*

Similarly to the case $q_v < \infty$, we can define the notion of maximal Q tree of $f \in C^\infty(G, M)$ as follows: a tree f_X is Q *maximal* if f_X is irreducible and any Q tree $f_{X'}$, $X \subsetneqq X'$, is reducible. In fact, for any $x \in \Sigma(f_l)$, $l \in L$, a maximal Q tree containing x as a vertex exists. However its uniqueness should not be expected.

2. Genericity of finite I_0 determinacy.

2.1. *Preliminary and some properties of pro sets in $\mathscr{E}(G, P)$ and $\mathscr{O}(G, P)$.* Let $G = (V, L, \Lambda)$ be a finite oriented graph, and let $J^k(G, P) = \prod_{l\in L} J^k(p_{\alpha(l)}, p_{\beta(l)})$ denote the real or complex k jet space, and $\pi^k\colon \mathscr{E}(G, P) \to J^k(G, P)$ (or $\pi^k\colon \mathscr{O}_{\mathbf{K}}(G, P) \to J^k(G, P)$, $\mathbf{K} = \mathbf{R}, \mathbf{C}$), $\pi_{kl}\colon J^k(G, P) \to J^l(G, P)$, $l < k$, the natural projections.

We call a subset $\Sigma \subset \mathscr{E}(G, P)$ (or $\Sigma \subset \mathscr{O}_{\mathbf{K}}(G, P)$) *a pro algebraic set* if there are algebraic sets Σ^r in $J^r(G, P)$ for $r = 0, 1, \ldots$ (or $r = r_1 < r_2 < \cdots$) such that

(1) $\Sigma^{r+1} \subset \pi_{r+1\,r}^{-1}(\Sigma^r)$ for all r and

(2) $\bigcap_r (\pi^r)^{-1}(\Sigma^r) = \Sigma$.

Sometimes we call Σ simply a pro set. We define

$$\operatorname{codim}\Sigma = \lim_{r\to\infty} \operatorname{codim}\Sigma^r.$$

We say a property P for $f \in \mathscr{E}(G, P)$ (resp. $f \in \mathscr{O}_{\mathbf{K}}(G, P)$) *holds in general* if it holds outside a pro set of infinite codimension.

To introduce some properties of pro sets, let Σ_i, $i = 1, 2, \dots$, be pro sets in (G, P) or $\mathscr{O}_{\mathbf{K}}(G, P)$. Then the countable intersection $\bigcap_{i=1}^{\infty} \Sigma_i$ is a pro set. In fact, this is a projective limit of the finite intersections $\Sigma^r = \bigcap_{i=1}^{r} \Sigma_i^r$ of algebraic sets Σ_i^r defining the pro sets Σ_i. Conversely a finite union of pro sets is a pro set. So we see immediately countable products and finite sums of generic properties are again generic.

LEMMA 2.1.1 (*Lemma* 0.4 **[P]**). *Let $\Sigma \subset \mathscr{E}(G, P)$ (resp. $\Sigma \subset \mathscr{O}_{\mathbf{K}}(G, P)$) be a pro algebraic set. Then Σ has infinite codimension if and only if any $z \in J^r(G, P)$ has a representative $f \notin \Sigma$.*

We recall Theorem 0.3.3 which says that there is an increasing positive integer-valued function e such that the property $C_{I_0}(f) \leq r$ of $f \in \mathscr{E}(G, P)$ (or $f \in \mathscr{O}_{\mathbf{K}}(G, P)$), in other words, having a stable r-parameter unfolding is dependent only on the $e(r)$-jet of f. So the set $\Sigma^{e(r)} \subset J^{e(r)}(G, P)$ of those which do not admit stable r-parameter unfoldings is an algebraic set. Conventionally, for all $r \in \mathbf{N}$, we define $\Sigma^r \subset J^r(G, P)$ appropriately so that $\pi_{rs}(\Sigma^r) \subset \Sigma^s$ for any $s < r$.

Then the set $\Sigma \subset \mathscr{E}(G, P)$ (or $\Sigma \subset \mathscr{O}_{\mathbf{K}}(G, P)$) of non-finitely I_0 determined diagrams f $(C_{I_0}(f) = \infty)$ is cut out by these sets Σ^r. So we have

PROPOSITION 2.1.2. *The set $\Sigma \subset \mathscr{E}(G, P)$ $(\mathscr{E}_{\mathbf{K}}(G, P))$ of non-finitely I_0 determined convergent diagrams is a pro algebraic set.*

Now we give briefly the definition of the number ${}^m\sigma(n, p)$. Let $W_d^r(n, p)$ denote the set of r jets in $J^r(n, p)$ with $\mathscr{K}^r$ codimensions $\geq d$. Then $W_d^r(n, p)$ is algebraic. Let ${}^mW_d^r(n, p)$ denote the union of all irreducible components of W_d^r with codimensions $\leq d - m$ and let ${}^mW^r(n, p) = \bigcup_{0 \leq d}^{m} W_d^r(n, p)$. It is easy to see that the codimension of ${}^mW^r(n, p)$ is decreasing to a constant value denoted ${}^m\sigma(n, p)$ as r tends to infinity (for the details of these definitions, see **[M1-2]**, **[Wl-2]**).

Let $Z_i \subset J^r(n, p)$ denote the set of jets of rank $\geq i$. Now we define a germ of a C^∞ submersion $\lambda^r \colon Z_i \to J^r(n-i, p-i)$ at any point $z \in Z_i$ with the property: any two jets $z', z'' \in Z_i$ close to z are $\mathscr{K}^r$ (contact) equivalent if and only if $\lambda^r(z'), \lambda^r(z'')$ are $\mathscr{K}^r$ equivalent. Here we may assume that $z \in Z_i$ is an r jet of map f of the form $f(x, u) = (f_u(x), u)$, $f_u \colon \mathbf{K}^{n-i}, 0 \to \mathbf{K}^{p-i}, 0$. Let

$H^i = 0 \times \mathbf{K}^i$, $H^{n-i} = \mathbf{K}^{n-i} \times 0$ and $H^{p-i} = \mathbf{K}^{p-i} \times 0$. Then any map germ f' with r jet z' close to z is transversal to H^i at the origin. Let $H_{f'} = f'^{-1}(H^i)$ and $\pi: H_{f'} \to H^{r-i}$ be a linear projection onto H^{r-i}. Then π is the germ of a diffeomorphism. We define $\lambda^r(x')$ to be the r jet of the map germ $(f' \mid H_{f'}) \circ \pi^{-1}: H^{n-i}, 0 \to H^{p-i}, 0$. Now it is an easy exercise to see λ^r is well defined and possesses the required properties.

By the above properties of λ^r, we have

$$Z_i \cap W_d^r(n, p) = (\lambda^r)^{-1}(W_d^r(n-i, p-i))$$

as germs at $z \in Z_i$ and consequently we have

LEMMA 2.1.3.

$$\begin{aligned} \operatorname{codim}^m W^r(n, p) \cap Z_i &= \operatorname{codim}^m W^r(n-i, p-i) \\ &= {}^m\sigma(n-i, p-i) \end{aligned}$$

and in particular

$${}^m\sigma(n-r, p-r) \leq {}^m\sigma(n, p).$$

2.2. *Proofs of the main theorems.* We begin by explaining the structure of the proofs of the theorems. In this section we will prove the "if" part of Theorem 3, the implication from the "if" part of Theorem 3 to the "if" part of Theorem 1, and the "only if" part of Theorem 1. The sets Σ in $\mathscr{E}(G, P)$, $\mathscr{O}_{\mathbf{K}}(G, P)$ are pro algebraic defined by common algebraic conditions (Proposition 2.1.1). So the "if" parts of Theorems 1, 2 and 3 for the real cases follow from the complex case, respectively. The "only if" part of Theorems 1, 3 for the other cases can be proved similarly to the complex cases, so we omit those proofs.

By Theorem 0.3.2 and Proposition 1.1.1, we see that whether a diagram $f \in \mathscr{E}(G, P)$ ($\mathscr{O}_{\mathbf{K}}(G, P)$) is trivial or not is determined by the jet of f of order k $(= i(G, P))$ depending only on (G, P). We denote $\Sigma_t^k(G, P)$, or simply by $\Sigma_t(G, P)$, the set of non-trivial jets in $J^k(G, P)$ germs, we see $k = p_{v_1} - p_{v_0} + 1$ if G is the graph $v_2 \to v_1 \to v_0$. We denote by $\Sigma_{\mathrm{uns}}(n, p)$ the set of non-stable jets in $J^{p+1}(n, p)$.

LEMMA 2.2.1. *Let G be the graph $v_2 \to v_1 \to v_0$. Let $P = (p_{v_2}, p_{v_1}, p_{v_0})$, $p_{v_0} \leq p_{v_1}, p_{v_2}$, be a triple of positive integers, and $k = p_{v_1} - p_{v_0} + 1$. Then the set of non-trivial jets $\Sigma_t^k(G, P) \subset J^k(G, P)$ is a union of constructible sets $\Sigma_t'(G, P)$, $\Sigma_1(G, P)$, $\Sigma_2(G, P)$ defined* *as follows*:

$$\Sigma_t'(G, P) = \{(J^k f(0), J^k g(0)) \in J^k(G, P) \mid \operatorname{rank} d(g \circ f)_0 = p_{v_0} \text{ and} (J^k f(0), J^k g(0)) \text{ is not trivial}\},$$
$$\Sigma_1(G, P) = \{(J^k f(0), J^k g(0)) \in J^k(G, P) \mid \operatorname{rank} dg_0 \neq p_{v_0}\},$$
$$\Sigma_2(G, P) = \{(J^k f(0), J^k g(0)) \in J^k(G, P) \mid \operatorname{rank} dg_0 = p_{v_0} \text{ and } \operatorname{rank} d(g \circ f)_0 \neq p_{v_0}\}.$$

And

$$\operatorname{codim} \Sigma_{t'} = \operatorname{codim} \Sigma_{\mathrm{uns}}(p_{v_2} - p_{v_0}, p_{v_1} - p_{v_0}),$$
$$\operatorname{codim} \Sigma_1 = p_{v_1} - p_{v_0} + 1 \quad \text{and} \quad \operatorname{codim} \Sigma_2 = p_{v_2} - p_{v_0} + 1.$$

Proof of Lemma 2.2.1. We denote $p_{v_i} = p_i$ for $i = 0, 1, 2$, and denote by Z_{p_0} the open set of k-jets $z = (J^k f(0), J^k g(0)) \in J^k(G, P)$ such that $\operatorname{rank} d(g \circ f)_0 = p_0$. In a similar way to the definition of λ^k in §2.1, we define a germ of C^∞-submersion $\lambda^k : Z_{p_0}(G, P) \to J^k(p_2 - p_0, p_1 - p_0)$ at $z \in Z_{p_0}(G, P)$ such that z' is trivial if and only if $\lambda^k(z')$ is stable.

Let $f = (f_1, f_2) : (\mathbf{K}^{p_2}, 0) \to (\mathbf{K}^{p_1}, 0) \to (\mathbf{K}^{p_0}, 0)$ be an analytic representative of $z = (z_2, z_1) \in J^k(G, P)$, let $H_2 = d(f_1 \circ f_2)_0^{-1}(0)$, $H_1 = d(f_1)_0^{-1}(0)$ and let $\pi_2 : \mathbf{K}^{p_2} \to H_2$, $\pi_1 : \mathbf{K}^{p_1} \to H_1$ be any linear projections. Let $f' = (f_2', f_1') : (\mathbf{K}^{p_2}, 0) \to (\mathbf{K}^{p_1}, 0) \to (\mathbf{K}^{p_0}, 0)$ be a map germ with k jet $(J^k f_2'(0), J^k f_1'(0))$ close to z. Then the restrictions $\pi_2 \mid (f_1' \circ f_2')^{-1}(0)$, $\pi_1 \mid f_1'^{-1}(0)$ are germs of diffeomorphisms at the origins. We define $\lambda^k(z')$ to be the k-jet of the map germ

$$\lambda(f_1', f_2') = \pi_2 \circ f_2' \mid (f_2' f_1')^{-1}(0) \circ (\pi_2 \mid (f_2' f_1')^{-1}(0))^{-1} : (\mathbf{K}^{p_2 - p_0}, 0) \to (\mathbf{K}^{p_1 - p_0}, 0).$$

We can see easily that λ^k is well defined and C^∞-submersion. Now we show the other properties of the submersion.

The germ of the restriction $f_2' \mid (f_1' \circ f_2')^{-1}(0) : (f_1' \circ f_2')^{-1}(0) \to f_1'^{-1}(0)$ at 0 is $\mathscr{A}$ equivalent to $\lambda(f_1', f_2')$.

By Proposition 1.1.1, (f_2', f_1') is trivial if and only if

$$f_2' \mid (f_1' f_2')^{-1}(0) : (f_1' f_2')^{-1}(0) \to f_1'^{-1}(0)$$

is infinitesimally stable if and only if $\lambda^k(z')$ is infinitesimally stable.

By the above properties of the submersion λ^k, we have the equality of germs at $z \in J^k(G, P)$

$$\begin{aligned}\Sigma'_t(G, P) &= \Sigma_t(G, P) \cap Z_{p_0}(G, P)\\ &= (\lambda^k)^{-1}(\Sigma_{\mathrm{uns}}(p_2 - p_0, p_1 - p_0)),\end{aligned}$$

from which we have $\operatorname{codim}\Sigma'_t(G, P) = \operatorname{codim}\Sigma_{\mathrm{uns}}(p_2 - p_0, p_1 - p_0)$. The complement of $Z_{p_0}(G, P)$ in $\Sigma_t(G, P)$ is the union of $\Sigma_1(G, P)$ and $\Sigma_2(G, P)$ which have obviously codimensions $p_2 - p_0 + 1$ and $p_1 - p_0 + 1$, respectively. This completes the proof of Lemma 2.2.1.

The following theorem is essentially due to Mather (for the proof, see **[P]**).

THEOREM 2.2.2. $\operatorname{codim}\Sigma_{\mathrm{uns}}(n, p) \geq n$ *if and only if* ${}^2\sigma(n, p) \geq n$.

Proof of the "only if" part of Theorem 2.1.1. We consider only the complex analytic case. Let $f \in \mathscr{O}_{\mathbf{C}}(G, P)$ be a diagram of complex analytic map germs. If the restriction $f_{G'}$ of f to a subgraph G' of G is not finitely I_0 determined then f is not finitely I_0 determined, since if f is finitely I_0 determined then f has a stable unfolding F (Theorem 0.3.2), and its restriction $F_{G'}$ is a stable unfolding of $f_{G'}$. So it suffices to prove that finite I_0 determinacy does not hold in general for the graph $G: v_2 \to v_1 \to v_0$ and triples $P = (p_{v_i}) = (p_i)$, if ${}^2\sigma(p_2 - p_0, p_1 - p_0) < p_2 - p_0$. Then we have

$$\operatorname{codim}\Sigma_{\mathrm{uns}}(p_2 - p_0, p_1 - p_0) < p_2 - p_0$$

by Theorem 2.2.2 and

$$\operatorname{codim}\Sigma_t(G, P) < p_2 - p_0$$

by Lemma 2.2.1.

Let Δ denote the diagonal set in $\mathbf{C}^{2p_1}$.

We write

$$J^k(G, \mathbf{C}^p) = \mathbf{C}^{p_2} \times J^k(p_2, p_1) \times \mathbf{C}^{2p_1} \times J^k(p_1, p_0) \times \mathbf{C}^{p_0}$$

and define for map germs $f = (f_2, f_1) \in \mathscr{O}_{\mathbf{C}}(G, P)$, $J^k(f_2, f_1): \mathbf{C}^{p_2} \times \mathbf{C}^{p_1} \to J^k(G, \mathbf{C}^p)$ by

$$\begin{aligned}&J^k(f_2, f_1)(x_2, x_1)\\ &\quad = (x_2, J_0^k f_2(x_2), f_2(x_2), x_1, J_0^k f_1(x_1), f_1(x_1)),\\ &\qquad\qquad (x_2, x_1) \in \mathbf{C}^{p_2} \times \mathbf{C}^{p_1}.\end{aligned}$$

By assumption we have

$$\operatorname{codim}(\mathbf{C}^{p_2} \times \Sigma_t(G, P) \times \Delta \times 0) = \operatorname{codim} \Sigma_t(G, P) + p_1 + p_0 < p_2 + p_1$$

so there is a map $f = (f_2, f_1) \in \mathscr{O}_{\mathbf{C}}(G, P)$ such that $(\pi^k(f_2), \pi^k(f_1)) \in \Sigma_t(G, P)$ and $\operatorname{codim}(\mathbf{C}^{p_2} \times \Sigma_t(G, P) \times \Delta \times 0)$ at $J^k(f_2, f_1)(0, 0)$ is less than $p_2 + p_1$. We denote

$$\Sigma_t(f_2, f_1) = J^k(f_2, f_1)^{-1}(\mathbf{C}^{p_2 1} \times \Sigma_t(G, P) \times \Delta \times 0).$$

Then we see $\dim \Sigma_t(f_2, f_1) \geq 1$, $P_2 \colon \Sigma_t(f_2, f_1) \to \mathbf{C}^{p_2}$ is an isomorphism into $\mathbf{C}^{p_2}$ and $P_1 = f_2 \circ P_2$ holds on $\Sigma_t(g_2, f_1)$, where $P_i \colon \mathbf{C}^{p_2} \times \mathbf{C}^{p_1} \to \mathbf{C}^{p_i}$ is the ith projection. If f is finitely I_0 determined then f_i are finitely $\mathscr{K}$ determined and the restrictions of f_i to the set $C_{v_i}(f)$ are proper and finite-to-one by Proposition 1.4.1 (for the definition of C_{v_i} see §1.3). So $\dim \Sigma_t(f_2, f_1) \geq 1$ implies $\dim f_2(P_2(\Sigma_t(f_2, f_1))) \geq 1$. By definition we have $f_2(P_2(\Sigma_t(f_2, f_1))) \subset C_{v_1}(f)$ and $f_1 \circ f_2 \circ P_2(\Sigma_t(f_2, f_1)) = 0$, so by Proposition 1.5.2, f is not finitely I_0 determined.

We have shown above that if the codimension of $\Sigma_t(G, P)$ at z is smaller than $p_2 - p_0$ then any representative f is not finitely I_0 determined. Then Lemma 2.1.1 shows that finite I_0 determinacy does not hold in general.

Proof of the implication: *"if" of Theorem* 3 ⇒ *"if" of Theorem* 1. We prove this for the complex case. It suffices to prove the genericity for the tree:

$$V = \{v_0, v_1, \alpha(l_i), \ i = 1, \dots, r\}, \quad L = \{l_0, l_1, \dots, l_r\},$$

$\Lambda(l_0) = (v_1, v_0)$, $\Lambda(l_i) = (\alpha(l_i), v_1)$ (because a union of diagrams $f \cup g$ with a common root is finitely I_0 determined if and only if f and g are).

First we assume that for any vertex $\alpha(l_i)$ the condition (3) holds. By the pro algebraicity of Σ (Lemma 2.1.1), we have only to find a finitely I_0 determined representative $(f_1, \dots, f_r, g) \in \mathscr{O}_{\mathbf{C}}(G, P)$ for a given k jet $z = (z_1, \dots, z_r, z_0)$. By the genericity of finite $\mathscr{K}$ determinacy of single map germs, $(z_1, \dots, z_r)$ admits finitely $\mathscr{K}$ determined polynomial representative $f = (f_1, \dots, f_r)$. Let $F = (F_1, \dots, F_r)$, $F_i \colon \mathbf{C}^{p_{\alpha(l_i)}+r}, 0 \to \mathbf{C}^{p_{v_1}+r}, 0$ be an infinitesimally stable unfolding of f, and let $\overline{g} \colon (\mathbf{C}^{p_{v_1}+r}, 0) \to (\mathbf{C}^{p_v+r}, 0)$ be the trivial unfolding of $g \colon \overline{g}(x, u) = (g(x), u)$. By Theorem 3, we can take another representative $\overline{g}' \colon (\mathbf{C}^{p_{v_1}+r}, 0) \to (\mathbf{C}^{p_{v_0}+r}, 0)$ of the k jets of

$\overline{g}$ such that $(F, \overline{g}') \in \mathscr{O}_{\mathbf{C}}(G, P+r)$ is finitely I_0-determined where $P+r=(p_v+r)_{v\in V}$. Since $\overline{g}\circ F$, $\overline{g}'\circ F$ are transversal to $\mathbf{C}^{p_{v_0}}\times 0$ in $\mathbf{C}^{p_{v_0}}\times \mathbf{C}^r$ their preimages are smooth.

Let

$$\pi_i\colon (\overline{g}'\circ F_i)^{-1}(\mathbf{C}^{p_{v_0}}\times 0) \\ \to (\overline{g}\circ F_i^{-1})(\mathbf{C}^{p_{v_0}}\times 0), \qquad i=1,\dots,r,$$

and

$$\pi_0\colon (\overline{g}'^{-1})(\mathbf{C}^{p_{v_0}}\times 0)\to(\overline{g}^{-1})(\mathbf{C}^{p_{v_0}}\times 0)$$

be some linear projections. Then $\pi_1,\dots,\pi_r$ and π_0 are germs of diffeomorphisms. We define the diagram $(f'', g'')\in\mathscr{O}_{\mathbf{C}}(G,P)$, $f''=(f_1'',\dots,f_r'')$ by

$$f_i''=\pi_0\circ F_i\circ\pi_i^{-1}\colon(\overline{g}\circ F_i)^{-1}(\mathbf{C}^{p_{v_0}}\times 0)\to\overline{g}^{-1}(\mathbf{C}^{p_{v_0}}\times 0)$$

and

$$g''=\overline{g}'\circ\pi_0^{-1}\colon\overline{g}^{-1}(\mathbf{C}^{p_{v_0}}\times 0)\to(\mathbf{C}^{p_{v_0}}\times 0).$$

It is easy to see that (f'', g'') has the same k-jets as (f, g). The composition $(F, \overline{g}')$ is a finitely I_0 determined unfolding of (f'', g''), so (f'', g'') also is finitely I_0 determined, because a stable unfolding of $(F, \overline{g}')$ is a stable unfolding of (f'', g''), and (f'', g'') is finitely I_0 determined if and only if (f'', g'') has a stable unfolding (Theorem 0.3.2). Now we have proven the "if" part of Theorem 1 for the case (3) holds for all vertices of height 2.

Secondly, we assume G has a vertex v_2' of height 2 such that $p_{v_2'}\le p_{v_0}$. By Theorem 2.2.4 (transversality theorem), a generic map germ $f_2'\in\mathscr{O}_{\mathbf{C}}(p_{v_2'}, p_{v_1})$ is transversal to $g^{-1}(0)-\{0\}$ and $0\in\mathbf{C}^{p_{v_1}}$ on a deleted neighbourhood of 0 in $\mathbf{C}^{p_{v_2'}}$ and $f_2'^{-1}(g^{-1}(0))$ is homeomorphic to an algebraic set. By comparing those dimensions, we see that 0 is an isolated point of $(g\circ f_2')^{-1}(0)$. By NST, there is an integer k such that

$$m(p_{v_2'})^k\subset(g\circ f_2')^*m(p_{v_0})\cdot\mathscr{O}(p_{v_2'}),$$

from which we have

$$(1)\qquad m(p_{v_2})^k\circ\theta(f_2')\subset(g\circ f_2')^*m(p_{v_0})\cdot\theta(f_2').$$

Thirdly we assume $p_{v_1'}\le p_{v_0}$. Then, for generic $g'\in\mathscr{O}_{\mathbf{C}}(p_{v_1'}, p_{v_0})$, there is an integer k such that

$$(2)\qquad m(p_{v_1'})^k\subset g^*m(p_{v_0})\cdot\mathscr{O}(p_{v_1}).$$

By the theory of single map germs (**[G1]**), for a generic map germ

$f_i' \in \mathscr{O}_{\mathbf{C}}(p_{\alpha(l_i)}, p_{v_1'})$, there is an integer k_i such that

$$\begin{aligned} m(p_{\alpha(l_i)})^{k_i} \cdot \theta(f_i') \\ \subset t f_i'(\theta(p_{\alpha(l_i)})) + f_i'^* m(p_{v_1'}) \cdot \theta(f_i'), \end{aligned} \tag{3}$$

from which together with (2), we have

$$\begin{aligned} m(p_{\alpha(l_i)})^{k_i k} \cdot \theta(f_i') \\ \subset t f_i'(\theta(p_{\alpha(l_i)})) + (g \circ f_i')^* m(p_{v_0}) \cdot \theta(f_i'). \end{aligned} \tag{4}$$

The above inclusions (1), (4) show the finiteness of I_0 codimension of (f, g) for the case where either of the conditions (1) or (2) occurs. This completes the proof of the implication.

Proof of Theorem 2. Let V_0 denote the set of vertices $v \in V$ for which condition (1) in Theorem 2 holds.

Firstly we show that $f_{vv_0}^{-1}(0) = 0$ for generic diagrams $f \in \mathscr{O}_{\mathbf{C}}(G, P)$ and $v \in V$, where f_{vv_0} denotes the composition of f_l along the oriented path joining v to the root v_0. Let $v \in V_0$, $v = v_k < v_{k-1} < \cdots < v_1 < v_0$ and let $f_i \in \mathscr{O}_{\mathbf{C}}(p_{v_i}, p_{v_{i-1}})$ for $i = 1, \dots, k$. The condition that $\dim \mathscr{O}(p_{v_k}) / f_{v_k v_0}^* m(p_{v_0}) \cdot \mathscr{O}(p_{v_k}) < r$ is an algebraic condition on finite jets of $f_1, \dots, f_k$, so by NST, the set K of those map germs such that

$$(f_1 \circ \cdots \circ f_k)^{-1}(0) = 0$$

is a pro algebraic set in $\prod_{i=1}^k \mathscr{O}_{\mathbf{C}}(p_{v_v i}, p_{v_{i-1}})$. By Lemma 2.1.1, to say K is of infinite codimension it suffices to show that any jet $z \in \prod_{i=1}^k J^r(p_{v_i}, p_{v_{i-1}})$ has a representative $(f_1, \dots, f_k)$ such that

$$(f_1 \circ \cdots \circ f_k)^{-1}(0) = 0 \in \mathbf{C}^{p_{v_k}}.$$

In general, for an analytic subset C of $\mathbf{C}^p$ and an r-jet z', there is a representative $f \in \mathscr{O}_{\mathbf{C}}(n, p)$ of z' such that $\operatorname{codim} f^{-1}(C) \geq \min\{n, \operatorname{codim} C\}$ by the transversality theorem (Theorem 2.2.4). Applying this to $f_1, \dots, f_k$, we see that z has a representative $(f_1, \dots, f_k)$ such that

$$\operatorname{codim}(f_1 \circ \cdots \circ f_i)^{-1}(0) > \min\{p_{v_0}, \dots, p_{v_i}\} = p_v,$$

which shows that $(f_1 \circ \cdots \circ f_k)^{-1}(0) = 0$.

Let $\overline{G}_v$ denote the subgraph of G consisting of all vertices $v' \leq v$ such that $p_v < p_{v''}$ for any vertex v'', $v' \leq v'' < v$, and of all edges in the paths joining such v' to v. By the conditions (1)–(3) of Theorem

2, the height of $\overline{G}_v$ is at most 2 for any vertex $v \in V$. Here we remark that $f \in \mathscr{O}_{\mathbf{C}}(G, P)$ is a union of those subgraphs $f_{\overline{G}_v}$.

As we have already seen above, for generic $f \in \mathscr{O}_{\mathbf{C}}(G, P)$ and vertices satisfying condition (1), we have $f_{vv_0}^{-1}(0) = 0$. By NST, there is an integer k_v such that

$$(1) \qquad m(p_v)^{k_v} \subset f_{v, v_0}^* m(p_{v_0}) \cdot \mathscr{O}(p_v).$$

By Theorem 1, the restriction $f_{\overline{G}_v}$ is finitely I_0-determined for generic $f \in \mathscr{O}_{\mathbf{C}}(G, P)$, so we have

$$(2) \qquad \dim \theta(f_{\overline{G}_v})/T(f_{\overline{G}_v}) \left(\bigoplus_{\substack{v' \in \overline{V}_v \\ \neq v}} \theta(p_{v'}) \right) + m(p_v) \cdot \theta(f_{\overline{G}_v}) < \infty,$$

where $\overline{V}_v$ denotes the set of vertices of $\overline{G}_v$ and $\theta(f_{\overline{G}_v})$ is regarded as an $\mathscr{E}(p_v)$ module via f^*. By (1) and (2), we have

$$(3) \qquad \dim \theta(f_{\overline{G}_v})/T(f_{\overline{G}_v}) \left(\bigoplus_{\substack{v' \in \overline{V}_v \\ \neq v}} \theta(p_{v'}) \right) \oplus m(p_{v_0}) \cdot \theta(f_{\overline{G}_v}) < \infty,$$

from which the finiteness of I_0 codimension of f follows.

This completes the proof of Theorem 2.2.2.

Proof of the "if" part of Theorem 3. The "only if" part follows that of Theorem 1. In this proof, we use an idea due to du Plessis. Firstly we introduce a result in the theory of stability of single map germs due to Mather, which is explicitly stated in **[P]**.

We write $p_{\alpha(l_i)} = n_i$, $i = 1, \dots, j$, $p_{v_1} = p$ and $p_{v_0} = p_0$. Let $f = (f_i)_{i=1,\dots,k}$, $f_i: (\mathbf{C}^{n_i}, 0) \to (\mathbf{C}^p, 0)$, be a stable polynomial map germ on the graph G (of height 1). Then there are germs of constructible stratifications A_i, $i = 1, \dots, k$, B of $\mathbf{C}^{n_i}$, $\mathbf{C}^p$ respectively, which possess the following properties:

(1) $S \in A_i$ are foliated by contact classes K_x, $x \in S$, and rank $df_{i\,x}$ is constant for $x \in S_i$, and $f_i \mid S$, $S \in A_i$, are non-singular (i.e., of full rank).

(2) $\Sigma(f_i)$ is a union of strata of positive codimension in A_i.

(3) f_i, $i = 1, \dots, k$, are multi transversal with respect to the foliations of $S \in A_i$, i.e., if $x_{i,j} \in \mathbf{C}^{n_i}$, $j = 1, \dots, a_i$, and $f_i(x_{i,j}) = y$ then $df_{x_{i,j}}(T_{x_{i,j}} K_{x_{i,j}})$ are in a general position at y.

(4) Let

$$B = \bigcap_{i,j} f_i(S_{i,j}), \qquad S_{i,j} \in A_i,$$

and

$$A'_i = \{S_i \in F_\iota^{-1}(S') \mid S_i \in A_i, \ S' \in B\}.$$

Then (A'_i, B) is a Thom regular stratification of $f = (f_i)_{i=1,\dots,k}$ and $S' \in B$ is foliated by constructible manifolds defined as follows:

$$L_y = \{y' \in S' \mid \text{the multi germ } (f_{i\, f_\iota^{-1}(y') \cap \Sigma(f_\iota)})_{i=1,\dots,k}$$
$$\text{is contact equivalent to } (f_{i\, f_\iota^{-1}(y) \cap \Sigma(f_\iota)})_{i=1,\dots,k}\}.$$

(5) Let $\iota: (\mathbf{C}^{p'}, 0) \to (\mathbf{C}^p, 0)$ be a germ of an imbedding transversal to f and let $f_\iota: (f_{\iota,i})_{i=1,\dots,k}$ be defined by the following diagram of fibre product

$$\begin{array}{ccc} f_{\iota,i}: (X, 0) & \longrightarrow & (\mathbf{C}^{p'}, 0) \\ \downarrow & & \downarrow \iota \\ f_i: (\mathbf{C}^{n_i}, 0) & \longrightarrow & (\mathbf{C}^p, 0), \end{array}$$

where X is a smooth submanifold in $\mathbf{C}^{n_i} \times \mathbf{C}^{p'}$ of codimension $n_i - p + p'$. Then f_ι is infinitesimally stable if and only if ι is transverse to the leaf L_0, $0 \in \mathbf{C}^p$.

Let f be as above and $\Sigma' \subset \mathscr{O}_{\mathbf{C}}(p, p_0)$ denote the set of map germs g for which the composition (f, g) is not finitely I_0 determined. By the same argument as in the proof of Proposition 2.1.2, we see that Σ' is a pro-algebraic set. By Lemma 2.1.1, we have only to show that any r-jet $z \in J^r(p, p_0)$ has a representative $g \in \mathscr{O}_{\mathbf{C}}(p, p_0)$ for which the composition (f, g) is finitely I_0 determined. Now we have the following lemma.

Lemma 2.2.3. *Let $\Sigma^* \in J^1(\mathbf{C}^p, \mathbf{C}^{p_0})$ denote the set of* 1 *jets $z = J^1 g(0)$ whose graphs* $\operatorname{graph}(g): \mathbf{C}^p \to \mathbf{C}^p \times \mathbf{C}^{p_0}$ *are not transversal to the leaves $L_y \times 0$, $y \in S$, $S \in B$. Then Σ^* is a constructible set and*

$$\operatorname{codim} \Sigma^* = \max\{p_0 + \operatorname{codim} S, p - \operatorname{codim} L_y + 1, S \in B, y \in S\}.$$

For the proof of Lemma 2.3.3, see **[P]**. Note that a connected tree $(f, g)_X$ of the composition (f, g) with one vertex x of height 1 in $g^{-1}(0)$ and root $0 \in \mathbf{C}^{p_0}$ is trivial if and only if $J^1 g(x) \notin \Sigma^*$ (by Proposition 1.1.1 and the property (5) of the stratification B).

Now we claim that $\operatorname{codim} \Sigma^* \geq p$. So now we assume that $\operatorname{codim} \Sigma^* < p$. By Lemma 2.2.3, this occurs only if $\operatorname{codim} L_y \geq 2$, $y \in S$, for some $S \in B$. This may occur only in the following two cases.

(i) There is a stratum $S' \in A'_i$ such that $f_i(S') = S$ and S' is foliated by contact classes of codim ≥ 2.
(ii) There are two strata $S' \in A'_i$, $S'' \in A'_j$ with $f_i(S') = f_j(S'') = S$ foliated by contact classes of codim ≥ 1.

First we assume that $\operatorname{rank} df_{i\,x_i}$, $df_{j\,x_j} \geq p_0$, for any $x_i \in S'$, $x_j \in S''$ and the condition (1) of the theorem holds for i, j. Then, in case (i), we have

$$\operatorname{codim} S \geq p - n_i +^2 \sigma(n_i - p_0, p - p_0) \geq p - p_0 \qquad \text{(Lemma 2.1.3)}$$

and in case (ii), by the property (4) of A_i, B, we have

$$\begin{aligned}\operatorname{codim} S \geq p - n_i + {}^1\sigma(n_i - p_0, p - p_0)\\ + p - n_j + {}^1\sigma(n_j - p_0, p - p_0),\end{aligned}$$

and by the inequality $p - n + {}^2\sigma(n, p) \geq 2 \cdot (p - n + {}^1\sigma(n, p))$ ((0.6) in [P]), we have

$$\geq \min\{p - n_i + {}^2\sigma(n_i - p_0, p - p_0), p - n_j + {}^2\sigma(n_j - p_0, p - p_0)\} \geq p - p_0.$$

Secondly we assume that $\operatorname{rank} df_{i\,x_i} \leq p_0 - 1$ for any point $x_i \in S'$. Then, in both cases (i) and (ii), we have $\dim S \leq \min\{n_i, p, p_0 - 1\}$, and by Lemma 2.2.3, $\operatorname{codim} \Sigma^* \geq p$. However this contradicts our assumption. So we have proven the claim.

Let S^* be a Whitney regular stratification of Σ^*. By Theorem 2.2.4, there is a pro algebraic set $\Sigma'' \subset \mathscr{O}_{\mathbf{C}}(p, p_0)$ with infinite codimension such that for any $g \notin \Sigma''$, $J^1 g$ is transversal to S^* off $0 \in \mathbf{C}^p$. For dimensional reasons, we have $(J^1 g)^{-1}(\Sigma^*) = 0$, which implies $C_v(f, g) \cap g^{-1}(0) = 0$ and, by Lemma 1.5.1, (f, g) is finitely I_0 determined.

This completes the proof of Theorem 3 for the case where f is polynomial and g is complex analytic. Infinitesimally stable map germs are finitely determined. So the statements for the other cases follow from the polynomial case.

THEOREM 2.2.4 (*Transversality theorem*). *Let S be a semi algebraic (resp. constructible, complex analytic) stratification of $J^k(\mathbf{R}^n, \mathbf{R}^p)$ (resp. $J^k(\mathbf{C}^n, \mathbf{C}^p)$). Then there is a pro semialgebraic (resp. pro-constructible, pro-complex analytic) subset Σ in $\mathscr{E}(n, p)$ (resp. $\mathscr{O}_{\mathbf{C}}(n, p)$) of infinite codimension with the following property: any $f \notin \Sigma$ admits a representative $\tilde{f}$ defined on an open neighbourhood U of 0 in $\mathbf{R}^n$ (resp. $\mathbf{C}^n$) for which the k jet section $J^k\tilde{f}$ is transversal* *to S outside 0 and the inverse image $(J^k\tilde{f})^{-1}S$ is homeomorphic to a semialgebraic (resp. constructible, complex analytic) stratified set. If the restrictions of the projection $\pi: J^k(\mathbf{R}^n, \mathbf{R}^p) \to \mathbf{R}^p$ to strata of S are submersive, then $J^k\tilde{f} \mid U - \{0\}$ is multi transversal to S.*

For the proof of Theorem 2.2.4, see [F].

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