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POINT SPECTRUM ON A QUASI HOMOGENEOUS TREE

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Dedicated to Professor Sh. Murakami on his 60th birthday

By using the algebraicity of the Green kernel it is shown that a linear operator of nearest neighbour type on a quasi homogeneous tree i.e. a tree admitting of a group of automorphism with finite quotient has no point spectrum on the space of square summable functions, provided the tree has a regular property and that the operator is invariant under the group of automorphism.

0. Introduction. This result is an extension of spectrum theorem on an anisotropic random walk on a homogeneous tree (see **[Ao1]** and **[Fi]**). In case of one dimensional lattice relevant results have been obtained in full generality (see **[Mo1]** and **[Mo2]**). See **[Ko]** for a similar problem on a Riemannian manifold. The author is indebted to the referee for various improvements of statements in this note. Among other things, in Theorem 1 the author has originally restricted himself to the graph Γ without loops and multiple edges. The referee has suggested the more complete present form with its proof.

1. Basic properties of the Green kernel. Let T be a connected locally finite tree with the set of vertices $V(T)$ and the set of edges $E(T)$. Let A be a symmetric operator on $l^2(T)$, the space of square summable complex valued functions on $V(T)$:

$$(1.1) \qquad Au(x) = \sum_{\langle x, x'\rangle} a_{x,x'}u(x') + a_{x,x}u(x)$$

for $u(x) \in l^2(T)$, with $a_{x,x}$ and $a_{x,x'} = a_{x',x} \in \mathbb{R}$. $\langle x, x'\rangle$ means that two vertices x, x' are adjacent to each other with respect to an edge $\overline{x, x'}$ binding x and x'.

We assume first that A is regular in the following sense:

$$(\mathscr{C}1) \qquad a_{x,x'} \neq 0 \quad \text{for all } \langle x, x'\rangle.$$

Suppose further that a discrete group of automorphism G of T acts fix point-freely on T:

$$(1.2) \qquad G \times V(T) \ni (g, x) \to g \cdot x \in V(T)$$

and that the quotient $\Gamma = G\backslash T$ is a finite graph. Recall that $V(\Gamma) = G\backslash V(T)$ and $E(\Gamma) = G\backslash E(T)$, where $G\overline{x, y}$ connects the vertices Gx and Gy of Γ. Observe that Γ may have loops and multiple edges. In particular, G must be a finitely generated free group (see **[S]** or **[T]**). Γ is locally homeomorphic to T. We call the tree T "quasi homogeneous". T can be regarded as the set of paths in T from a base point $* \in V(T)$ to points in $V(T)$.

We set up the following condition:

$(\mathscr{C}2)$ *A is invariant under the action of G.*

Then A becomes a bounded and hence self-adjoint operator with domain $\mathscr{D}(A) = l^2(T)$. So the resolvent $(z - A)^{-1}$ is uniquely defined for $z \in \mathbb{C} - \mathbb{R}$. We denote by $G(x, y|z)$, $x, y \in V(T)$ and $z \in \mathbb{C} - \mathbb{R}$, the Green kernel for A, i.e., the matrix elements of the resolvent: $(e_x, (z - A)^{-1} e_y)$ for $e_x, e_y \in l^2(T)$, where $(\ ,\ )$ and e_x denote the inner product on $l^2(T)$ and the function on $V(T)$ which is equal to 1 at x and zero elsewhere respectively.

It is obvious that

$$G(g \cdot x, g \cdot y|z) = G(x, y|z) \tag{1.3}$$

for an arbitrary $g \in G$. We denote by $W_x(z)$ the inverse $G(x, x|z)^{-1}$. As a function of x, W_x depends only on the coset $G \cdot x \in V(\Gamma) = G\backslash V(T)$.

We shall frequently use the following lemma which has been proved in our previous paper **[Ao2]**.

LEMMA 1.1. *$W_x(z)$, $x \in V(T)$, satisfy the basic equations:*

$$z - a_{x,x} - W_x = \sum_{\substack{\langle x, y\rangle \\ y \in V(T)}} \frac{1}{2}\left(-W_x + \sqrt{W_x^2 + 4a_{x,y}^2 W_x / W_y}\right) \tag{$\mathscr{C}$}$$

and

$$W_x(z) \sim z \quad \textit{for } \operatorname{Im} z \to \pm\infty. \tag{1.4}$$

$W_x(z)$ are uniquely determined by $(\mathscr{C})$ and (1.4).

Since W_x depends only on the coset $G \cdot x \in G\backslash V(T)$, there are N algebraic equations for the unknown W_x, $x \in V(\Gamma)$, where N denotes the number of vertices in $V(\Gamma)$. Hence $W_x(z)$ are all algebraic functions in z.

We shall also write $W_{\overline{x}}(z) = G(\overline{x}, \overline{x}|z)^{-1}$ in place of $W_x(z) = G(x, x|z)^{-1}$ in the case where $G \cdot x = \overline{x} \in G\backslash V(T)$. This will not lead to any confusion. The following is an easy consequence of the spectral representation of the Green kernel (see [**Ak** or **Ca**]).

LEMMA 1.2. *For each* $x \in V(T)$,

$$\text{(1.5)} \qquad \operatorname{Im} W_x(z) \cdot \operatorname{Im} z > 0 \quad \textit{for } z \in \mathbb{C} - \mathbb{R}.$$

The following two were proved in [**Ao2**]:

LEMMA 1.3. *For each adjacent pair* $x, y \in V(T)$, *the multiplier* $\alpha(\binom{x}{y}|z) = G(\omega, y|z)/G(\omega, x|z)$ *is expressed as*

$$\frac{-W_x + \sqrt{W_x^2 + 4a_{x,y}^2 W_x/W_y}}{2a_{x,y}} = W_x \cdot \frac{-1 + \sqrt{1 + 4a_{x,y}^2/(W_x W_y)}}{2a_{x,y}}$$

provided x *lies on the geodesic line from* ω *to* y. *We have thereby*

$$\text{(1.6)} \qquad 2a_{x,y}G(x, y|z) = -1 + \sqrt{1 + 4a_{x,y}^2 G(x, x|z) \cdot G(y, y|z)}.$$

LEMMA 1.4. *For each adjacent pair* $x, y \in V(T)$, *the function* $\{-W_x + \sqrt{W_x^2 + 4a_{x,y}^2 W_x/W_y}\}$ *is holomorphic in* z *and its imaginary part has the opposite sign to* $\operatorname{Im} z$, *provided* $z \in \mathbb{C} - \mathbb{R}$.

2. Statement of main theorems. We take and fix a real number λ. Since $W_x(z)$, $x \in V(\Gamma)$, are algebraic in z, they are expressed in Puiseux expansions. There exists a minimal exponent ρ_x such that

$$\text{(2.1)} \qquad \lim_{\substack{z \to \lambda \\ \operatorname{Im} z > 0}} W_x(z) \cdot (z - \lambda)^{-\rho_x} = c_x \neq 0.$$

Owing to Lemma 1.2, we have $-1 \le \rho_x \le 1$ and $\rho_x \in \mathbb{Q}$. Indeed if $|\rho_x| > 1$, then

$$\left| \lim_{z \to \lambda + 0} \arg W_x(z) - \lim_{z \to \lambda - 0} \arg W_x(z) \right| > \pi$$

from (2.1), which contradicts Lemma 1.2, $\rho_x \in \mathbb{Q}$ follows from the algebraicity of the function $W_x(z)$ in z. We denote by $V_\alpha(\Gamma)$ the set of vertices $x \in V(\Gamma)$ such that $\rho_x = \alpha$. λ is an eigenvalue of the operator A if and only if there exists at least one $x \in V(\Gamma)$ such that $\rho_x = 1$, i.e., $V_1(\Gamma) \neq \varnothing$. In fact the spectral kernel $d\theta(x, y|\lambda)$ of the operator A is positive definite:

$$\text{(2.2)} \qquad d\theta(x, x|\lambda) \cdot d\theta(y, y|\lambda) \ge d\theta(x, y|\lambda)^2$$

and therefore λ is an eigenvalue if and only if

$$\theta(x, x|\lambda+0) - \theta(x, x|\lambda-0) \neq 0$$

for some $x \in V(\Gamma)$ (see [Ca] for details). We denote by N_α the number of vertices in $V_\alpha(\Gamma)$ for each $\alpha \in \mathbb{R}$ where $-1 \leq \alpha \leq 1$. Then we have $\sum_{-1\leq\alpha\leq 1} N_\alpha = N$.

The set of vertices $V_\alpha(\Gamma)$ and edges connecting vertices in $V_\alpha(\Gamma)$ define a finite subgraph Γ_α of Γ such that $V(\Gamma_\alpha) = V_\alpha(\Gamma)$. We denote by N'_α the number of edges and L_α the number of loops in Γ_α. A proper circuit in Γ is a sequence $x_0, x_1, \dots, x_k$ of successive adjacent vertices such that $k \geq 3$, $x_0 = x_k$ and $x_i \neq x_j$ for $0 \leq i < j < k$. Then the main theorem can be stated as follows.

THEOREM 1. (i) Γ_1 *has no proper circuit. Thus,* Γ_1 *is a disjoint union of pseudo-trees, i.e., trees where loops and multiple edges are allowed.* (ii) *If* $-1 < \rho < 1$, *then no vertex of* Γ_ρ *is adjacent to any vertex of* Γ_1. (iii) *We have the equality*:

$$\sum_{x\in V_1(\Gamma)} \frac{1}{c_x} = N_1 - \left(N'_1 - \frac{1}{2}L_1\right) - N_{-1}. \tag{2.3}$$

Hence λ *is an eigenvalue of* A *if and only if* $N_1 - (N'_1 - \frac{1}{2}L_1) - N_{-1} > 0$.

REMARK 1. The sum in the left-hand side in (2.3) is equal to $\sum_{x\in V(\Gamma)} \mathrm{Res}_{z=\lambda} G(x, x|z)$.

THEOREM 2. *If* Γ *is regular, i.e., if there is an equal number* (*greater than* 1) *of edges in* Γ *emanating from each vertex in* $V(\Gamma)$, *then the operator* A *is point spectrum free on* $l^2(T)$.

COROLLARY. *If* T *is a Cayley graph of a free group with finitely many generators and* G *is a subgroup of finite index, then* A *is point spectrum free.*

REMARK 2. In this corollary the operator A has at most N bands of continuous spectra in view of the projection freeness theorem for reduced C^*-algebras of free groups (see [Cu] and references therein). It seems likely that there appear exactly N bands for generic A.

The following question raised by the referee seems very likely.

Question 1. Does the set of exponents $\{\rho_x\}$ consist of only $\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$?

3. Proofs of the theorems. We start to prove

LEMMA 3.1. *Γ_1 has no proper circuit.*

Proof. Γ_1 is decomposed into connected components $\Gamma_1^{(1)}, \dots, \Gamma_1^{(K)}$. For each k, $1 \le k \le K$, we denote by $T_1^{(k)}$ the set of paths in $\Gamma_1^{(k)}$ starting from a base point $\overline{x} \in V(\Gamma_1^{(k)})$ and ending in points of $V(\Gamma_1^{(k)})$. Then $T_1^{(k)}$ can be regarded as a subtree of T. We take an arbitrary adjacent pair $x, y \in V(T_1^{(k)})$. We have Puiseux expansions at $z = \lambda$:

$$W_x(z) = c_x(z - \lambda) + (\text{higher degree terms}), \tag{3.1}$$

$$W_y(z) = c_y(z - \lambda) + (\text{higher degree terms}), \tag{3.2}$$

where c_x and c_y are both positive as is seen from Lemma 1.2. In the same way we have

$$G(x, y|z) = \frac{c_{x,y}}{z - \lambda} + (\text{higher degree terms}) \tag{3.3}$$

for $c_{x,y} \in \mathbb{R}$. Let $\{u_j(x)\}_{1 \le j \le M}$, $1 \le M \le +\infty$, be an orthonormal system of λ-eigenfunctions for A. The matrix $((c_{x,y}))$ defines the projection operator from $l^2(T)$ onto the λ-eigenspace. Since $c_{x,x} = 1/c_x$ and $c_{y,y} = 1/c_y$, we have

$$\frac{1}{c_x} = \sum_{j=1}^{M} u_j(x)^2, \tag{3.4}$$

$$\frac{1}{c_y} = \sum_{j=1}^{M} u_j(y)^2, \quad \text{and} \tag{3.5}$$

$$c_{x,y} = \sum_{j=1}^{M} u_j(x) \cdot u_j(y). \tag{3.6}$$

The relation (1.6) shows that

$$\left\{ \sum_{j=1}^{M} u_j(x) \cdot u_j(y) \right\}^2 = \sum_{j=1}^{M} u_j(x)^2 \cdot \sum_{j=1}^{M} u_j(y)^2. \tag{3.7}$$

As a result of the Schwarz inequality, this implies that the two M dimensional vectors $\{u_j(x)\}_{1 \le j \le M}$ and $\{u_j(y)\}_{1 \le j \le M}$ are linearly dependent: $u_j(x) = t(x, y) u_j(y)$ for $t(x, y) \in \mathbb{R}$. Let $x \in V(T_1^{(k)})$ be an arbitrary point such that $G \cdot x = \overline{x} \in V(\Gamma_1^{(k)})$. Applying the above relation successively we have

$$u_j(x) = t(x) \cdot u_j(x^*) \tag{3.8}$$

for an arbitrary $x^* \in V(T_1^{(k)})$ and j such that $u_j(x^*) \neq 0$, where $t(x)$ is a real function on $V(T_1^{(k)})$ such that $\sum_{x \in V(T_1^{(k)})} t(x)^2 < +\infty$. But then

$$\sum_{j=1}^{M} \sum_{x \in V(T_1^{(k)})} u_j(x)^2 = \sum_{j=1}^{M} u_j(x^*)^2 \cdot \sum_{x \in V(T_1^{(k)})} t(x)^2 < +\infty. \tag{3.9}$$

Suppose $\Gamma_1^{(k)}$ has a proper circuit. Then this represents a non-trivial action of an element g of G on T and it follows from [T] that g must have infinite order. Indeed, by [T], g has either infinite order or order 2. Now one has to verify that the second case is impossible: if $g^2 =$ identity and the circuit is even then g fixes a vertex of T, in contradiction with fixed point-freeness. If the circuit is odd, then g inverts an edge $\overline{x, y}$ in T whose G-orbit lies on the circuit. But then $G\overline{x, y}$ is a loop, in contradiction with properness of the circuit. Since $1/c_x$ is invariant under this action: $1/c_x = 1/c_{g \cdot x}$, we have

$$+\infty = \sum_{-\infty}^{+\infty} \frac{1}{c_{g^l \cdot x}} \leq \sum_{j=1}^{M} \sum_{x \in V(T_1^{(k)})} u_j(x)^2 \tag{3.10}$$

which is finite from (3.9). This is a contradiction. Hence each $\Gamma_1^{(k)}$ has no proper circuit and it is a pseudo-tree. T_1 is itself a disjoint union of pseudo-trees. Lemma 3.1 has thus been proved.

LEMMA 3.2. *Let $x \in V_1(\Gamma)$ and $y \in V_{-1}(\Gamma)$ which are adjacent to each other. We have Puiseux expansions at $z = \lambda$ for $W_x(z)$ as in (3.1) and for $W_y(z)$ as follows:*

$$W_y(z) = \frac{c_y}{z - \lambda} + (\textit{higher degree terms}) \tag{3.11}$$

where $c_y < 0$ from Lemma 1.2. Then

$$c_x \cdot c_y \leq -4a_{x,y}^2 \quad \textit{and} \tag{3.12}$$

$$b_{x,y} = -1 \pm \sqrt{1 + \frac{4a_{x,y}^2}{c_x \cdot c_y}} < 0, \tag{3.13}$$

where we take the positive root $\sqrt{}$ in the right hand side.

Proof. From Lemma 1.4, for $\operatorname{Im} z > 0$,

$$\begin{aligned}(3.14)\qquad &\operatorname{Im}(-W_x + \sqrt{W_x^2 + 4a_{x,y}^2 W_x/W_y}) \\ &= \operatorname{Im}\{c_x(-1 \pm \sqrt{1 + 4a_{x,y}^2/(c_x c_y)})(z-\lambda) \\ &\qquad + (\text{higher degree terms})\} < 0.\end{aligned}$$

c_x being positive, $-1 \pm \sqrt{1 + 4a_{x,y}^2/(c_x c_y)}$ must be negative, i.e.

$$(3.15)\qquad 0 \le \sqrt{1 + 4a_{x,y}^2/(c_x c_y)} < 1.$$

Lemma 3.2 has thus been proved.

We shall denote by $\varepsilon_{x,y}$ the sign $\pm$ appearing in the right hand side of (3.14). Note that $\varepsilon_{x,y}$ is symmetric in x, y from Lemma 1.3.

In the sequel, whenever we speak of a sum $\sum_{\langle x,y\rangle}$, where $x \in V(\Gamma)$, the sum refers to all edges from x to y, each carrying the weight inherited from T.

LEMMA 3.3. *Suppose $x \in V_1(\Gamma) = V(\Gamma_1)$. Then there is no $y \in V_\rho(\Gamma)$ adjacent to x in Γ for $-1 < \rho < 1$.*

Proof. W_x has a Puiseux expansion (3.1). Suppose that there exists one $y \in V_\rho(\Gamma)$, $-1 < \rho < 1$, which is adjacent to x. Let α be the greatest exponent among these ρ. Then comparing the constant term and the term $(z-\lambda)^{(1-\alpha)/2}$ respectively, we have from $(\mathscr{E})$

$$(3.16)\qquad \lambda - a_{x,x} = \sum_{\substack{\langle x,y\rangle \\ y \in V_1(\Gamma)}} \varepsilon_{x,y} a_{x,y} \sqrt{c_x/c_y},$$

$$(3.17)\qquad 0 = \sum_{\substack{\langle x,y\rangle \\ y \in V_\alpha(\Gamma)}} \varepsilon_{x,y} a_{x,y} \sqrt{c_x/c_y}(z-\lambda)^{(1-\alpha)/2}.$$

Here we take $\sqrt{c_x/c_y}$ as positive. Since, as a result of Lemma 1.4, the imaginary part of each term in the right hand side satisfies

$$(3.18)\qquad \operatorname{Im}\{\varepsilon_{x,y} a_{x,y} \sqrt{c_x/c_y}(z-\lambda)^{(1-\alpha)/2}\} < 0$$

for $\operatorname{Im} z > 0$, which is a contradiction to (3.17). Hence the set of $y \in V_\alpha(\Gamma)$ which is adjacent to x must be empty. Namely there is no $y \in V_\rho(\Gamma)$, $-1 < \rho < 1$, which is adjacent to x.

Therefore for $x \in V_1(\Gamma)$, the equation $(\mathscr{E})$ becomes

$$(3.19)\quad z - W_x - a_{x,x} = \frac{1}{2} \sum_{\substack{\langle x,y\rangle \\ y \in V_1(\Gamma) \cup V_{-1}(\Gamma)}} \left(-W_x + \sqrt{W_x^2 + 4a_{x,y}^2 W_x / W_y}\right).$$

Comparing the term $z - \lambda$, we have

$$(3.20)\quad 1 - c_x = \frac{1}{2} \sum_{\substack{\langle x,y\rangle \\ y \in V_{-1}(\Gamma)}} \left(-c_x + \varepsilon_{x,y} c_x \sqrt{1 + 4a_{x,y}^2/(c_x c_y)}\right) + \frac{1}{2} \sum_{\substack{\langle x,y\rangle \\ y \in V_1(\Gamma)}} \left\{-c_x + 2\varepsilon_{x,y} a_{x,y} \sqrt{c_x/c_y} \left(\frac{c_x'}{c_x} - \frac{c_y'}{c_y}\right)\right\},$$

where $c_x'(z-\lambda)^2$ denotes the quadratic term in the Puiseux expansion of $W_x(z)$. (3.20) is reexpressed as

$$(3.21)\quad \frac{1}{c_x} - 1 = \frac{1}{2} \sum_{\substack{\langle x,y\rangle \\ y \in V_{-1}(\Gamma)}} \left(-1 + \varepsilon_{x,y} \sqrt{1 + 4a_{x,y}^2/(c_x c_y)}\right) + \frac{1}{2} \sum_{\substack{\langle x,y\rangle \\ y \in V_1(\Gamma)}} \left\{-1 + \varepsilon_{x,y} \frac{2a_{x,y}}{\sqrt{c_x c_y}} \left(\frac{c_x'}{c_x} - \frac{c_y'}{c_y}\right)\right\}.$$

Summing up both sides over the vertices of $V_1(\Gamma)$ and seeing that the terms

$$\varepsilon_{x,y} \frac{2a_{x,y}}{\sqrt{c_x c_y}} \left(\frac{c_x'}{c_x} - \frac{c_y'}{c_y}\right)$$

are alternating in x and y for the proper edges $\overline{x,y}$ of Γ_1 (while they are zero for loops), we have that they cancel. We get

$$(3.22)\quad \sum_{x \in V_1(\Gamma)} \frac{1}{c_x} - N_1 = \frac{1}{2} \sum_{\substack{x \in V_1(\Gamma),\, y \in V_{-1}(\Gamma) \\ \langle x,y\rangle}} b_{x,y} - \left(N_1' - \frac{1}{2} L_1\right).$$

To compute the right hand side of (3.22), we observe

LEMMA 3.4. *Let* $x \in V_{-1}(\Gamma)$. *Then*

$$\sum_{\substack{\langle x, y\rangle \\ y \in V_1(\Gamma)}} b_{x,y} = -2. \tag{3.23}$$

Proof. W_y has a Puiseux expansion as in (3.11)

$$W_x = \frac{c_x}{z-\lambda} + (\text{higher degree terms}) \quad \text{for } c_x < 0. \tag{3.24}$$

We compare the term $(z-\lambda)^{-1}$ in both sides of the equations $(\mathscr{E})$ and obtain

$$\begin{aligned} -c_x = {} & \frac{1}{2} \sum_{\substack{\langle x, y\rangle \\ y \in V_1(\Gamma)}} \left(-c_x + \varepsilon_{x,y} c_x \sqrt{1 + 4a_{x,y}^2/(c_x c_y)}\right) \\ & + \frac{1}{2} \sum_{-1 \leq \rho < 1} \sum_{\substack{\langle x, y\rangle \\ y \in V_\rho(\Gamma)}} (-c_x + \varepsilon_{x,y} c_x), \end{aligned} \tag{3.25}$$

i.e.,

$$-2 = \sum_{\substack{\langle x, y\rangle \\ y \in V_1(\Gamma)}} b_{x,y} + \sum_{-1 \leq \rho < 1} \sum_{\substack{\langle x, y\rangle \\ y \in V_\rho(\Gamma)}} (-1 + \varepsilon_{x,y}). \tag{3.26}$$

Hence the sum

$$\sum_{\substack{\langle x, y\rangle \\ y \in V_1(\Gamma)}} b_{x,y}$$

must be an even integer at least equal to -2. Since every $b_{x,y}$ is negative by Lemma 3.2, this sum is just equal to -2. Lemma 3.4 has been proved.

Proof of Theorem 1. (i) follows from Lemma 3.1. (ii) follows from Lemma 3.3. (iii) is an immediate consequence of (3.22) and (3.23). Theorem 1 has thus been proved.

Proof of Theorem 2. We have only to show that there never occurs $N_1 - (N_1' - \frac{1}{2}L_1) - N_{-1} > 0$ for any $\lambda \in \mathbb{R}$. Suppose λ is an eigenvalue of A. Let Γ_1 be defined as at the beginning of the preceding section. For $x \in V_1(\Gamma)$, let $\deg(x)$ be the number of edges of Γ_1 incident with x. Assume that m (≥ 2) edges emanate from each vertex in T. Let

K be the number of edges between Γ_1 and Γ_{-1}. Then $K \le mN_{-1}$. On the other hand, according to Lemma 3.3,

$$K = \sum_{x \in V_1(\Gamma)} (m - \deg(x)) = mN_1 - (2N_1' - L_1). \tag{3.27}$$

Hence, $mN_{-1} \ge mN_1 - 2(N_1' - \frac{1}{2}L_1) \ge mN_1 - m(N_1' - \frac{1}{2}L_1)$, and $N_1 - (N_1' - \frac{1}{2}L_1) - N_{-1} \le 0$. Theorem 2 has thus been proved.

REMARK 3. In order to show that A is point spectrum free, it is not sufficient that Γ is finite. The following example is very illuminating as a counter example.

Let Γ be a complete bipartite graph consisting of $(p+q)$ points $\{1, 2, \cdots, p+q\}$, $p > q \ge 2$, such that each vertex $\{j\}$, $1 \le j \le p$, is adjacent to the points $\{p+k\}$, $1 \le k \le q$. We assume that $a_{j,p+k} = a_{p+k,j} = 1$, and other $a_{x,y}$ all vanish. The group of automorphisms G is isomorphic to the free group of $(p-1)(q-1)$ generators. Then the equations $(\mathscr{E})$ reduce to $W_1 = \cdots = W_p$, $W_{p+1} = \cdots = W_{p+q}$ and

$$z - W_1 = \frac{q}{2}\left(-W_1 + \sqrt{W_1^2 + 4W_1/W_{p+1}}\right), \tag{3.28}$$

$$z - W_{p+1} = \frac{p}{2}\left(-W_{p+1} + \sqrt{W_{p+1}^2 + 4W_{p+1}/W_1}\right), \tag{3.29}$$

i.e.,

$$\frac{z}{W_1} - 1 = \frac{q}{2}\left(-1 + \sqrt{1 + 4/(W_1 W_{p+1})}\right), \tag{3.28$'$}$$

$$\frac{z}{W_{p+1}} - 1 = \frac{p}{2}\left(-1 + \sqrt{1 + 4/(W_1 W_{p+1})}\right). \tag{3.29$'$}$$

Hence

$$\frac{1}{q}G(1, 1|z) - \frac{1}{p}G(p+1, p+1|z) = \left(\frac{1}{q} - \frac{1}{p}\right)\frac{1}{z} \ne 0. \tag{3.30}$$

This shows that $G(1, 1|z)$ or $G(p+1, p+1|z)$ has a pole at $z = 0$. 0 is an eigenvalue. Indeed in Theorem 1 we have $V_1(\Gamma) = \{1, 2, \cdots, p\}$ and $V_{-1}(\Gamma) = \{p+1, p+2, \cdots, p+q\}$ so that $N_1 - N_1' - N_{-1} \risingdotseq p - q > 0$. Hence

$$\sum_{j=1}^{p} \frac{1}{c_j} = \frac{p}{c_1} = p - q, \quad \text{i.e.} \quad 1/c_1 = 1 - \frac{q}{p} > 0.$$

The spectrum of A^2 in this case coincides with a part of the one of an operator obtained from a random walk on a barycentric subdivision of polygonal graphs investigated by G. Kuhn-P. M. Soardi, J. Farault-M. A. Picardello or J. M. Cohen-A. R. Trenholme (see **[Ku]**, **[Fa]** or **[Co]**). By using this result one can also compute the point spectrum of A as above. The author is indebted to the referee for having informed it to us. See also **[B]**.

Question 2. Assume that Γ is fixed. It seems to be an interesting question to ask whether the existence of point spectrum really depends or not on the data $\{a_{x,y}\}_{x,y\in V(\Gamma)}$ under the conditions $(\mathscr{C}1)$ and $(\mathscr{C}2)$.

REMARK 4. A modified version of Theorems 1 and 2 is probably true even if the action of G is not necessarily free, provided the quotient $G\backslash T$ is finite. But the author does not know any answer.

REMARK 5. Theorem 1 (iii) remains valid when T is a finite connected tree and A is a linear operator on $l^2(T)$ defined as in (1.1). In this case $W_x(z)$, $x \in T$, are rational in z. Hence there occur only $V_1(T)$ and $V_{-1}(T)$. Theorem 1 can then be modified as follows:

THEOREM 1a. *For an eigenvalue λ of A, we denote by N_1 and N_{-1} the numbers of $V_1(T)$ and $V_{-1}(T)$ respectively, and by N_1' the number of edges in T connecting vertices in $V_1(T)$. Then*

$$\sum_{x\in V(T)} \operatorname{Res}_{z=\lambda} G(x, x|z) = N_1 - N_1' - N_{-1}. \tag{3.31}$$

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