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TWO APPLICATIONS OF THE UNIT NORMAL BUNDLE OF A MINIMAL SURFACE IN R^N

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Dedicated to Professor Shingo Murakami on his sixtieth birthday

A Gauss parametrization of a minimal surface in R^3 is well known. We prove a generalization.

THEOREM A. *Let U be an open set of $S^N(1)$ and f a function on U such that*

$$\Delta_{S^N(1)} f = -Nf$$

and 0 is an eigenvalue of $\operatorname{Hess} f + f\langle\ ,\ \rangle$ *of multiplicity $N-2$, where $\langle\ ,\ \rangle$ is the metric of $S^N(1)$ and $\Delta_{S^N(1)}$ is the Laplacian of $S^N(1)$. Then the map of U into R^{N+1} defined by*

$$(*) \qquad f\eta + \operatorname{grad} f$$

is of rank 2 and gives a minimal surface, where η is the identity map on $S^N(1)$. Conversely, for a minimal surface M in R^{N+1}, a neighborhood of each point of M without geodesic points has this representation.

If M is a complete orientable minimal surface of finite total curvature, then there is a global representation $(*)$ of M. Using this idea, we obtain the following.

THEOREM B. *Let M be a complete orientable minimal surface of finite total curvature in R^{N+1}. Then there exist a positive real number $c(N)$ depending on N such that*

$$\operatorname{index}(M) \leq c(N) \int (-K) * 1_M,$$

where K is the Gauss curvature of M and $*1_M$ is the area form of M.

Theorem B gives an answer for an open question posed by Cheng and Tysk in **[CY1]**. After this paper was submitted, the author learned that Cheng and Tysk in **[CT2]** obtained a similar result as Theorem B by using another Gauss map (generalized Gauss map).

Finally we consider a generalization of minimal herissons **[RT]**.

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2. Second variation formula. Let M be a minimal surface in R^{N+1} and χ the immersion. Let $U(M)$ be the unit normal bundle of the normal bundle $N(M)$. Then we define a Gauss map G of $U(M)$ into the N-dimensional unit sphere $S^N(1)$ by $G(x, \eta) = \eta$ for $(x, \eta) \in U(M)$. G induces a degenerate Riemannian metric of constant curvature 1 on $U(M)$. Let ξ be a section of $N(M)$ with compact support. Then a function F_ξ on $U(M)$ is defined by

$$F_\xi(x, \eta) = \langle \xi, \eta \rangle,$$

where $(x, \eta) \in U(M)$. Let $I(\xi, \xi)$ be the second variation of the area functional in the direction of ξ. Then we get

PROPOSITION 2.1.

$$I(\xi, \xi) = ((N-1)/\omega) \int (|\nabla F_\xi|^2 - N F_\xi^2) * 1_{U(M)},$$

where ω is the volume of $S^{N-2}(1)$ and $*1_{U(M)}$ is the volume form of $U(M)$.

This is well known in the case of $N = 2$.

Proof. Let x be a point of M and e_α for $\alpha = 3, \ldots, N+1$ be a local orthonormal framing of $N(M)$ such that

$$\nabla^\perp_X e_\alpha = 0 \quad \text{for all tangent vectors } X \text{ at } x,$$

where $\nabla^\perp$ is the normal connection of $N(M)$. Furthermore we may consider that the second fundamental form A_η in the direction of η is diagonal and given by

$$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}.$$

Then we get $G_*(\tilde{e}_1) = -\lambda e_1$, $G_*(\tilde{e}_2) = \lambda e_2$ and $G_*(\zeta) = \zeta$, where $\tilde{e}_1, \tilde{e}_2$ are horizontal lifts of principal vectors e_1, e_2 at x to the tangent space of $U(M)$ at (x, η) and ζ is a normal vector with $\langle \eta, \zeta \rangle = 0$. Thus the induced metric is given by

$$\begin{pmatrix} \lambda^2 & & & \\ & \lambda^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

and the volume form is $\lambda^2 * 1_M * 1_{S^N(1)}$. Note that λ^2 is $(1/2)|A_\eta|^2$. Since

$$\tilde{e}_1 F_\xi = \langle \nabla^\perp_{e_i}\xi , \eta\rangle \quad \text{and} \quad \zeta F_\xi = \langle \xi , \zeta\rangle ,$$

we have

$$|\nabla F_\xi|^2 = \left((1/\lambda)^2 \sum \langle \nabla^\perp_{e_i}\xi , \eta\rangle^2\right) + |\xi|^2 - F_\xi^2 ,$$

which implies

$$\int |\nabla F_\xi|^2 * 1_{U(M)} = \int (1/2)|\nabla F_\xi|^2 |A_\eta|^2 * 1_M * 1_{S^{N-2}(1)}.$$

Now we have the integral over the fibre at x as follows:

$$\int (1/2)|\nabla F_\xi^2|\,|A_\eta^2| * 1_{S^{N-2}(1)}$$
$$= \int \left\{ \sum \langle \nabla^\perp_{e_i}\xi , \eta\rangle^2 + (1/2)|A_\eta|^2|\xi|^2 - (1/2)|A_\eta|^2 F_\xi^2 \right\} * 1_{S^{N-2}(1)}.$$

When we put $\eta = \sum y^\alpha e_\alpha$, we have

$$\int \left\{ \sum \langle \nabla^\perp_{e_i}\xi , \eta\rangle^2 \right\} * 1_{S^{N-2}(1)}$$
$$= \int \left\{ \sum y^\alpha y^\beta \langle \nabla^\perp_{e_i}\xi , e_\alpha\rangle \langle \nabla^\perp_{e_i}\xi , e_\beta\rangle \right\} * 1_{S^{N-2}(1)}.$$

It follows from

$$\int y^\alpha y^\beta * 1_{S^{N-2}(1)} = (\omega/(N-1))\delta_{\alpha\beta}$$

that we obtain

$$\int \left\{ \sum \langle \nabla^\perp_{e_i}\xi , \eta\rangle^2 \right\} * 1_{S^{N-2}(1)} = (\omega/(N-1))|\nabla^\perp \xi|^2$$

and

$$\int (1/2)|A_\eta|^2|\xi|^2 * 1_{S^{N-2}(1)}$$
$$= (1/2)\int \left\{ \sum h^\alpha_{ij} h^\beta_{ij} y^\alpha y^\beta |\xi|^2 \right\} * 1_{S^{N-2}(1)}$$
$$= (\omega/2(N-1))|\sigma|^2|\xi|^2 ,$$

where $h^\alpha_{ij} = \langle A_{e_\alpha} e_i , e_j\rangle$ and $|\sigma|^2 = \sum h^\alpha_{ij} h^\alpha_{ij}$. On the other hand, since

$$\int (1/2)|A_\eta|^2 F_\xi^2 * 1_{S^{N-2}(1)}$$
$$= (1/2)\int \left\{ \sum h^\alpha_{ij} h^\beta_{ij} y^\alpha y^\beta y^\gamma y^\delta \langle e_\gamma , \xi\rangle \langle e_\delta , \xi\rangle \right\} * 1_{S^{N-2}(1)}$$

holds and we may consider $e_3 = \xi/|\xi|$, by

$$\int y^\alpha y^\beta (y^3)^2 * 1_{S^{N-2}(1)} = (\omega/(N+1)(N-1))(\delta_{\alpha\beta} + 2\delta_{3\alpha}\delta_{3\beta}),$$

we obtain

$$\begin{aligned}(1/2)\int |A_\eta|^2 F_\xi^2 * 1_{S^{N-2}(1)} &= (\omega/2(N+1)(N-1))|\sigma|^2|\xi|^2 \\ &\quad + (\omega(N+1)(N-1))\left\{\sum \langle \xi, \sigma_{ij}\rangle\langle \xi, \sigma_{ij}\rangle\right\},\end{aligned}$$

where $\sigma_{ij} = \sum h_{ij}^\alpha e_\alpha$. Thus we have

$$\begin{aligned}\int (|\nabla F_\xi|^2 - NF_\xi^2) * 1_{U(M)} \\ = (\omega/(N-1))\int \left(|\nabla^\perp \xi|^2 - \sum \langle \xi, \sigma_{ij}\rangle\langle \xi, \sigma_{ij}\rangle\right) * 1_M.\end{aligned}$$ □

PROPOSITION 2.2. *Let ξ be a normal vector field of $N(M)$. Then ξ is a Jacobi field if and only if*

$$\Delta_{U(M)} F_\xi = -NF_\xi.$$

Proof. We fix a point (x, η) of $U(M)$. Let $\gamma(s)$ be a geodesic with arc length parameter s such that $\gamma(0) = x$. We denote by X the tangent of $\gamma(s)$ at x. Let e_1 and e_2 be the principal vectors of A_η such that $A_\eta e_1 = \lambda e_1$ and $A_\eta e_2 = -\lambda e_2$ and $e_1(s)$ and $e_2(s)$ the parallel vector fields along $\gamma(s)$ with respect to the connection of $T(M)$ such that $e_1(0) = e_1$ and $e_2(0) = e_2$. Let e_α, $\alpha = 3, \dots, e_{N+1}$ be an orthonormal basis of $N_x(M)$ and $e_\alpha(s)$ the parallel vector fields along $\gamma(s)$ with respect to $\nabla^\perp$ such that $e_\alpha(0) = e_\alpha$. We may set $e_3(0) = \eta$. Then $(\gamma(s), e_3(s))$ is the horizontal lift of $\gamma(s)$ through (x, η) in $U(M)$. By the definition of G, we obtain

$$G_*(\tilde{\gamma}_*(s)) = -A_{e_3(s)}\gamma_*(s).$$

Let $\tilde{\nabla}$ be the covariant differentiation with respect to the degenerate metric induced by G. Then we have

$$G_*(\tilde{\nabla}_{\tilde{\gamma}_*(0)}\tilde{\gamma}_*(s)) = \text{the component of } [-dA_{e_3(s)}\gamma_*(s)/ds]_{s=0} \text{ orthogonal to } \eta.$$

It follows that

$$\begin{aligned}\tilde{\nabla}_{\tilde{\gamma}_*(0)}\tilde{\gamma}_*(s) = &((\langle \eta, (\nabla_X \sigma)(X, e_1)\rangle/\lambda)\tilde{e}_1 \\ &- ((\langle \eta, (\nabla_X \sigma)(X, e_2)\rangle/\lambda)\tilde{e}_2 \\ &- \sum_{\alpha=4}^{N-1}\sum_{k=1}^{2}\langle \eta, \sigma(X, e_K)\rangle\langle e_\alpha, \sigma(X, e_K)\rangle_\alpha.\end{aligned}$$

It is easy to extend e_α for $\alpha = 4, \ldots, N-1$ to the vertical vector fields $\tilde{e}_\alpha$ on $U(M)$ such that

$$\tilde{\nabla}_{\tilde{e}_\alpha}\tilde{e}_\alpha = 0 \quad \text{at } (x, \eta).$$

Furthermore, for the horizontal lift $\tilde{Y}$ of a vector field Y defined on a neighborhood at x, we have

$$\tilde{\nabla}_{\tilde{e}_\alpha}\tilde{Y} = ((\langle A_{e_\alpha}Y, e_1\rangle/\lambda)\tilde{e}_1 - ((\langle A_{e_\alpha}Y, e_2\rangle/\lambda)\tilde{e}_2 \quad \text{at } (x, \eta).$$

Using these vector fields, we obtain the following for each point $(x, \eta) \in U(M)$.

$$\begin{aligned}\operatorname{Hess} F_\xi(X, X) = &\left\langle \eta, \operatorname{Hess}\xi(X, X) + \sum \langle \sigma(X, e_k), \xi\rangle\sigma(X, e_k)\right\rangle \\ &- \langle \eta, (\nabla_X\sigma)(X, e_1)\rangle\langle \eta \cdot \nabla^\perp_{e_1}\xi\rangle/\lambda \\ &+ \langle \eta, (\nabla_X\sigma)(X, e_2)\rangle\langle \eta \cdot \nabla^\perp_{e_2}\xi\rangle/\lambda \\ &- \sum \langle \eta, \sigma(X, e_k)\rangle^2 F_\xi,\end{aligned}$$

$$\begin{aligned}\operatorname{Hess} F_\xi(e_\alpha, e_\alpha) &= -F_\xi \quad \text{for } \alpha = 4, \ldots, N-1 \\ \operatorname{Hess} F_\xi(X, e_\alpha) &= \langle e_\alpha, \nabla^\perp_X\xi - ((\langle \eta, \nabla^\perp_{e_1}\xi\rangle/\lambda)\sigma(X, e_1) \\ &\qquad + ((\langle \eta, \nabla^\perp_{e_2}\xi\rangle/\lambda)\sigma(X, e_2)\rangle.\end{aligned}$$

Thus we have

$$\Delta_{U(M)}F_\xi = -(1/\lambda)^2\langle \eta, J(\xi)\rangle - NF_\xi,$$

where J is the Jacobi operator of $N(M)$. □

We know that $\chi^\perp = \sum\langle \chi, e_\alpha\rangle e_\alpha$ is a Jacobi field, where χ is the position vector of M. By the calculation as in Proposition 2.2, we obtain

LEMMA 2.1. *Hess* $F^\perp_\chi + F^\perp_\chi\langle\ ,\ \rangle$ *has an eigenvalue* 0 *of multiplicity* $N-2$ *at* $(x, \eta) \in U(M)$ *such that* $\det A_\eta \neq 0$.

Now we may consider that $F_\chi^\perp$ is locally a function on an open set U of $S^N(1)$. Then we define a map of U into R^{N+1} such that

$$F_\chi^{\perp\eta} + \operatorname{grad} F_\chi^\perp.$$

By a simple calculation, it is just χ. Conversely let f be an eigenfunction of eigenvalue N on an open set U in $S^N(1)$ such that the eigenvalue of the $\operatorname{Hess} f + f\langle\ ,\ \rangle$ has 0 of multiplicity $N-2$. Then

$$f\eta + \operatorname{grad} f$$

is a map of rank 2 and hence gives a minimal surface. Thus we obtain a Gauss parametrization of a minimal surface in R^{N+1}.

As a generalization of Theorem A, we easily obtain the following.

PROPOSITION 2.3. *Let U be an open set of $S^N(1)$ and f a function on U such that* $\operatorname{Hess} f + f\langle\ ,\ \rangle$ *has an eigenvalue 0 of multiplicity $N-m$. Then $f\eta + \operatorname{grad} f$ is a map of U into R^{N+1} of rank m and furthermore gives an m-dimensional submanifold such that the $(m-1)$st mean curvature vector vanishes. We call the representation the Gauss parametrization by an eigenfunction. Conversely let M be an m-dimensional submanifold in R^{N+1} such that the $(m-1)$st mean curvature vector vanishes, then a neighborhood of each point such that* $\det A_\eta \neq 0$ *for some normal vector η the Gauss parametrization by an eigenfunction.*

REMARK. In **[DG]**, similar constructions are presented.

COROLLARY 2.1. *Let M be a complex m-dimensional Kaehler submanifold in C^{N+1}. Then a neighborhood of each point such that* $\det A_\eta \neq 0$ *for some normal vector η admits the Gauss parametrization by an eigenfunction.*

Proof. It is well known that the $(2m-1)$st mean curvature vector vanishes on M.

Let M be a minimal surface in R^{N+1} and ξ a Jacobi field. Then Proposition 2.2 implies that F_ξ is an eigenfunction of eigenvalue N. We define the rank γ_ξ of Jacobi field by $N-\mu$, where μ is the multiplicity of eigenvalue 0 of

$$\operatorname{Hess} F_\xi + F_\xi\langle\ ,\ \rangle.$$

By Proposition 2.3, we have a γ_ξ-dimensional submanifold with zero $(\gamma_\xi - 1)$st mean curvature vector. For example, let M be a minimal surface in R^3. Then $\gamma_\xi = 0$ or 2 holds for a Jacobi field ξ and if $\gamma_\xi = 2$ holds, then we obtain a minimal surface

$$\xi - \sum (A_\eta^{-1})^{ij} \langle \nabla_{e_i}^\perp \xi , \eta \rangle e_j ,$$

which gives a minimal deformation of $M - \{\text{geodesic points}\}$ whose normal variation vector field is ξ. In fact

$$\chi + s \left\{ \xi \sum (A_\eta^{-1})^{ij} \langle \nabla_{e_i}^\perp \xi , \eta \rangle e_j \right\}$$

is a one parameter family of minimal surfaces, where χ is the immersion of M into R^3.

Next let M be a minimal surface in R^4 and ξ a Jacobi field. Then γ_ξ is 0, 2 or 3. In the case of $\gamma_\xi = 3$, we have a hypersurface of zero second mean curvature in R^4, which implies zero scalar curvature. Thus the first given minimal surface is a limit of deformation of hypersurfaces of zero scalar curvature in R^4.

3. The index of minimal surfaces. Let M be a complete orientable minimal surface of finite total curvature in R^{N+1}. Then there exists a compact orientable Riemann surface $\overline{M}$ and finite points $p_1, \dots, p_q \in \overline{M}$ such that M is conformally equivalent to $\overline{M} - \{p_1, \dots, p_q\}$ and the generalized Gauss map of M into $G_2(R^{N+1})$ is extendable over $\overline{M}$. Let L be the tautological vector bundle over $G_2(R^{N+1})$ with rank $N-1$. Then the restriction of the induced bundle over $\overline{M}$ to M is the normal bundle $N(M)$. So the unit sphere bundle $U(\overline{M})$ over $\overline{M}$ gives a compactification of $U(M)$ such that the ends are fibres at p_i. It is clear that the map G is extendable on $U(\overline{M})$ and we denote by $\overline{G}$ the map. Note that $\overline{G}$ is real analytic.

LEMMA 3.1. *The degenerate set S for $\overline{G}$ is an analytic set of codimension ≥ 2 if M is not in some R^3.*

Proof. It is clear that S is an analytic set. Assume that S has an open set of $U(M)$. Then as analytic function $|A_\eta|^2$ on $U(M)$ is zero on some open set, which implies that M is plane. Assume that S has codimension 1. Then we note that the rank of $\theta|_S$ is 1 or 2, where θ is the projection of $U(M)$ onto M. If the rank is 2, there is an open set U of M such that each fibre at $x \in U$ has an $(N-3)$-dimensional submanifold where $|A_\eta^2| \equiv 0$. For each $x \in U$, we have an orthonormal basis $e_3, \dots, e_{N+1}$ such that, for all $\alpha \geq 5$,

$$A_{e_3} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, \quad A_{e_\alpha} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and hence, for any unit normal vector $\eta = ae_3 + be_4 +$ others, $\det A_\eta = 0$ holds if and only if $a^2\lambda^2 + b^2\mu^2 = 0$, which implies that if $\lambda \neq 0$ and $\mu \neq 0$, then the set where $\det A_\eta = 0$ is an $(N-4)$-dimensional sphere. It is a contradiction and hence λ or $\mu = 0$, which implies the first normal space on U is at most 1-dimensional. It is easy to see that M is in some R^3. Next assume that the rank of $\theta|_S$ is 1. Then, on the image $\theta(S)$, the second fundamental form of M vanishes. On the other hand, it is well known that totally geodesic points are isolated. It is contradiction. □

By the result in **[H]**, we can have a stratification of S such that if a stratum T satisfies $T \cap S \neq \emptyset$, then $S \supset T$. So $\overline{G}(S)$ has a stratification and $\overline{G}^{-1}(\overline{G}(S))$ is a sum of finite stratums of codimension ≥ 2. By a simple argument, we get

LEMMA 3.2.

$$\overline{G}\colon U(\overline{M}) \backslash \overline{G}^{-1}(\overline{G}(S)) \to S^N(1) \backslash \overline{G}(S)$$

is a k-sheeted covering map, where k is the total curvature of $M/2\pi$.

From Proposition 2.1, we obtain the following:

$$\text{index}(M) \leq \text{the number of eigenvalues of } \Delta_{U(\overline{M})} \text{ that are strictly less than } N.$$

Let $\{\lambda_i\}_{i=0}^\infty$ and $\{\mu_i\}_{i=0}^\infty$ be eigenvalues of $\Delta_S N_{(1)}$ and $\Delta_{U(\overline{M})}$, respectively. A theorem in **[5]**, together with Lemma 3.2 implies

$$\sum e^{-\mu_i t} \leq k \left(\sum e^{-\lambda_i t}\right).$$

Thus we conclude that

$$(\text{index}(M))e^{-Nt} \leq \sum_{\mu_i < N} e^{-\mu_i t} \leq \sum e^{-\mu_i t} \leq k \left(\sum e^{\lambda_i t}\right).$$

Hence

$$\text{index}(M) \leq e^{Nt} \left(\sum e^{-\lambda_i t}\right) k.$$

Note that if M is not in some R^3, then $c(N)$ is given by

$$2\pi \inf_{t>0} \left\{ e^{Nt} \left(\sum e^{-\lambda_i t}\right)\right\}.$$

4. A generalization of minimal herissons. Recently Rosenberg and Toubiana **[RT]** give some results on complete minimal finite branched surfaces in R^3 of finite total curvature 4π, which are called minimal herissons and parametrized by their Gauss image.

Let M be an m-dimensional submanifold of zero $(m-1)$st mean curvature vector in R^{N+1}. We consider the following condition $(**)$.

$(**)$ *There exist finite stratum S of $U(M)$ and S' of $S^N(1)$ such that codimensions of elements of S and $S' \geq 2$ and*
$G: U(M)\backslash S \to S^N(1)\backslash S'$
is a k-sheeted covering.

Let $\mathfrak{M}$ denote the space of m $(2 \leq m \leq N)$-dimensional submanifolds of zero $(m-1)$st mean curvature vector in R^{N+1} which satisfy $(**)$. Following as in **[RT]**, we can define a sum operation in $\mathfrak{M}$:

$$M_1 + M_2 = \left\{ \sum \theta(x_i) + \sum \theta(y_i) : G_1^{-1}(z) = \{x_i\}, \right.$$
$$\left. G_2^{-1}(z) = \{y_i\}, \text{ where } z \in S^N(1)\backslash S'_1 \cup S'_2 \right\},$$

where G_1 and G_2 are the Gauss map of M_1 and M_2, respectively and S'_1 and S'_2 satisfy $(**)$ for G_1 and G_2. Note that the equality of dimensions of M_1 and M_2 is not necessary. This operation may be considered as follows: for $z \in S^N(1)\backslash S'_1 \cup S'_2$, we define a function f by

$$f(z) = \sum F^{\perp}_{\chi_1}(x_i) + \sum F^{\perp}_{\chi_2}(y_i),$$

where χ_1 and χ_2 are immersions of M_1 and M_2 into R^{N+1}, respectively. It is clear that

$$\Delta_{S^N(1)} f = -Nf$$

on $U = S^N(1)\backslash S'_1 \cup S'_2$ and hence f is analytic on U. By the analyticity of f on U, the multiplicity of the eigenvalue 0 of $\operatorname{Hess} f + f\langle\ ,\ \rangle$ is constant $N-m$ on some open dense set of U. Thus we get an m-dimensional submanifold of zero $(m-1)$st mean curvature vector in R^{N+1} which gives $M_1 + M_2$.

PROPOSITION 4.1. *Assume that $M_1 + M_2$ is of dimension m. Then $M_1 + M_2$ is of zero $(m-1)$st mean curvature vector and parametrized by Gauss image. In particular, the total absolute curvature is the volume of $S^N(1)$.*

REMARK. The study of f which satisfies $\Delta_{S^N(1)} f = -Nf$ has a relation to N-dimensional space-like minimal submanifolds of constant curvature 1 in an $(N+2)$-dimensional deSitter space time **[K]**.

In [N], Nayatani proves that, *if* M *be a complete orientable minimal surface of finite total curvature, then* M *has a finite index.* But it does not imply the existence of $c(N)$.

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