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DIVISION ALGEBRAS OVER NONLOCAL HENSELIAN SURFACES

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Let R be the coordinate ring of an integral affine algebraic surface, $\widetilde{R}$ the henselization of R along a reduced, connected curve and $\widetilde{K}$ the quotient field of $\widetilde{R}$. Then every central $\widetilde{K}$-division algebra $\widetilde{D}$ of exponent n in $B(\widetilde{K})$ is cyclic of degree n. If K is the quotient field of R and D is a central K-division algebra of exponent n with ramification divisor Z on $\operatorname{Spec} R$, then there is an étale neighborhood $U \to \operatorname{Spec} R$ of Z such that upon restriction to $K(U)$, D is a cyclic algebra of exponent n and index n.

In this paper we continue to investigate the structure of division algebras D finite dimensional over their center K, where K has transcendence degree 2 over an algebraically closed field k of characteristic 0. The motivating question behind this work, which remains unanswered, is whether the exponent of the class $[D]$ in the Brauer group $\mathrm{B}(K)$ is equal to the degree $\sqrt{(D:K)}$ of the division algebra. This question has been addressed in the works [2], [3] and [8]. In [2] it was shown that exponent(D) = degree(D) when exponent(D) has prime factorization $2^n 3^m$. It was shown in [3] that exponent(D) = degree(D) when K is the quotient field of the henselization at a closed point on a normal algebraic surface. Division algebras over such fields K were also studied in [8]. In [8] it was shown that every central K-division algebra is also cyclic. That is, if exponent(D) = n, there is a cyclic Galois extension L/K of degree n which splits D (see for example, [15, §30]). Thus a structure theory for division algebras was obtained which is similar to that of global fields. The purpose of this paper is to extend the results of [8] to the case where K is the function field of a ring $\widetilde{R}$ obtained by henselizing an affine algebraic surface along a curve. The line of proof used here pretty nearly follows that of [8]. As another parallel to [8] we point out in Remark 8 that [8, §2] can be adapted to construct the algebra D as a symbol $(\alpha, \beta)_n$ over K in the special case that D ramifies on a curve Z whose normalization $\overline{Z}$ is simply connected.

The results of this paper are mainly concerned with surfaces that have been henselized along a curve. For the basic properties of henselian couples, the reader is referred to [**14**]. Let R be the coordinate ring of a normal, integral, 2-dimensional affine variety over k. Let I be an ideal in R such that R/I is reduced and connected. Let $\widehat{R}$ be the completion of R in the I-adic topology. Then $\widehat{R}$ is a normal domain. To see this, note first that R is a G-ring [**12**, Theorem 77, p. 254]. Therefore $\widehat{R}$ is a normal ring [**12**, Theorem 79, p. 258]. Since R/I is connected, it follows that $\widehat{R}$ is connected. Thus $\widehat{R}$ is a normal domain. Let $(\widetilde{R}, \widetilde{I})$ be the henselization of R along I. By [**6**, Proposition 1.5], $\widetilde{R}$ is also a normal domain. We now state our main result.

THEOREM 1. *Let $\widetilde{K}$ be the quotient field of either $\widehat{R}$ or $\widetilde{R}$ and $\widetilde{D}$ a central, finite dimensional $\widetilde{K}$-division algebra with* exponent$(\widetilde{D}) = n$. *Then $\widetilde{D}$ is a cyclic algebra of degree n.*

Before starting the proof of Theorem 1 we mention an important consequence for algebras over K, the quotient field of R. For simplicity let us assume $\mathrm{B}(R) = 0$ and R is regular. The sequence

$$0 \to \mathrm{B}(K) \xrightarrow{a} \bigoplus_{C} \mathrm{H}^1(\mathrm{K}(C), \mathbb{Q}/\mathbb{Z}) \tag{1}$$

is exact, where the summation is over all irreducible curves C on $\operatorname{Spec} R$ [**5**, §3]. Therefore the class $[D]$ in $\mathrm{B}(K)$ is completely determined by the ramification data $a([D])$ in $\bigoplus \mathrm{H}^1(\mathrm{K}(C), \mathbb{Q}/\mathbb{Z})$. The irreducible curves Z_i where $a([D]) \neq 0$ make up the ramification divisor $Z = Z_1 \cup \cdots \cup Z_m$ of D. Denote $a([D])$ by $(L_1, \ldots, L_m)$ where L_i is a cyclic Galois extension of the function field $\mathrm{K}(Z_i)$ of Z_i. Again, for the sake of simplicity, assume Z is connected. Suppose I is a radical ideal for Z and let $(\widetilde{R}, \widetilde{I})$ be the henselization of R along I. Let $(R, I) \to (A, J)$ be an étale neighborhood of (R, I). Then we can assume A is a domain. Let $\mathrm{K}(A)$ be the quotient field of A. Let $\operatorname{Spec} A/J = W$. Then $W \cong Z$. In fact we may write W as a union of irreducibles $W = W_1 \cup \cdots \cup W_m$ where $W_i \cong Z_i$ for each i. The diagram

$$\begin{array}{ccc} \mathrm{B}(\mathrm{K}(A)) & \longrightarrow & \bigoplus_{i=1}^m \mathrm{H}^1(\mathrm{K}(W_i), \mathbb{Q}/\mathbb{Z}) \\ \uparrow & & \uparrow \gamma \\ \mathrm{B}(K) & \longrightarrow & \bigoplus_{i=1}^m \mathrm{H}^1(\mathrm{K}(Z_i), \mathbb{Q}/\mathbb{Z}) \end{array} \tag{2}$$

commutes. Since $R \to A$ is unramified on Z, the vertical arrow γ is an isomorphism. Up to the isomorphism γ, the ramification data for $D \otimes \mathrm{K}(A)$ on $\operatorname{Spec} A$ agrees with that for D on $\operatorname{Spec} R$. So $D \otimes \mathrm{K}(A)$ has exponent n. Therefore, upon restriction to $\widetilde{K}$, $\widetilde{D} = D \otimes \widetilde{K}$ has exponent n. By Theorem 1, $\widetilde{D} = D \otimes \widetilde{K}$ has index n. More specifically, $\widetilde{D}$ is split by a cyclic extension $L/\widetilde{K}$ of degree n. Therefore, for some (A, J), $D \otimes \mathrm{K}(A)$ is a cyclic algebra with index = exponent. This proves

Corollary 2. *Let R be the affine coordinate ring of a smooth surface with quotient field K and $\mathrm{B}(R) = (0)$. Let D be a central K-division algebra. There is an étale R-algebra A such that upon restriction to $\mathrm{K}(A)$ the ramification data of D are preserved and D becomes a cyclic central simple algebra with index = exponent.* □

We now begin the proof of Theorem 1. We begin with some general results about splitting the ramification of central simple algebras on surfaces. Let S be a normal, integral, algebraic surface with function field F. Let L be a finite extension field of F and $Y \to S$ the integral closure of S in L.

$$\begin{array}{ccc} & & L \\ & \diagup & | \\ Y & & F \\ \downarrow & \diagup & \\ S & & \end{array}$$

Let $\pi\colon Y' \to Y$ be any desingularization of Y. That is, Y' is a nonsingular surface and π is a proper, birational morphism. There is a complex

$$(3)\qquad \begin{aligned} 0 \to \mathrm{B}(Y') \to \mathrm{B}(L) \xrightarrow{a} \bigoplus_C \mathrm{H}^1(\mathrm{K}(C), \mathbb{Q}/\mathbb{Z}) \xrightarrow{r} \bigoplus_P \mu(-1) \\ \xrightarrow{s} \mathrm{H}^4(Y', \mu) \to 0 \end{aligned}$$

which is exact except possibly at the term $\bigoplus \mathrm{H}^1(\mathrm{K}(C), \mathbb{Q}/\mathbb{Z})$. This follows by combining sequences (3.1) and (3.2) of **[5]**. If $\mathrm{H}^3(Y', \mu) = 0$, (3) is exact. The first summation is over all irreducible curves $C \subseteq Y'$, the second over all closed points $P \in Y'$. Let D be a central F-division algebra and $D_L = D \otimes L$, the restriction of D to L. We say that L *splits the ramification of D on S* if there exists a desingularization $\pi\colon Y' \to Y$ such that $[D_L]$ is in the image of the map $\mathrm{B}(Y') \to \mathrm{B}(L)$. As was shown in **[3]**, it is possible to find a desingularization $\sigma: S' \to S$ such that the ramification divisor Z of D on S' has normal crossings. As was pointed out in [**8**, §1], it is technically easier to test whether L splits the ramification of D on S' than on S. The following proposition was implicitly proved in the text immediately preceding Theorem 1.6 of [**8**]. We will make use of the construction used in the proof; hence we give it here for reference.

PROPOSITION 3. *With the preceding notation, if the exponent of* $[D]$ *in* $\mathrm{B}(F)$ *is* n, *then there exists a cyclic Galois extension* L/F *of degree* n *that splits the ramification of* D *on* S'.

Proof. Let Z be the ramification divisor of D on S'. Using [**11**, §V.1] we can find nonsingular curves D_1, D_2 on S' such that $Z \sim D_1 - D_2$ and the curve $Z \cup D_1 \cup D_2$ is a divisor with normal crossings. So there is a function $\alpha \in F$ such that the principal divisor (α) has underlying curve $Z \cup D_1 \cup D_2$ and α has valuation $+1$ on each irreducible component of Z. Let $L = F(\alpha^{1/n})$. Let Y' be the integral closure of S' in L. Let $Y'' \to Y'$ be any resolution of the singularities of Y'. Since (α) has normal crossings Y' has only rational singularities [**8**, Theorem 1.2]. We want to show that the algebra $D_L = D \otimes L$ is unramified along each prime divisor of Y'', or that D_L extends to an Azumaya algebra on Y''. Let σ be the composite morphism $Y'' \to Y' \to S'$. Then σ has ramification index n at the prime components of Z. If E is the exceptional divisor of $Y'' \to Y'$, then D_L is unramified on $Y'' - E$ by [**8**, diagram (4)]. Since Y' has rational singularities E is simply connected so $B(Y'') \cong B(Y'' - E)$ [**8**, Corollary 0.2]. Thus D_L is unramified on Y'' and L splits the ramificiation of D on S'. □

EXAMPLE 4. This is an example of a field extension L/F that splits the ramification of D but does not split the Brauer class of D. In the setting of Theorem 1 above, this phenomenon cannot occur because the surface $X = \operatorname{Spec} \widetilde{R}$ is henselized. Let $S = A^2$, the affine plane over k, $F = k(x, y)$ and D the symbol algebra $(x, y)_2$. Let L be the quadratic extension $F\sqrt{xy(x^2-1)(y^2-1)}$. The ramification divisor of D on S is the curve $xy = 0$. Now L splits the ramification of D on S since the ramification index of $Y' \to S$ is 2 at the primes (x) and (y). So D_L is unramified on the surface defined by the equation $z^2 = xy(x^2-1)(y^2-1)$. We claim D_L is not split. This is because D_L remains unsplit upon restriction to the field $M = F(\sqrt{x(x^2-1)}, \sqrt{y(y^2-1)})$. In fact the symbol algebra $(x, y)_2$

is a generator of ${}_2B(C_1 \times C_2)$ where C_1 and C_2 are the elliptic curves defined by $u^2 = x(x^2-1)$ and $v^2 = y(y^2-1)$ respectively (see [7, Example 9]).

As in Theorem 1, let R be the affine coordinate ring of a normal, integral, 2 dimensional variety over k. Let I be an ideal in R such that R/I is reduced and connected. Let $(\widetilde{R}, \widetilde{I})$ be the henselization of R along I. Let $\widetilde{K}$ be the quotient field of $\widetilde{R}$ and $X = \operatorname{Spec} \widetilde{R}$. Let $\widetilde{D}$ be a central $\widetilde{K}$-division algebra with exponent n in $\mathrm{B}(\widetilde{K})$. Let $\pi: X' \to X$ be a resolution of the singularities of X. Let $Z \subseteq X'$ be the ramification divisor of $\widetilde{D}$ on X'. If necessary, blow up points on X' so that the ramification divisor of $\widetilde{D}$ on X' is a divisor with normal crossings.

COROLLARY 5. *Let $\pi: X' \to X$, $\widetilde{K}$, $\widetilde{D}$, n be as above. Then there exists a cyclic extension L of $\widetilde{K}$ of degree n that splits the ramification of $\widetilde{D}$ on X'.*

Proof. Since $\widetilde{R}$ is the direct limit of integral domains A_i of finite type over K there is an étale neighborhood A of (R, I) and a central simple algebra Λ over $F = \mathrm{K}(A)$ such that $\widetilde{D} = \Lambda \otimes_F \widetilde{K}$. Since $U = \operatorname{Spec} A$ is an algebraic surface we apply Proposition 3 to find a cyclic splitting field E/F for the ramification of Λ on U. Let $L = \widetilde{K}E$ and let Y' be the integral closure of X' in L. By the construction in the proof of Proposition 3 we see that $Y' \to X'$ has ramification index n along each of the prime components of Z, where Z is the ramification divisor of $\widetilde{D}$ on X'. Thus $\widetilde{D}_L = \widetilde{D} \otimes L = \Lambda \otimes E \otimes L$ is unramified on any desingularization of Y'. The construction of E also makes it clear that $L/\widetilde{K}$ is cyclic of degree n. □

Proof of Theorem 1. By approximation techniques [**6**] it suffices to assume $\widetilde{K}$ is the quotient field of $\widetilde{R}$. We use the notation introduced immediately before Corollary 5. By Corollary 5 there is a cyclic extension L of degree n that splits the ramification of $\widetilde{D}$ on X'. If Y' is the integral closure of X' in L and $Y'' \to Y'$ is a resolution of the singularities of Y', then Lemma 6 below shows that $B(Y'') = 0$. Thus $\widetilde{D}_L$ is split. □

LEMMA 6. *Let $X = \operatorname{Spec} \widetilde{R}$ be as above. Let $\pi: X' \to X$ be a resolution of the singularities of X. Then $\mathrm{B}(X') = \mathrm{H}^2(X', \mathrm{G}_m) = 0$, $\mathrm{H}^3(X', \mathrm{G}_m) = \mathrm{H}^3(X', \mu) = 0$ and $\mathrm{H}^4(X', \mathrm{G}_m) = \mathrm{H}^4(X', \mu) = 0$.*

Before proving the lemma we state a corollary which follows immediately from (3) and Lemma 6.

COROLLARY 7. *Let $\pi: X' \to X$ be a resolution of the singularities of $X = \operatorname{Spec} \widetilde{R}$. Let $\widetilde{K}$ be the quotient field of $\widetilde{R}$. The sequence*

$$0 \to \mathrm{B}(\widetilde{K}) \xrightarrow{a} \bigoplus_{C} \mathrm{H}^1(\mathrm{K}(C), \mathbb{Q}/\mathbb{Z}) \xrightarrow{r} \bigoplus_{P} \mu(-1) \to 0$$

is exact where the first summation is over all irreducible curves $C \subseteq X'$ and the second over all closed points $P \in X'$. □

Proof of Lemma 6. First we note that since X' is smooth, $\mathrm{H}^p(X', \mathrm{G}_m)$ is torsion for $p \geq 2$ [**10**, p. 71]. Thus $\mathrm{H}^p(X', \mathrm{G}_m) = \mathrm{H}^p(X', \mu)$ for $p \geq 3$ by Kummer theory. Since X' is not complete, $\mathrm{H}^4(X', \mu) = 0$ [**13**, Cor. VI.11.5]. Since X is normal it has finitely many singular points say $\xi_1, \dots, \xi_m$. Let $g_i: \xi_i \hookrightarrow X$ be the closed immersion, for each i. Let $\Omega = \{\xi_1, \dots, \xi_m\}$. Then on $X' - \pi^{-1}(\Omega)$, π is an isomorphism; hence the sheaves $\mathrm{R}^q\pi_*(\mu_n)$ have support on Ω for $q \geq 0$. By proper base change each stalk $\mathrm{R}^q\pi_*(\mu_n)_{\xi_i}$ is canonically isomorphic to $\mathrm{H}^q(X'_{\xi_i}, \mu_n)$ where $X'_{\xi_i} = X' \times \xi_i$ is the fiber of π over ξ_i. So $\mathrm{R}^q\pi_*(\mu_n)$ is the direct image sheaf $\bigoplus_{i=1}^m g_{i*}(F_i)$ where F_i is a sheaf on ξ_i [**13**, Cor. II.3.11]. Since $\xi_i = \operatorname{Spec} k$ and k is algebraically closed, F_i is the constant sheaf $\mathrm{R}^q\pi_*(\mu_n)_{\xi_i} = \mathrm{H}^q(X'_{\xi_i}, \mu_n)$. The spectral sequence for $g_i: \xi_i \hookrightarrow X$ is $\mathrm{H}^p(X, \mathrm{R}^q g_{i*}(F_i)) \Rightarrow \mathrm{H}^{p+q}(\xi_i, F_i)$. Since g_i is a closed immersion $\mathrm{R}^q g_{i*}(F_i) = 0$ for $q > 0$. Thus $\mathrm{E}_0^j = \mathrm{H}^j(\xi_i, F_i) = \mathrm{E}_1^j = \cdots = \mathrm{E}_j^j = \mathrm{H}^j(X, g_{i*}F_i)$. Again, k is algebraically closed, so $\mathrm{H}^j(X, g_{i*}F_i) = 0$ for $j > 0$. This proves Step 1.

Step 1. $\mathrm{H}^p(X, \mathrm{R}^q\pi_*(\mu_n)) = 0$ for $p > 0, q > 0$.
Step 2. Let $Z = \operatorname{Spec} \widetilde{R}/\widetilde{I}$. Then $\operatorname{Pic} X \cong \operatorname{Pic} Z$.
This follows from [**16**].
Step 3. $\mathrm{B}(X) = 0$ and $\mathrm{H}^2(X, \mu_n) = 0$.
Since Z is an affine curve, $\operatorname{Pic} Z$ is divisible. This follows from the exact Kummer sequence

$$\operatorname{Pic} Z \xrightarrow{n} \operatorname{Pic} Z \to \mathrm{H}^2(Z, \mu_n)$$

and the fact that $\mathrm{H}^2(Z, \mu_n) = 0$ since Z is not complete [**13**, Cor. VI.11.5]. By Step 2, $\operatorname{Pic} X$ is also divisible. Now $\mathrm{B}(X) \cong \mathrm{B}(Z) = 0$ [**9**] or [**16**] since Z is 1 dimensional over K. Kummer theory gives the exact sequence

$$(4) \qquad \operatorname{Pic} X \xrightarrow{n} \operatorname{Pic} X \to \mathrm{H}^2(X, \mu_n) \to \mathrm{H}^2(X, \mathrm{G}_m) \xrightarrow{n} \cdots .$$

But ${}_n\mathrm{B}(X) = {}_n\mathrm{H}^2(X, \mathrm{G}_m)$ by Gabber's theorem, so $\mathrm{H}^2(X, \mu_n) = 0$. This completes Step 3.

Now let C denote the fiber $X' \times_X \Omega$ over the singular points of X. Let C_{red} denote the reduced fiber and write $C_{\mathrm{red}} = C_1 \cup \cdots \cup C_t$ as a union of irreducible curves. We may assume C_{red} has pure codimension one. The closed immersion $C_{\mathrm{red}} \hookrightarrow X'$ induces a homomorphism $\operatorname{Pic} X' \to \operatorname{Pic} C_{\mathrm{red}}$. The Kummer map is $\operatorname{Pic} C_{\mathrm{red}} \to \mathrm{H}^2(C_{\mathrm{red}}, \mu_n)$.

Step 4. The composite map $\operatorname{Pic} X' \to \operatorname{Pic} C_{\mathrm{red}} \to \mathrm{H}^2(C_{\mathrm{red}}, \mu_n)$ is surjective.

For each irreducible component C_i of C_{red} choose a point P_i such that P_i is not in the singular set of C_{red}. We can also assume each P_i is not on the curve $\pi^{-1}(Z)$. Since each C_i is nonsingular and X' is nonsingular we can find a prime divisor V_i for each i such that V_i intersects C_i transversally at P_i:

So V_i is prime, disjoint from $\pi^{-1}(Z)$, hence is a henselian curve. Thus V_i is geometrically unibranched and intersects C_{red} exactly at P_i. Consider the diagram

$$(5) \qquad \begin{array}{ccccc} 1 & \mapsto & P_i & & \\ \mathbb{Z} & \to & \operatorname{Pic} C_i & \to & \operatorname{Pic}(C_i - P_i) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z}/n & \to & H^2(C_i, \mu_n) & \to & H^2(C_i - P_i, \mu_n) \end{array}$$

Now $\mathrm{H}^2(C_i - P_i, \mu_n) = 0$ and $\mathbb{Z}/n \to \mathrm{H}^2(C_i, \mu_n)$ is an isomorphism. Thus (5) shows that the class of P_i in $\operatorname{Pic} C_i$ maps to a generator of $\mathrm{H}^2(C_i, \mu_n)$. The composite $\operatorname{Pic} X' \to \operatorname{Pic} C_i$ takes the class of V_i to the class of P_i. This proves Step 4 since $\mathrm{H}^2(C_{\mathrm{red}}, \mu_n) = \bigoplus \mathrm{H}^2(C_i, \mu_n)$.

Step 5. $\mathrm{H}^2(X', \mu_n) \cong \mathrm{H}^0(X, \mathrm{R}^2\pi_*(\mu_n))$.

Consider the spectral sequence for $\pi\colon X' \to X$,

$$(6) \qquad \mathrm{H}^p(X, \mathrm{R}^q\pi_*(\mu_n)) \Rightarrow \mathrm{H}^{p+q}(X', \mu_n).$$

From Steps 1 and 3 the sequence looks like

$$\begin{array}{lll} H^0(X, R^2\pi_*\mu_n) & 0 & 0 \\ H^0(X, R^1\pi_*\mu_n) & 0 & 0 \\ H^0(X, \pi_*\mu_n) & H^1(X, \pi_*\mu_n) & H^2(X, \pi_*\mu_n) = 0 \end{array}$$

So $H^2(X', \mu_n) = E_0^2 \supseteq E_1^2 \supseteq E_2^2 = 0$. Since $E_1^2 = 0$, $E_0^2 = E_\infty^{0,2} = H^0(X, R^2\pi_*\mu_n)$ and the map $H^2(X', \mu_n) \to H^0(X, R^2\pi_*\mu_n)$ is an isomorphism.

Step 6. The Kummer theory map $\mathrm{Pic}\, X' \to H^2(X', \mu_n)$ is surjective.

The spectral sequence $H^p(X, R^q\pi_*(G_m)) \Rightarrow H^{p+q}(X', G_m)$ yields the exact sequence of lower degree terms

$$(7) \qquad 0 \to \mathrm{Pic}\, X \to \mathrm{Pic}\, X' \to H^0(X, R^1\pi_*(G_m)).$$

Combining (7) with the Kummer theory maps (4) and Step 5 we get the commutative diagram

$$\begin{array}{ccccccc} & \downarrow n & & \downarrow n & & & \\ 0 \to & \mathrm{Pic}\, X & \to & \mathrm{Pic}\, X' & \to & H^0(X, R^1\pi_* G_m) & \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & \to & H^2(X', \mu_n) & \to & H^0(X, R^2\pi_*\mu_n) & \to 0 \end{array}$$

Now

$$\begin{aligned} H^0(X, R^2\pi_*\mu_n) &= \bigoplus_{i=1}^{m} H^0(X, g_{i^*}(H^2(X'_{\xi_i}, \mu_n))) \\ &= \bigoplus H^2(X'_{\xi_i}, \mu_n) = H^2(C, \mu_n). \end{aligned}$$

The inclusion $C_{\mathrm{red}} \hookrightarrow C$ is defined by a sheaf of nil ideals so $H^2(C, \mu_n) \to H^2(C_{\mathrm{red}}, \mu_n)$ is an isomorphism [**4**, VIII, Cor. 1.2]. By Step 4 we see that the composite $\mathrm{Pic}\, X' \to H^2(X', \mu_n) \to H^2(C_{\mathrm{red}}, \mu_n)$ is surjective. Combining the above results gives $\mathrm{Pic}\, X' \to H^2(X', \mu_n)$ surjective.

Step 7. $B(X') = H^2(X', G_m) = 0$.

Since X' is a smooth surface, $B(X') = H^2(X', G_m)$. By Kummer theory,

$$\mathrm{Pic}\, X' \xrightarrow{n} \mathrm{Pic}\, X' \to H^2(X', \mu_n) \to {}_nB(X') \to 0$$

is exact. By Step 6, $B(X') = 0$.

Step 8. $H^3(X', G_m) = H^3(X', \mu_n) = 0$.

As pointed out at the beginning of this proof, $H^3(X', G_m) = H^3(X', \mu_n)$. The spectral sequence (6) yields $H^3(X', \mu_n) = E_0^3 = E_1^3 = E_2^3 = E_3^3 = H^3(X, \mu_n) = 0$ since X is an affine surface. □

REMARK 8. Let (R, I), $(\widetilde{R}, \widetilde{I})$ be as in Theorem 1 except assume moreover that R is regular and $Z = \operatorname{Spec} R/I$ has simply connected desingularization. That is, if $\overline{Z}$ is the desingularization of Z, then $\mathrm{H}^1(\overline{Z}, \mathbb{Q}/\mathbb{Z}) = 0$. Let $\widetilde{K}$ be the quotient field of $\widetilde{R}$ and $\widetilde{D}$ a central $\widetilde{K}$-division algebra with exponent n in $\mathrm{B}(\widetilde{K})$. Then Theorem 1 shows that $\widetilde{D}$ is cyclic, hence is a symbol algebra $(\alpha, \beta)_n$ over $\widetilde{K}$. Following the steps of [**8**, §2] one can give an explicit description of α and β. The details are omitted.

REMARK 9. We close with some comments on the possibility of globalizing the above techniques to an affine rational surface with trivial Brauer group (e.g. A^2). In Corollary 2, suppose one can find the étale R-algebra A such that $(\mathrm{K}(A)\colon K)$ is prime to $\operatorname{index}(D)$. Then, upon restriction to $\mathrm{K}(A)$ the index of D remains constant by [**1**, p. 60]. So $\operatorname{index}(D) = \operatorname{exponent}(D)$. To prove that such an algebra A always exists does not appear to be possible in the near future. The henselian property was used in a critical way in Step 4 to lift Picard group elements from the ramification divisor. Suppose an étale neighborhood A of the ramification divisor can be constructed such that (1) the $\operatorname{degree}(\mathrm{K}(A)\colon K)$ is prime to $\operatorname{degree}(D)$ and (2) on $\operatorname{Spec} A$, the composite map $\operatorname{Pic}(\operatorname{Spec} A \times X') \to \operatorname{Pic} C_{\mathrm{red}} \to \mathrm{H}^2(C_{\mathrm{red}}, \mu_n)$ of Step 4 is surjective. Then, upon restriction to $\mathrm{K}(A)$ D will be a cyclic algebra with index = exponent. Again, this means $\operatorname{exponent}(D) = \operatorname{index}(D)$.

REFERENCES

[1] A. A. Albert, *Structure of Algebras*, Amer. Math. Soc., New York, 1939, revised 1961.

[2] M. Artin, *Brauer-Severi varieties*, in *Brauer Groups in Ring Theory and Algebraic Geometry*, Lecture Notes in Math., Vol. 917, Springer-Verlag, Berlin (1982), 194–210.

[3] ——, *Two dimensional orders of finite representation type*, preprint.

[4] M. Artin, A. Grothendieck and J. L. Verdier, SGA 4, *Théorie des Topos et Cohomologie Etale des Schémas*, Lecture Notes in Math., Vols. 269, 270, 305, Springer-Verlag, Berlin/New York, 1972–1973.

[5] M. Artin and D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London Math. Soc., **25** (1972), 75–95.

[6] J. Denef and D. Harbater, *Global approximation in dimension two*, J. Algebra, **129** (1990), 159–193..

[7] T. Ford, *On the Brauer group and the cup product map*, in: *Perspectives in Ring Theory* (F. van Oystaeyen and Lieven Le Bruyn, Eds.), NATO ASI series, Reidel, Dordrecht, 1988.

[8] T. Ford and D. Saltman, *Division algebras over henselian surfaces*, Israel Mathematical Conference Proceedings 1 (1989), 320–336.

[9] S. Greco, *Algebras over nonlocal Hensel rings* II, J. Algebra, **13** (1969), 48–56.
[10] A. Grothendieck, *Le groupe de Brauer*, I, II, III, in: *Dix Exposés sur la Cohomologie des Schémas*, North Holland, Amsterdam, 1968.
[11] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977.
[12] H. Matsumura, *Commutative Algebra*, 2nd. ed., Benjamin/Cummings, Reading, Mass., 1980.
[13] J. Milne, *Etale Cohomology*, Princeton University Press, Princeton, N.J., 1980.
[14] M. Raynaud, *Anneaux Locaux Henséliens*, Lecture Notes in Math. Vol. 169, Springer-Verlag, Berlin 1970.
[15] I. Reiner, *Maximal Orders*, Academic Press, New York 1975.
[16] R. Strano, *On the étale cohomology of Hensel rings*, Comm. Algebra, **12** (1984), 2195–2211.

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