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THE HOMOLOGY OF A FREE LOOP SPACE

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Denote by X^{S^1} the space of all continuous maps from the circle into a simply connected finite CW complex, X. THEOREM: Let $\Bbbk$ be a field and suppose that either $\operatorname{char}\Bbbk > \dim X$ or that X is $\Bbbk$-formal. Then the betti numbers $b_q = \dim H_q(X^{S^1}; \Bbbk)$ are uniformly bounded above if and only if the $\Bbbk$-algebra $H^*(X; \Bbbk)$ is generated by a single cohomology class. COROLLARY: If, in addition, X is a smooth closed manifold and $\Bbbk$ is as in the theorem, and if $H^*(X; \Bbbk)$ is not generated by a single class then X has infinitely many distinct closed geodesics in any Riemannian metric.

1. Introduction. In this paper (co)homology is always singular and $b_q(-; \Bbbk) = \dim H_q(-; \Bbbk)$ denotes the *qth betti number* with respect to a field $\Bbbk$. The *free loop space*, X^{S^1}, of a simply connected space, X, is the space of all continuous maps from the circle into X.

The study of the homology of X^{S^1} is motivated by the following result of Gromoll and Meyer:

THEOREM **[16]**. *Assume that X is a simply connected, closed smooth manifold, and that for some field $\Bbbk$ the betti numbers $b_q(X^{S^1}; \Bbbk)$ are unbounded. Then X has infinitely many distinct closed geodesics in any Riemannian metric.*

(The proof in **[16]** is for $\Bbbk = \mathbb{R}$, but the arguments work in general.)

The Gromoll-Meyer theorem raises the problem of finding simple criteria on a topological space X which imply that the $b_q(X^{S^1}; \Bbbk)$ are unbounded for some $\Bbbk$. This problem was solved for $\Bbbk = \mathbb{Q}$ by Sullivan and Vigué-Poirrier **[28]**. They considered simply connected spaces X such that $\dim H^*(X; \mathbb{Q})$ was finite, and they showed that then the $b_q(X^{S^1}; \mathbb{Q})$ were unbounded if and only if the cohomology algebra $H^*(X; \mathbb{Q})$ was not generated by a single class. And they drew the obvious corollary following from the Gromoll-Meyer theorem.

It is generally conjectured that the same phenomenon should hold in any characteristic; explicitly:

Conjecture. Suppose X is simply connected and, for some field $\Bbbk$, $H^*(X;\Bbbk)$ is finite dimensional. Then the $b_q(X^{S^1};\Bbbk)$ are unbounded if and only if the $\Bbbk$-algebra $H^*(X;\Bbbk)$ is not generated by a single class.

One direction of the conjecture is trivial:

REMARK. If $H^*(X;\Bbbk)$ is generated by a single class then the $b_q(X^{S^1};\Bbbk)$ are uniformly bounded. Indeed, consider the Eilenberg-Moore spectral sequence [12], [25] for the fibre square

$$\begin{array}{ccc} X^{S^1} & \longrightarrow & X^I \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[\Delta]{} & X\times X \end{array}, \qquad \pi f = (f(0), f(1)), \ \Delta x = (x, x).$$

It converges from $\mathrm{Tor}^{H\otimes H}(H, H)$ to $H^*(X^{S^1};\Bbbk)$, where $H = H^*(X;\Bbbk)$ is considered as a module over $H\otimes H$ via $(\alpha\otimes\beta)\cdot\gamma = (-1)^{\deg\beta\deg\gamma}\alpha\beta\gamma$.

Now if H is generated by a single class then it is easy to compute $\mathrm{Tor}^{H\otimes H}(H, H)$ explicitly and to see that $b_q(X^{S^1};\Bbbk)\le 2$, all q. □

In this paper we establish the conjecture under an additional hypothesis; in particular we prove it for any X if $H^i(X;\Bbbk)=0$ for all $i>\mathrm{char}\,\Bbbk$. It was already known in some cases: for instance it was shown by L. Smith [26] in characteristic two when $H^*(X;\mathbb{Z}_2)$ has the form $\bigotimes_i \mathbb{Z}_2[x_i]/x_i^{n_i}$ and $Sq^1=0$. And McCleary and Ziller [20] and Ziller [30] have proved it for homogeneous spaces in all characteristics. Results have also been obtained by Anick [4] and Roos [24]. And McCleary [19] has established a weaker form of the conjecture: if ΩX denotes the classical loop space of based maps $S^1\to X$ then the $b_q(\Omega X;\Bbbk)$ are unbounded if and only if $H^*(X;\Bbbk)$ is not generated by a single class.

To state our theorem we first set (for a given field $\Bbbk$)

$$r_X+1=\inf\{i\ge 2 \mid H^i(X;\Bbbk)\ne 0\} \quad \text{and}$$
$$n_X=\sup\{i \mid H^i(X;\Bbbk)\ne 0\}.$$

Then we have

THEOREM I. *Let X be a simply connected space and let $\Bbbk$ be a field such that $H^*(X;\Bbbk)$ is finite dimensional. Then the conjecture holds for X and for $\Bbbk$ if either:*

(A) $\operatorname{char}\Bbbk \geq n_X/r_X$ *or* (B) *X is $\Bbbk$-formal* ([3], [13]).

The Gromoll-Meyer theorem then implies the

COROLLARY. *Let X be a simply connected closed manifold and let $p > 0$ be a prime. If $H^*(X;\mathbb{Z}_p)$ is not generated by a single class, and if either $p \geq n_X/r_X$ or X is p-formal then X has infinitely many distinct closed geodesics in any Riemannian metric.*

The definition of $\Bbbk$-formal will be recalled in §3. Here we limit ourselves to giving:

Examples of $\Bbbk$-formal spaces. The class of $\Bbbk$-formal spaces includes suspensions, and those spaces X for which $\widetilde{H}_i(X;\Bbbk)$ is zero if i is outside an interval of the form $[k+1, 3k+1]$, and this class is closed under products and wedges—for all this see [3]. Manifolds X are $\Bbbk$-formal if $\widetilde{H}_i(X;\Bbbk)$ is zero outside an interval of the form $[k+1, 4k+2]$ ([13]) if $\operatorname{char}\Bbbk \neq 2, 3$. And if X is a simply connected finite complex such that $\widetilde{H}_i(X, \Bbbk)$ is zero outside an interval of the form $[k+1, 2k]$ then the boundary of a regular neighbourhood of X (embedded in a large $\mathbb{R}^N$) is a $\Bbbk$-formal manifold. □

We turn now to the proof of Theorem I, which we shall outline here, the details following in §§2, 3, 4. We work henceforth over a fixed field $\Bbbk$ and denote $\bigotimes_k$ and Hom_k simply by $\bigotimes$ and Hom. The tensor algebra on a vector space, V, is denoted by $T(V)$. We adopt the convention "$V^k = V_{-k}$" to raise and lower degrees in graded vector spaces, V; in a *differential graded vector space* (DGV) the differential maps $V_k \to V_{k-1}$ (and hence $V^k \to V^{k+1}$). *Differential graded algebras* are called DGA's and a DGA morphism which induces an isomorphism of cohomology is called a DGA *quism* and denoted by $\xrightarrow{\simeq}$.

Recall now that the *Hochschild homology* $\mathrm{HH}_*(A)$ of an algebra, A, is given by $\mathrm{HH}_*(A) = \operatorname{Tor}^{A\otimes A^{\mathrm{opp}}}(A, A)$. If A is a DGA we shall use the same terminology:

$$\mathrm{HH}_*(A) = \operatorname{Tor}^{A\otimes A^{\mathrm{opp}}}(A, A)$$

denotes the *Hochschild homology of* A, where now Tor is the differential tor of Eilenberg-Moore **[21]**. When we want to emphasize that we are in the DGA case we write $\mathrm{HH}_*(A, d)$. (Some authors call this Hochschild hyperhomology.)

The starting point for the proof of Theorem I is a result of Burghelea-Fiedorowicz **[8]** and Cohen **[11]** which asserts that

$$H_*(X^{S^1}; \Bbbk) = \mathrm{HH}_*(C_*(\Omega X; \Bbbk), d), \tag{1.1}$$

where $C_*(\Omega X; \Bbbk)$ is the DGA of singular chains on the Moore loop space of X. Thus if $(T(V), d) \xrightarrow{\simeq} (C_*(\Omega X; \Bbbk), d)$ is an Adams-Hilton model **[2]** for X then we have

$$H_*(X^{S^1}; \Bbbk) \cong \mathrm{HH}_*(T(V), d), \tag{1.2}$$

because DGA quisms induce isomorphisms of Hochschild homology, as follows from the Eilenberg-Moore comparison theorem **[21**; Theorem 2.3].

Let (Ω^*, d) be the DGA obtained by dualizing the bar construction on $(T(V), d)$—we recall the definition in §2. The main result (Theorem II) of §2 will show that

$$\mathrm{HH}^*(\Omega^*, d) \cong \mathrm{Hom}(\mathrm{HH}_*(T(V), d), \Bbbk). \tag{1.3}$$

In §3, on the other hand, we observe that either of conditions (A) and (B) gives a DGA quism $(\Omega^*, d) \xrightarrow{\simeq} (A, d)$, where (A, d) is a commutative differential graded algebra (CDGA). In the case of condition (A) this follows from a deep theorem of Anick **[4]**; in the case of condition (B) it is a consequence of one of the equivalent definitions of $\Bbbk$-formal (**[3]**, **[13]**). In either case we again apply the comparison theorem of **[21]** to obtain

$$\mathrm{HH}^*(\Omega^*, d) \cong \mathrm{HH}^*(A, d). \tag{1.4}$$

The isomorphisms (1.1), (1.2), (1.3) and (1.4) combine to yield

$$\mathrm{H}^*(X^{S^1}; \Bbbk) \cong \mathrm{HH}^*(A, d). \tag{1.5}$$

As we note in §3, the CDGA (A, d) satisfies $H(A) = H^*(X; \Bbbk)$. Indeed when X is $\Bbbk$-formal $(A, d) = (H^*(X; \Bbbk), 0)$ and so (1.5) becomes

$$\mathrm{H}^*(X^{S^1}; \Bbbk) \cong \mathrm{HH}^*(H^*(X; \Bbbk)),$$

in this case. This answers a question of Anick **[3]** in positive characteristic; in characteristic zero it has been proved by Vigué-Poirrier **[29]** and Anick **[3]**.

The last step in the proof of Theorem I is the proof, in §4 of

THEOREM III. *Let* (A, d) *be a CDGA such that* $H^{<0}(A) = 0$, $H^0(A) = \Bbbk$, $H^1(A) \doteq 0$ *and* $H(A)$ *is finite dimensional. Then the integers* $b_q = \dim \mathrm{HH}^q(A, d)$ *are unbounded if and only if* $H(A)$ *is not generated by a single class.*

The proof of Theorem III follows the lines of the proof in **[28]** when $\Bbbk = \mathbb{Q}$ via the construction of a Sullivan model for (A, d), but with additions and modifications to cover the problems caused by positive characteristic.

2. Hochschild homology. In this section we prove a result which implies (1.3), namely

THEOREM II. *Suppose* (R, d) *is an augmented DGA such that* $H_{<0}(R) = 0$, $H_0(R) = \Bbbk$ *and each* $H_i(R)$ *is finite dimensional. If* $(\Omega^*(R), d)$ *is the DGA dual to the bar construction on* (R, d) *then*

$$\mathrm{HH}^*(\Omega^*(R), d) \cong \mathrm{Hom}(\mathrm{HH}_*(R, d), \Bbbk).$$

Before starting the proof, however, we recall some definitions and facts from or about:

(a) differential homological algebra, (b), the opposite of a DGA, (c) differential coalgebras and comodules and (d) bar constructions.

(a) *Differential homological algebra* (**[21]**, **[5]**, **[14]**). An (R, d)-*module* is a DGV, (V, d), together with an R-module structure on V such that $d(r \cdot v) = dr \cdot v + (-1)^{\deg r} r \cdot dv$. It is *semi-free* if it is the increasing union of submodules $V(0) \subset V(1) \subset \cdots$ such that $V(0)$ and each $V(i+1)/V(i)$ is R-free on a basis of cycles. For any (R, d)-module, (M, d) there is a morphism $\phi\colon (V, d) \to (M, d)$ from a semi-free module (V, d) such that $H(\phi)$ is an isomorphism; such a morphism is called a *semi-free resolution* of (M, d). Given any such resolution and any second (R, d)-module, (N, d), we have

$$\mathrm{Tor}^R(M, N) = H(V \otimes_R N).$$

(b) *The opposite DGA.* The opposite DGA, (R^{opp}, d), has the same underlying differential graded vector space as (R, d), but the product " $\circ$ " is given by: $r \circ r' = (-1)^{\deg r \deg r'} r' r$. The *enveloping* DGA (R^e, d), is then defined by $(R^e, d) = (R, d) \otimes (R^{\mathrm{opp}}, d)$ so that

$$(r_1 \otimes r_2)(r_3 \otimes r_4) = (-1)^{\deg r_2 (\deg r_3 + \deg r_4)} r_1 r_3 \otimes r_4 r_2 .$$

Notice that multiplication makes (R, d) into a left (R^e, d)-module: $(r_1 \otimes r_2) \cdot r = (-1)^{\deg r \deg r_2} r_1 r r_2$; similarly we can make (R, d) into a right (R^e, d)-module.

(c) *Differential comodules* [**21**, §6]. A *comodule* over a differential graded coalgebra (DGC), (C, d) is a DGV, (W, d), together with a DGV morphism $(W, d) \xrightarrow{\gamma} (W, d) \otimes (C, d)$ which makes W into a graded C-comodule. If (W, d) is also an (R, d)-module via $\alpha: (R, d) \otimes (W, d) \to (W, d)$ then these structures are *compatible* if γ is an R-module map (equivalently α is a C-comodule map).

If M and N are respectively a right and left (C, d)-comodule then their *cotensor product*, $M \square_C N$ is the kernel of the DGV morphism $\gamma_M \otimes 1 - 1 \otimes \gamma_N: M \otimes N \to M \otimes C \otimes N$. If M has a compatible left (R, d)-module structure and if Q is any right (R, d)-module then a natural DGV map

$$\omega: Q \otimes_R (M \square_C N) \to (Q \otimes_R M) \square_C N \tag{2.1}$$

is constructed as follows:

Observe that $M \square_C N$ is a sub (R, d)-module of $M \otimes N$, so that the inclusion induces $\varphi: Q \otimes_R (M \square_C N) \to Q \otimes_R (M \otimes N)$. Since clearly $\gamma_{Q \otimes_R M} \otimes 1 - 1 \otimes \gamma_N$ vanishes on $\operatorname{Im} \phi$, we have $\operatorname{Im} \phi \subset (Q \otimes_R M) \square_C N$, and so (2.2) is defined by ϕ.

(d) *Bar constructions.* Denote the augmentation ideal of R by $\overline{R}$ and define a graded vector space $s\overline{R}$ by $(s\overline{R})_n = \overline{R}_{n-1}$. The *bar construction* ([**21**], [**29**]) on (R, d), denoted by (BR, δ), is the DGC defined (modulo signs) by: BR is the tensor coalgebra on $s\overline{R}$ (as usual $sr_1 \otimes \cdots \otimes sr_n$ is written $[sr_1 | \cdots | sr_1]$) and

$$\delta[sr_1 | \cdots | sr_n] = \sum_{i=1}^{n} \pm [sr_1 | \cdots | sdr_i | \cdots | sr_n] + \sum_{i=1}^{n-1} \pm [sr_1 | \cdots | s(r_i r_{i+1}) | \cdots | sr_n].$$

The dual DGA, $\operatorname{Hom}((BR, \delta); \Bbbk)$, is denoted by $(\Omega^*(R), d)$.

From the bar construction one builds the classic acyclic construction $(R \otimes BR, \nabla)$ given by $\nabla = d \otimes 1 + 1 \otimes \delta + \tau$ with

$$\tau(r \otimes [sr_1 | \cdots | sr_n]) = \pm rr_1 \otimes [sr_2 | \cdots | sr_n].$$

It is in an obvious way a left (R, d)-module and a right (BR, δ)-comodule. Finally, we have the two-sided bar construction $(R \otimes BR \overline{\otimes} R^{\mathrm{opp}}, D)$ with $D = d \otimes 1 \otimes 1 + 1 \otimes \delta \otimes 1 + 1 \otimes 1 \otimes d + \theta$, and

$$\theta(r \otimes [sr_1 | \cdots | sr_n] \otimes r') = \pm rr_1 \otimes [sr_2 | \cdots | sr_n] \otimes r' \pm r \otimes [sr_1 | \cdots | sr_{n-1}] \otimes r_n r'.$$

It is straightforward ([21; §6]) that the augmentation $\varepsilon\colon BR \to \Bbbk$, together with the multiplication map $R \otimes R^{\mathrm{opp}} \to R$ defines an (R^e, d)-semi-free resolution $(R \otimes BR \otimes R^{\mathrm{opp}}, D) \to (R, d)$. Thus

(2.2) $H(R \otimes_{R^e} (R \otimes BR \otimes R^{\mathrm{opp}})) = \mathrm{Tor}^{R \otimes R^{\mathrm{opp}}}(R, R) = \mathrm{HH}_*(R, d)$,

and indeed this was the original definition of Hochschild homology.

These constructions may also be applied to (R^{opp}, d) to yield the DGC $(B(R^{\mathrm{opp}}, d), \delta)$ and the acyclic construction $(R^{\mathrm{opp}} \otimes B(R^{\mathrm{opp}}), \nabla)$. Moreover a DGC isomorphism, $\omega\colon (B(R^{\mathrm{opp}}), \delta) \to ((BR)^{\mathrm{opp}}, \delta)$, onto the opposite DGC is defined by

$$\omega[sr_1|\cdots|sr_n] = (-1)^k[sr_n|\cdots|sr_1], \quad k = \sum_{i<j}(\deg sr_i)(\deg sr_j).$$

Thus $1 \otimes \omega$ converts $(R^{\mathrm{opp}} \otimes B(R^{\mathrm{opp}}), \nabla)$ into a DGV, $(R^{\mathrm{opp}} \otimes (BR)^{\mathrm{opp}}, \nabla')$, which is both an (R^{opp}, d)-module and an $((BR^{\mathrm{opp}}, \delta)$-comodule.

We come now to the

Proof of Theorem II. As in [6] there is DGA quism of the form $(T(V), d) \xrightarrow{\simeq} (R, d)$ with $v_i = 0$, $i \le 0$, and each V_i finite dimensional. By the Eilenberg-Moore comparison theorem [21; Theorem 2.3] Ω^* preserves quisms and HH^* converts quisms to isomorphisms. We may thus replace (R, d) by $(T(V), d)$ and assume that

(2.3) $R = R_{\ge 0}$, $R_0 = \Bbbk$ and each R_i is finite dimensional.

Now let $((BR)^e, \delta)$ denote the DGC $(BR, \delta) \otimes ((BR)^{\mathrm{opp}}, \delta)$ and set

$$M(R) = (R \otimes BR \otimes R^{\mathrm{opp}} \otimes (BR)^{\mathrm{opp}}, \nabla \otimes 1 + 1 \otimes \nabla').$$

Evidently $M(R)$ has compatible left (R^e, d)-module and right $((BR)^e, \delta)$-comodule structures. Moreover, we have

LEMMA 2.4. *For any right (R^e, d)-module, Q, and any left $((BR)^e, \delta)$-comodule N the natural DGV map*

$$\omega\colon Q \otimes_{R^e} (M(R) \,\square_{(BR)^e} N) \to (Q \otimes_{R^e} M(R)) \,\square_{(BR)^e} N$$

is an isomorphism.

Proof of (2.4). We may ignore differentials and write $M(R) = R^e \otimes (BR)^e$. The standard isomorphism $(BR)^e \,\square_{(BR)^e} N \cong N$ gives an isomorphism

(2.5) $$M(R) \,\square_{(BR)^e} N \cong R^e \otimes N$$

of R^e modules. Analogously, we have a $(BR)^e$-comodule isomorphism

$$Q \otimes_{R^e} M(R) \cong Q \otimes (BR)^e . \tag{2.6}$$

Using (2.5) and (2.6) one easily identifies ω with the identity of $Q \otimes N$. □

We apply Lemma 2.4 with $Q = (R, d)$ and $N = (BR, \delta)$, the module (resp., comodule) structures being defined by multiplication (resp., comultiplication) as described in (b) above. Notice that (2.5) becomes

$$M(R) \square_{(BR)^e} BR \cong R^e \otimes BR \cong R \otimes BR \otimes R^{\text{opp}} ;$$

according to **[17**; Lemma 2.01] the differential induced thereby in $R \otimes BR \otimes R^{\text{opp}}$ is that of the two-sided bar construction. Thus (cf. (2.2))

$$H(R \otimes_{R^e} (M(R) \square_{(BR)^e} BR)) \cong \operatorname{Tor}^{R \otimes R^{\text{opp}}}(R, R) .$$

For simplicity denote the graded dual of a graded vector space by $V^\# = \operatorname{Hom}(V, \Bbbk)$. Thus $(\Omega^*(R), d) = (BR, \delta)^\#$. Because of our assumption (2.3) both R and BR are concentrated in degrees ≥ 0, and are finite dimensional in each degree. For such spaces # commutes with $\otimes$ so that, for instance, $([\Omega^*(R)]^e, d) = ((BR)^e, \delta)^\#$. Thus we deduce from Lemma 2.4 that

$$\operatorname{HH}_*(R, d)^\# \cong H^*\{[(R \otimes_{R^e} M(R)) \square_{(BR)^e} BR]^\#\} . \tag{2.7}$$

Write $Y = [R \otimes_{R^e} M(R)]^\#$. We shall show that Y is an $(\Omega^*(R)^e, d)$ semi-free resolution of $\Omega^*(R)$. Since

$$[(R \otimes_{R^e} M(R)) \square_{(BR)^e} BR]^\# = Y \otimes_{\Omega^*(R)^e} \Omega^*(R) ,$$

it will then follow from (2.7) that $\operatorname{HH}_*(R, d)^\# = \operatorname{HH}^*(\Omega^*(R), d)$, as desired.

That Y is $(\Omega^*(R)^e, d)$-semi-free can be seen by filtering it by the spaces F_j of functions vanishing on $[R_{\geq j} + d(R_j)] \otimes_{R^e} M(R)$. And a homology isomorphism $Y \to \Omega^*(R)$ of $(\Omega^*(R)^e, d)$-modules is defined by dualizing the diagonal $BR \to BR \otimes BR$, regarded as a map

$$BR \to 1 \otimes (BR)^e \subset R \otimes_{R^e} M(R) .$$

3. Reduction to the commutative case. Let

$$(T(V), d) \xrightarrow{\simeq} (C_*(\Omega X ; \Bbbk), d)$$

be an Adams-Hilton model **[2]** for a space X satisfying the conditions of the conjecture, and denote the dual of the bar construction on

$(T(V), d)$ by (Ω^*, d). In this section we prove

PROPOSITION 3.1. *If X satisfies condition* (A) *or condition* (B) *of Theorem* I *then there is a DGA quism* $(\Omega^*, d) \xrightarrow{\simeq} (A, d)$ *with* (A, d) *a CDGA and* $H(A) \cong H^*(X; \Bbbk)$. *If condition* (B) *holds,* $(A, d) = (H^*(X; \Bbbk), 0)$.

Proof. The main result of [4] asserts that if (A) holds then the differential in the Adams-Hilton model may be chosen so as to map V into the sub Lie algebra $L \subset T(V)$ generated by V. This identifies $(T(V), d)$ as the universal enveloping algebra, $U(L, d)$ of the DGL (*differential graded Lie algebra*) (L, d).

Recall that the bar construction is a tensor coalgebra, and in particular contains the sub-coalgebra, S, of symmetric (in the graded sense) tensors. In particular, we have $S(sL) \subset S(s(UL_+)) \subset B(UL)$. As in the case of characteristic zero ([23; Appendix B], [10]), $S(sL)$ is a sub DGC of $B(UL)$ and the inclusion $S(sL) \to B(UL)$ is a homology isomorphism [22]. Dualizing this gives a quism from (Ω^*, d) to the CDGA $S(sL)^\#$. On the other hand [1] $(C_*(\Omega X; \Bbbk), d)$ is connected by DGA quisms to the cobar construction on $(C_*(X; \Bbbk), d)$, and hence to $\Omega^*(C^*(X; \Bbbk), d)$. Thus $\Omega^*(T(V), d)$ is connected by quisms to $\Omega^*\Omega^*(C^*(X; \Bbbk), d)$, and so by [21; Theorem 6.2] we have $H(A) \cong H(\Omega^*(T(V), d)) \cong H^*(X; \Bbbk)$.

Now suppose X satisfies condition (B); i.e., X is $\Bbbk$-formal. One of the equivalent definitions of this is ([3], [13]) that X have an Adams-Hilton model which is the dual of the bar construction on $H^*(X; \Bbbk)$: $(T(V), d) = \Omega^*(H^*(X; \Bbbk), 0)$. Thus $(\Omega^*, d) = \Omega^*(\Omega^*(H^*(X; \Bbbk), 0))$ and by [21; Theorem 6.2] this maps by a quism to $(H^*(X; \Bbbk), 0)$: $(\Omega^*, d) \xrightarrow{\simeq} (H^*(X; \Bbbk), 0)$.

4. The commutative case. In this section we prove

THEOREM III. *Let (A, d) be a CDGA such that $H^{<0}(A) = 0$, $H^0(A) = \Bbbk$, $H^1(A) = 0$ and $H(A)$ is finite dimensional. Then the integers $b_q = \dim HH^q(A, d)$ are unbounded if and only if $H(A)$ is not generated by a single class.*

Proof. As in the rational case ([27], [7], [18]) it is straightforward to construct a DGA quism of the form

$$(\Lambda V, d) \xrightarrow{\simeq} (A, D)$$

in which: $V = V^{\geq 2}$ is a graded vector space, $\Lambda V =$ exterior algebra $(V^{\text{odd}})\otimes$ symmetric algebra (V^{even}) and $\operatorname{Im} d \subset (\Lambda V)^+ \cdot (\Lambda V)^+$. Using the Eilenberg-Moore comparison theorem **[21**; Theorem 2.3] we replace (A, d) by $(\Lambda V, d)$.

The same argument as given in **[28]** for $\Bbbk = \mathbb{Q}$ now establishes

LEMMA 4.1. *The algebra* $H(\Lambda V)$ *is generated by a single class if and only if* $\dim V^{\text{odd}} \leq 1$.

If, moreover, $H(\Lambda V)$ is generated by a single class then the hypothesis $\dim H(\Lambda V) < \infty$ implies, in view of (4.1) that the only possibilities for $(\lambda V, d)$ are: $V = 0$, $V = (x)$ with $\deg x$ odd, or $V = (x, y)$ with $dy = x^k$ and $\deg y$ odd. In all these cases there is an obvious quism $(\Lambda V, d) \xrightarrow{\simeq} (H(\Lambda V), 0)$, which induces an isomorphism of Hochschild homology. Now a direct calculation shows $\dim \mathrm{HH}^q(H(\Lambda V), 0) \leq 2$ for all q.

It remains to show that the $\mathrm{HH}^q(\Lambda V, d)$ have unbounded dimensions if $\dim V^{\text{odd}} \geq 2$. Recall that sV is the graded space given by $(sV)_{k+1} = V_k$; thus $(sV)^k = V^{k+1}$. Denote by $\Gamma(sV)$ the free divided powers algebra on sV, **[9]**, and denote the ith divided power of sx by $\gamma_i(sx)$.

Consider the multiplication homomorphism,

$$\phi\colon (\Lambda V, d) \otimes (\Lambda V, d) \to (\Lambda V, d).$$

According to **[15**; Proposition 1.9], ϕ extends to a DGA quism of the form

$$(4.2) \qquad \phi\colon (\Lambda V \otimes \Lambda V \otimes \Gamma(sV), D) \xrightarrow{\simeq} (\Lambda V, d)$$

in which

$$(4.3) \qquad \phi(\Gamma(sV)^+) = 0,$$

$$(4.4) \qquad \operatorname{Im} D \subset (\Lambda V \otimes \Lambda V)^+ \otimes \Gamma(sV) \quad \text{and}$$

$$(4.5) \qquad D(\gamma_i(sx)) = D(sx) \cdot \gamma_{i-1}(sx).$$

For ease of notation denote the algebra $\Lambda V \otimes \Lambda V \otimes \Gamma(sV)$ by $\Sigma(V)$, and for $\Phi \in \Lambda V$ write $\Phi' = \Phi \otimes 1 \otimes 1$ and $\Phi'' = 1 \otimes \Phi \otimes 1$. Then the model (4.2) also satisfies:

$$(4.6) \qquad \text{For } x \in V^n, \quad Dsx - (x' - x'') \in \Sigma(V^{<n}).$$

Now choose a basis $x_1, x_2, \ldots, x_m, y, x_{m+1}, \ldots, x_i, \ldots$ in which $\deg x_1 \leq \cdots \leq \deg x_m \leq \deg y \leq \cdots \leq \deg x_i \leq \cdots$, and y is the first basis element of odd degree. (All other basis elements are denoted by x_j, some j.)

Lemma 4.7. *The differential D in $\Sigma(V)$ can be chosen so that $Dsy-(y'-y'')\in\Sigma(x_1,\dots,x_m)$ and for all i, $Dsx_i-(x_i'-x_i'')$ is in the ideal generated by the x_j', x_j'' and $\Gamma(sx_j)^+$, $j<i$.*

Proof. D is constructed inductively on n; if it has already been defined in $s(V^{\le n})$ then there is always a linear map of degree zero,

$$f\colon V^{n+1}\to\Sigma(V^{\le n})\cap\ker\phi$$

such that $dv'-dv''-Df(v)\equiv 0$ and given any such f, D may be extended to $\Sigma(V^{\le n+1})$ by setting

$$D(sv)=v'-v''-f(v),\qquad v\in V^{n+1}.$$

Now notice that because $V^1=0$ and $\operatorname{Im} d\subset(\Lambda V)^+\cdot(\Lambda V)^+$ it follows that $dy\in\Lambda(x_1,\dots,x_m)$ and dx_i is in the ideal generated by the x_j, $j<i$. Moreover, that $Dsy-(y'-y'')\in\Sigma(x_1,\dots,x_m)$ is immediate from (4.6) as is $Dsx_i-(x_i'-x_i'')\in\Sigma(x_1,\dots,x_{i-1})$ for $i\le m$.

Suppose then that the lemma is proved for some $x_1,\dots,x_i$, $i\ge m$. Let $I\subset\Sigma(x_1,\dots,x_m,y,\dots,x_i)$ be the ideal generated by the $\Sigma(x_j)^+$, $j\le i$. Since

$$dx_j\in\Lambda^+(x_1,y,\dots,x_{j-1})\cdot\Lambda^+(x_1,\dots,y,\dots,x_{j-1})$$

it follows from our induction hypothesis on Dsx_j and from (4.5) that D maps I to itself. Dividing by I gives us a CDGA of the form $(\Sigma(y),\overline{D})$ and a commutative diagram of CDGA morphisms

$$\begin{array}{ccc}(\Sigma(x_1,\dots,y,\dots,x_i),D) & \xrightarrow[\simeq]{\phi} & (\Lambda(x_1,\dots,y,\dots,x_i),d)\\ \big\downarrow{\scriptstyle\rho} & & \big\downarrow{\scriptstyle\rho}\\ (\Sigma y,\overline{D}) & \xrightarrow[\phi]{\simeq} & (\Lambda y,0)\end{array}$$

in which

$$\phi(y')=\phi(y'')=y,\ \ \phi(\gamma_i(sy))=0\quad\text{and}$$
$$\overline{D}(\gamma_i(sy))=(y'-y'')\gamma_{i-1}(sy).$$

As described at the start of the proof, there is always an element $w\in\Sigma(x_1,\dots,y,\dots,x_i)\cap\ker\phi$ such that $dx_{i+1}'-dx_{i+1}''-Dw=0$, and D may be extended to $\Sigma(x_1,\dots,x_{i+1})$ by setting $Dsx_{i+1}=x_{i+1}'-x_{i+1}''-w$. And for any such w,

$$\overline{D}\rho w=\rho Dw=\rho(dx_{i+1}'-dx_{i+2}'')=0,$$

since dx'_{i+1} and dx''_{i+1} are in I. Moreover $\phi\rho w = \rho\phi w = 0$. Since ϕ: $(\Sigma y, \overline{D}) \to (\Lambda y, 0)$ is a surjective quism it follows that $\rho w = \overline{D}u$, some $u \in \ker\phi \cap \Sigma y$.

Regard u as an element of $\ker\phi \cap \Sigma(x_1, \dots, y, x_i)$ via the inclusion of Σy. Then $\rho(w - Du) = 0$, $\phi(w - Du) = D\phi u = 0$ and so we may define $Dsx_{i+1} = x'_{i+1} - x''_{i+1} - w + Du$. Now we have $\rho(w - Du) = \rho w - \overline{D}u = 0$ and so $Dsx_{i+1} - (x'_{i+1} - x''_{i+1}) \in I$, as desired. □

We now return to the proof of Theorem III. It follows from (4.5) and (4.6) that the quism ϕ: $(\Lambda V \otimes \Lambda V \otimes \Gamma(sV), D) \to (\Lambda V, d)$ is a $(\Lambda V, d) \otimes (\Lambda V, d)$-semi-free resolution. Hence

$$\mathrm{HH}^*(\Lambda V, d) = H(\Lambda V \otimes_{\Lambda V \otimes \Lambda V} (\Lambda V \otimes \Lambda V \otimes \Gamma(sV))) = H(\Lambda V \otimes \Gamma(sV)).$$

Denote the differential in $\Lambda V \otimes \Gamma(sV)$ by δ. Lemma 4.7 shows that $\delta(sx_i)$ is in the ideal generated by the x_j and $\Gamma(sx_j)$, $j < i$. Let $z = x_{n+1}$ $(n \geq m)$ be the first x_i of odd degree and divide $\Lambda V \otimes \Gamma(sV)$ by the ideal generated by the x_j, $j \leq n$.

This produces a CDGA of the form $(\Lambda(y, z, x_{n+2}, \dots) \otimes \Gamma(sV), \overline{\delta})$. The same argument as given in **[28]** shows that if this CDGA has unbounded betti numbers then so does $(\Lambda V \otimes \Gamma(sV), \delta)$, as desired. But by Lemma 4.7, $\overline{\delta}sx_i$ is in the ideal generated by $sx_1, \dots, sx_{i-1}$, for $i \leq n$. Moreover $\Gamma(sx_i) =$ the exterior algebra $\Lambda(sx_i)$ because $\deg sx_i$ is odd. Hence $sx_1 \wedge \cdots \wedge sx_n$ is a cycle.

And since $\delta(sy)$ and $\delta(sz)$ are also in the ideal generated by $sx_1, \dots, sx_n$ it follows from (4.5) that the elements $sx_1 \wedge \cdots \wedge sx_n \wedge \gamma_i(sy) \wedge \gamma_j(sz)$ are all $\overline{\delta}$-cycles. Under the projection $\Lambda V \otimes \Gamma(sV) \to \Gamma(sV)$ these elements map to linearly independent homology classes, since the differential included in $\Gamma(sV)$ is zero, by (4.4). Thus they represent linearly independent classes in $H(\Lambda(y, z, \dots) \otimes \Gamma(sV), \overline{\delta})$, and hence the betti numbers of $(\Lambda(y, z, \dots) \otimes \Gamma(sV), \overline{\delta})$ are indeed unbounded. □

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