PACIFIC JOURNAL OF MATHEMATICS
Vol. 147, No. 2, 1991

POLYNOMIAL HULLS OF GRAPHS

H. ALEXANDER

We shall consider the polynomially convex hull of the graph of a continuous complex-valued function on the boundary of the unit ball. We show first that the hull covers the closed unit ball and then consider several of its properties. In particular, when is the hull also a graph; i.e. single sheeted? When the hull is a graph we show, in some cases, that it contains analytic structure. We also consider the graph in $\mathbf{C}^2$ of a real-valued continuous function on the boundary of a 3-cell which is contained in a real hyperplane in $\mathbf{C}^2$ and partially extend some results of Bedford and Klingenberg who studied the case of smooth functions.

Introduction. We employ standard notation of uniform algebras. For a compact set X in $\mathbf{C}^n$, $\widehat{X}$ will denote the polynomially convex hull, $C(X)$ the algebra of all continuous complex-valued functions on X, and $P(X)$ the closure in $C(X)$ of the polynomials in the uniform norm. B_n will be the open unit ball in $\mathbf{C}^n$ and bB_n its boundary.

THEOREM 1. *Let $f \in C(bB_2)$. Let $G(f)$ be the graph in $\mathbf{C}^3$ of f. Then $\widehat{G(f)}$ covers B_2; i.e., the projection to $\mathbf{C}^2$ of the set $\widehat{G(f)}$ is $\overline{B}_2$.*

As a consequence, $\widehat{G(f)}$ has real dimension at least 4 and the closure of $\widehat{G(f)}\backslash G(f)$ contains $G(f)$; cf. **[1]**. As a statement about Banach algebras the theorem says the following: let $\mathscr{B}$ be the closed subalgebra of $C(bB_2)$ generated by f and the polynomials, then $P(bB_2) \subsetneq \mathscr{B}$ and every homomorphism of $P(bB_2)$ extends ("lifts") to $\mathscr{B}$. The corresponding result for the graph of a function over the unit circle in $\mathbf{C}^1$ is false. In fact, by the Wermer maximality theorem **[16]**, if f is a continuous function on the circle then either f extends to be analytic on the open disk, in which case the hull of its graph is the graph of its extension to the closed disk or f does not extend, in which case the graph of f is polynomially convex. Similarly the following result does not hold in one variable.

COROLLARY. *Suppose that f_n converges to f uniformly on bB_2 and that $\widehat{G(f)}$ is a graph over $\overline{B}_2$; i.e., $\widehat{G(f)}$ is one-sheeted. Then $\widehat{G(f_n)}$ converges to $\widehat{G(f)}$ in the Hausdorff metric.*

This raises the question of when $\widehat{G(f)}$ is a graph; if it is the graph of a function (i.e., $\widehat{G(f)}$ is one-sheeted over B_2) then the function must be continuous on $\overline{B}_2$ as its graph is compact.

THEOREM 2. *Let $f \in C(bB_2)$ and suppose that there exists $F \in C(\overline{B}_2)$ such that f is the restriction of F to bB_2 and that either*
(a) *the real and imaginary parts of F are pluriharmonic on B_2 or*
(b) *$F = |g|$ where $g \in P(\overline{B}_2)$ and g is nowhere zero on $\overline{B}_2$.*
Then $\widehat{G(f)} = G(F)$.

Part (b) is false if g is allowed to have zeros; e.g., if $g(\lambda) = \lambda_1$, the hull of $G(f)$ is not a graph. Part (a) applies in particular if f extends to be holomorphic in B_2 or is the complex conjugate of such a function. A local version of this is valid. According to a result of Stout **[14]** and Lupacciolu **[9]** a continuous function on bB_2 which is weakly CR on $bB_2 \backslash K$, for K compact in bB_2, extends to a holomorphic function on $B_2 \backslash \widehat{K}$.

THEOREM 3. *Suppose that $f \in C(bB_2)$ and that f is a (weakly) CR function on an open subset W of bB_2. Let $K = bB_2 \backslash W$. Then*

$$\widehat{G(f)} \cap ((B_2 \backslash \widehat{K}) \times \mathbf{C}) = G(\tilde{f})$$

where $\tilde{f}$ is the holomorphic extension of f to $B_2 \backslash \widehat{K}$. In particular, $\widehat{G(f)}$ is one-sheeted over $B_2 \backslash \widehat{K}$.

Graphs which are hulls have been considered in a similar setting by Bedford and Klingenberg **[4]**. Let S^2 be the unit sphere in $\mathbf{R}^3 = \{(z, t) : z \in \mathbf{C}, t \in \mathbf{R}\}$ and let f be a smooth real function on S^2. Let $G(f) = \{(z, t, f(z, t)) : (z, t) \in S^2\}$, the graph of f in $\mathbf{C}^2 = \mathbf{R}^3 \times \mathbf{R}$. Bedford and Klingenberg showed that the polynomial hull of $G(f)$ in $\mathbf{C}^2$ is the graph of a Lipschitz function F on E_3, the closed unit ball in $\mathbf{R}^3$. Moreover, $G(F)$ is a disjoint union of analytic disks. We can show that part of this holds for merely continuous functions.

THEOREM 4. *Let $f \in C(S^2)$ be real-valued, then there exists a real-valued $F \in C(E_3)$ such that $\widehat{G(f)} = G(F)$.*

We do not know if $G(F)$ contains analytic disks in the continuous case, as it must in the smooth case. In fact, in view of the examples of Stolzenberg **[13]** and Wermer **[17]** on hulls without analytic structure, it would be interesting to find $f \in C(S^2)$ such that $\widehat{G(f)}$ contains no analytic structure.

Returning to graphs in $\mathbf{C}^3$, we can exhibit some cases of analytic structure in hulls. In the statement of Theorem 6, π denotes the projection of $\mathbf{C}^3$ $(= \mathbf{C}^2 \times \mathbf{C})$ to $\mathbf{C}^2$.

THEOREM 5. *Let $F \in C(B_2)$ be as in Theorem 2(a). Then through every point of $G(F)$ there is a one-dimensional subvariety of $B_2 \times \mathbf{C}$ which contains the point and which is contained in $G(F)$.*

The same conclusion is obvious for F of Theorem 2(b); the variety is just $V \times \{c\}$ where V is a level set $\{g = a\}$ and $c = |a|$.

THEOREM 6. *Let F be a real continuous function on $\overline{B}_2$ which is smooth on B_2 and let f be the restriction of F to bB_2. Suppose that $G(F)$ is polynomially convex; i.e., that $G(F) = \widehat{G(f)}$. Let $\lambda_0 \in B_2$ and suppose that $F(\lambda_0)$ is a regular value of F. Then there exists a connected Riemann surface R_0 and an injective holomorphic imbedding into $\mathbf{C}^3$ such that the image R of R_0 contains $(\lambda_0, F(\lambda_0))$ and is contained in $G(F)$. Moreover the boundary of R with respect to the polynomials in $\mathbf{C}^3$ lies over bB_2 in the sense that R is contained in the polynomial hull of $\overline{R} \cap \pi^{-1}(bB_2)$.*

After giving the proofs, we shall discuss some examples and open questions. In particular, we shall see that fibers over $\lambda \in B_2$: $\widehat{G(f)}_\lambda :\equiv \{w : (\lambda, w) \in \widehat{G(f)}\}$, are not in general convex (as contrasted with the case of hulls of sets in $\mathbf{C}^2$ which lie over the unit circle in $\mathbf{C}$, cf. **[2]**, **[11]**) and shall give some examples relating to the Corollary to Theorem 1 and to Theorem 2.

Rather than prove Theorem 1, we shall prove the following direct generalization to higher dimensions.

THEOREM 7. *Let f be a continuous map of bB_n to $\mathbf{C}^k$ for $1 \le k \le n-1$. Let $G(f)$ be the graph of f in $\mathbf{C}^{n+k}$. Then $\widehat{G(f)}$ covers B_n; i.e., the projection $G(f)$ to $\mathbf{C}^n$ equals $\overline{B}_n$.*

Note that Theorem 7 is not valid for $k = n$; e.g., if $f(z) = \overline{z}$, then $G(f)$ is polynomially convex in $\mathbf{C}^{2n}$.

1.

1.1. As noted, Theorem 1 is a special case of Theorem 7 which we now prove. Let $X = G(f) \subseteq \mathbf{C}^{n+k}$ and let $Q = \pi(\widehat{X})$ where

$\pi\colon \mathbf{C}^{n+k} \to \mathbf{C}^n$ is the projection $\pi(\lambda, w) = \lambda$. We must show that $\pi(Q) = \overline{B}_n$. We argue by contradiction and suppose that Q is a proper subset of $\overline{B}_n$. From the map $\pi\colon (\widehat{X}, X) \to (Q, bB_n)$ we get the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
\check{H}^{2n-1}(\widehat{X}, \mathbf{C}) & \to & \check{H}^{2n-1}(X, \mathbf{C}) & \to & \check{H}^{2n}(\widehat{X}, X, \mathbf{C}) & \to & \check{H}^{2n}(\widehat{X}, \mathbf{C}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\check{H}^{2n-1}(Q, \mathbf{C}) & \to & \check{H}^{2n-1}(bB_n, \mathbf{C}) & \to & \check{H}^{2n}(Q, bB_n, \mathbf{C}) & \to & \check{H}^{2n}(Q, \mathbf{C}).
\end{array}$$

Since $2n-1 \geq n+k$, $\check{H}^{2n-1}(\widehat{X}, \mathbf{C}) = 0$ by a result of A. Browder [6]. Since Q is a compact subset of $\mathbf{C}^n$, it follows that $\check{H}^{2n}(Q, \mathbf{C}) = 0$. We thus have

$$\begin{array}{ccccccc}
0 & \to & \check{H}^{2n-1}(X, \mathbf{C}) & & & & \\
\uparrow & & \cong \uparrow \beta & & & & \\
\check{H}^{2n-1}(Q, \mathbf{C}) & \underset{\alpha}{\to} & \check{H}^{2n-1}(bB_n, \mathbf{C}) & \underset{\gamma}{\to} & \check{H}^{2n}(Q, bB_n, \mathbf{C}) & \to & 0.
\end{array}$$

Since π maps X homeomorphically to bB_n, β in the diagram is an isomorphism. Since $\beta \circ \alpha$ is the zero map by commutativity, it follows that α is the zero map. Hence γ is an isomorphism. Thus

$$\check{H}^{2n}(Q, bB_n, \mathbf{C}) \cong \check{H}^{2n-1}(bB_n, \mathbf{C}) = \mathbf{C}. \tag{1}$$

Let Z be the cone on bB_n. By excision

$$\check{H}^{2n}(Q, bB_n, \mathbf{C}) \cong \check{H}^{2n}(Z \cup Q, Z, \mathbf{C}). \tag{2}$$

We have the exact sequence

$$\check{H}^{2n-1}(Z, \mathbf{C}) \to \check{H}^{2n}(Z \cup Q, Z, \mathbf{C}) \to \check{H}^{2n}(Z \cup Q, \mathbf{C}). \tag{3}$$

Since Q is assumed to be a proper subset of $\overline{B}_n$, $Z \cup Q$ is a proper subset of $Z \cup B_n \approx S^{2n}$ and therefore $\check{H}^{2n}(Z \cup Q, \mathbf{C}) = 0$. Also Z is a $2n$-cell and so $\check{H}^{2n-1}(Z, \mathbf{C}) = 0$. It follows from (3) that $\check{H}^{2n}(Z \cup Q, Z, \mathbf{C}) = 0$. This contradicts (1) and (2), and proves the theorem.

1.2. An alternate proof of Theorem 7 which is due to J. P. Rosay is the following. Suppose, by way of contradiction, that $\widehat{G(f)}$ does not cover the ball. Then, without loss of generality, we may assume that the origin is not covered. Hence we can choose a Runge domain D in $\mathbf{C}^n \times \mathbf{C}^k$ such that

$$\widehat{G(f)} \subseteq D \subseteq (\mathbf{C}^n \backslash \{0\}) \times \mathbf{C}^k.$$

Approximate f on bB_n uniformly by a smooth map g such that $G(g) \subseteq D$.

Consider the Bochner-Martinelli form

$$\omega = \sum_{j=1}^{n}(-1)^{j+1}\frac{\overline{z}_j}{|z|^{2n}}d\overline{z}_1\wedge\cdots\wedge\widehat{d\overline{z}_j}\wedge\cdots\wedge d\overline{z}_n\wedge dz_1\wedge\cdots\wedge dz_n.$$

Then ω is a closed form of degree $2n-1$ on $\mathbf{C}^n\backslash\{0\}$. Let σ be the pull back of ω to $\mathbf{C}^n\backslash\{0\}\times\mathbf{C}^k$ by the projection of $\mathbf{C}^n\times\mathbf{C}^k$ to $\mathbf{C}^n$. Then σ is a closed $2n-1$ form on D. By a result of Serre (see [**8**, Thm. 2.7.11]), $H^{2n-1}(D, \mathbf{C}) = 0$, since $2n-1\geq n+k$. Therefore σ is exact in D and so, by Stokes,

$$\int_{G(g)}\sigma = 0.$$

But

$$\int_{G(g)}\sigma = \int_{bB_n}\omega.$$

And this last integral by a simple computation equals const $\times$ (vol. of unit ball in $\mathbf{C}^n$) and in particular is *not* zero. This is a contradiction.

2. Proof of the Corollary to Theorem 1. For $X\subseteq\mathbf{C}^3$ and $\varepsilon>0$ we define

$$X_\varepsilon = \{(\lambda, w)\in\mathbf{C}^3=\mathbf{C}^2\times\mathbf{C}\colon \exists(\lambda, w')\in X \text{ with } |w-w'|\leq\varepsilon\}.$$

We need to show that for $\varepsilon>0$, there exists k_0 such that (a) $\widehat{G(f_k)}\subseteq G(F)_\varepsilon$ for $k\geq k_0$ and (b) $G(F)\subseteq\widehat{G(f_k)}_\varepsilon$ for $k\geq k_0$. For (a), since $G(F)$ is polynomially convex, there is a compact polynomially convex set $N\subseteq G(F)_\varepsilon$ such that N contains a neighborhood of $G(f)$ in $bB_2\times\mathbf{C}$. Hence there exists a k_0 such that $G(f_k)\subseteq N$ for $k\geq k_0$. Then $\widehat{G(f_k)}\subseteq\widehat{N}=N\subseteq G(F)_\varepsilon$. This gives (a).

Let $\lambda_0\in B_n$. By Theorem 1 there exists $w_k\in\mathbf{C}$ such that $(\lambda_0, w_k)\in\widehat{G(f_k)}$. By (a), $|w_k-F(\lambda_0)|\leq\varepsilon$ if $k\geq k_0$. Thus $(\lambda_0, F(\lambda_0))\in\widehat{G(f_k)}_\varepsilon$ for $k\geq k_0$. Hence $G(F)\subseteq\widehat{G(f_k)}_\varepsilon$ for $k\geq k_0$.

3.

3.1. Let Ω be a Runge domain in $\mathbf{C}^n$, K compact in Ω and v plurisubharmonic on Ω. Then by [**8**, Theorem 4.3.4],

$$\sup\{v(z)\colon z\in\widehat{K}\} = \sup\{v(z)\colon z\in K\}.$$

We need a version of this fact.

LEMMA. *Let v be continuous on $\overline{B}_2 \times \mathbf{C}$ and plurisubharmonic on $B_2 \times \mathbf{C}$. Let K be compact in $\overline{B}_2 \times \mathbf{C}$. Then* $\sup\{v(z) \colon z \in \widehat{K}\} = \sup\{v(z) \colon z \in K\}$.

Proof. Let $r < 1$. Set $v_r(\lambda, w) = v(r\lambda, w)$. Then v_r is push on the Runge domain $\{(\lambda, w) \in \mathbf{C}^2 \times \mathbf{C} \colon \|\lambda\| < \frac{1}{r}\}$. By the above, $\sup\{v_r(z) \colon z \in \widehat{K}\} = \sup\{v_r(z) \colon z \in K\}$. Letting $r \to 1$ gives the lemma.

3.2. *Proof of Theorem* 2(a). Write $F = \varphi + i\psi$ where φ and ψ are continuous on $\overline{B}_2$ and pluriharmonic on B_2. Set $u(\lambda, w) = \varphi(\lambda) - \operatorname{Re} w$. Then $u = 0$ on $G(f)$. The lemma implies $u \leq 0$ on $G(f)^\frown$. Repeating the argument with $-u$ gives $u \equiv 0$ on $\widehat{G(f)}$; i.e., $\operatorname{Re} w = \varphi(\lambda)$ on $\widehat{G(f)}$. In the same way we see that $\operatorname{Im} w = \psi(\lambda)$ on $\widehat{G(f)}$. Hence $\widehat{G(f)} = G(F)$.

3.3. *Proof of Theorem* 2(b). For $(\lambda, w) \in G(f)$, $w = |g(\lambda)| > 0$. Since intervals of the real axis are polynomially convex we conclude that the coordinate function w is invertible in $P(\widehat{G(f)})$. Also g is invertible in $P(\overline{B}_2)$. Hence g/w and w/g are in $P(\widehat{G(f)})$. As $|g/w| = 1$ and $|w/g| = 1$ on $G(f)$, we have $|g/w| \leq 1$ and $|w/g| \leq 1$ on $\widehat{G(f)}$. Hence $w = \operatorname{Re} w = |w| = |g(\lambda)|$ on $\widehat{G(f)}$, i.e., $\widehat{G(f)} = G(F)$.

4. Proof of Theorem 3. Let $\lambda_0 \in B_2 \backslash \widehat{K}$. Let $\pi \colon \mathbf{C}^2 \times \mathbf{C} \to \mathbf{C}^2$ be the projection $\pi(\lambda, w) = \lambda$. We must show that $\pi^{-1}(\lambda_0) \cap \widehat{G(f)}$ consists of the one point $(\lambda_0, \tilde{f}(\lambda_0))$. We argue by contradiction and suppose that there exists $(\lambda_0, w_0) \in \widehat{G(f)}$ with $w_0 \neq \tilde{f}(\lambda_0)$.

As $\lambda_0 \notin \widehat{K}$ there exists a polynomial $P(\lambda)$ such that $|P(\lambda_0)| > 1 > \|P\|_{\widehat{K}} \equiv \sup\{|P(\lambda)| \colon \lambda \in \widehat{K}\}$. Set $E = \{\lambda \in \overline{B}_2 \colon |P(\lambda)| \geq 1\}$. Then $\tilde{f}$ is continuous on E and holomorphic on $E \cap B_2$.

Let $X = \widehat{G(f)} \cap \pi^{-1}(E)$ and let $X_1 = \widehat{G(f)} \cap \pi^{-1}(bE)$. Let $\mathscr{A}$ be the subalgebra of $C(X)$ consisting of all functions h such that h is locally $P(X)$-holomorphic at each point of π^{-1} (interior (E)). By the local maximum modulus principle the Shilov boundary of $\mathscr{A}$ is contained in X_1 (cf. [7, Lemma 9.1]). The function $w - \tilde{f}(\lambda)$ is contained in $\mathscr{A}$.

Let μ be a Jensen measure for (λ_0, w_0) for the algebra $\mathscr{A}$ such that μ has support in the Shilov boundary X_1 (**[5]**, **[15]**). Then $-\infty \neq \log|w_0 - \tilde{f}(\lambda_0)| \leq \int_{X_1} \log|w - \tilde{f}(\lambda)|\, d\mu$. Since $w - \tilde{f}(\lambda) = 0$

on $X_1 \cap \pi^{-1}(W)$ and so $\log|w - (\tilde{\lambda})| \equiv -\infty$ there, it follows that $\mu(\pi^{-1}(W)) = 0$. Hence μ is concentrated on $\pi^{-1}(bE \cap B_2)$. Let $d\nu = \pi_*(d\mu)$. Then $d\nu$ is concentrated on $bE \cap B_2$ and represents evaluation at λ_0 for polynomials in $\mathbf{C}^2$. Since $|P(\lambda)| = 1$ for $\lambda \in bE \cap B_2$, we get $1 < |P(\lambda_0)| = |\int P\,d\nu| \leq \int |P|\,d\nu = 1$. This gives the desired contradiction.

5.

5.1. For $A \subseteq \mathbf{C}^n$ and $p \in \mathbf{C}^n$, $A+p$ denotes, as usual, the translate of A. The following is well-known and easy to prove using Rouché's theorem.

LEMMA. *Let V_1 and V_2 be analytic curves in $\mathbf{C}^2$ such that z_0 is an isolated point of $V_1 \cap V_2$. Then $V_1 \cap (V_2 + p)$ is non-empty for all p sufficiently small in $\mathbf{C}^2$.*

5.2. Let f_1 and f_2 be smooth real-valued functions on S^2 and let F_1 and F_2 be the corresponding Lipschitz functions on E_3 given by the theorem of Bedford and Klingenberg [**4**].

LEMMA. *If $f_1 < f_2$ on S^2 then $F_1 < F_2$ on E_3.*

Proof. Suppose that $G(F_2)$ has a non-empty intersection with $G(F_1)$. Choose $t \geq 0$ maximal such that $G(F_2)+(0, it)$ meets $G(F_1)$. The intersection contains no points over S^2 since $f_1 < f_2$.

Let P be a point in the intersection. By [**4**] there exists an analytic disk V_1 in $G(F_1)$ containing P and likewise an analytic disk V_2 in $G(F_2) + (0, it)$ containing P. Clearly P is isolated in $V_1 \cap V_2$. By choice of t, $V_1 \cap (V_2 + (0, it'))$ is empty for $t' > 0$. This contradicts the Lemma of §5.1.

Thus $G(F_1)$ and $G(F_2)$ are disjoint. Since $F_1(\lambda) < F_2(\lambda)$ for $\lambda \in E_3$ near S^2 we conclude that $F_1(\lambda) < F_2(\lambda)$ for all $\lambda \in E_3$.

5.3. As before let f_1 and f_2 be real continuous smooth functions on S^1 with corresponding F_1 and F_2. Clearly, if $c \in \mathbf{R}$, $F_1 + c$ corresonds to $f_1 + c$.

COROLLARY. $\|F_1 - F_2\|_{E_3} = \|f_1 - f_2\|_{S^2}$.

Proof. The norms are sup norms. Let $q > \|f_1 - f_2\|_{S^2}$. Then $f_2 < f_1 + q$ on S^2. Hence $F_2 < F_1 + q$ on E_3. By symmetry, $\|F_2 - F_1\|_{E_3} < q$. Hence $\|F_2 - F_1\|_{E_3} \leq \|f_1 - f_2\|_{S^2}$.

5.4. *Proof of Theorem* 4. Given f a continuous real-valued function on S^2 choose smooth $\{f_n\}$ converging uniformly to f. By the previous corollary the F_n corresponding to f_n converge uniformly to some function F on E_3. It remains to show that $\widehat{G(f)} = G(F)$.

The following general fact about hulls is easy to check: if $X_n \to X$ and $\widehat{X}_n \to Y$ in the Hausdorff metric, then $Y \subseteq \widehat{X}$. Since $f_n \to f$ and $F_n \to F$ uniformly we get $G(f_n) \to G(f)$ and $\widehat{G(f_n)} = G(F_n) \to G(F)$. Hence $G(F) \subseteq \widehat{G(f)}$.

To prove the opposite inclusion we argue by contradiction and suppose that there exists a point $(p', t) \in \widehat{G(f)} \backslash G(F)$ where $p' \in E_3$, $t \in \mathbf{R}$.

First we claim that $p' \notin S^2$. In fact, if $p' \in S^2$ there is an entire function in $\mathbf{C}^2$ which peaks on $E_3 \times \mathbf{R}$ on the set $\{p'\} \times \mathbf{R}$; it arises from the tangent plane to S^2 at p'. Let $\pi\colon \mathbf{C}^2 \to \mathbf{R}^3$ be the projection $\pi(q', t) = q'$. Then $\widehat{G(f)} \cap \pi^{-1}(p')$ is a peak set and it follows that $\widehat{G(f)} \cap \pi^{-1}(p') = (G(f) \cap \pi^{-1}(p'))^{\widehat{}} =$ the singleton $\{(p', f(p'))\}$. As this point is in $G(F)$, the claim is valid.

As $t \neq F(p')$ we can assume that $t > F(p')$. Choose n such that $\|f_n - f\|_{S^2} < t - F_n(p')$ and $F_n(p') < t$. Let V be the analytic disk in $G(F_n)$ through $(p', F_n(p'))$ given by **[4]** since $p' \notin S^2$. V is a one-dimensional subvariety of $\Omega = E_3^0 \times \mathbf{R}$. Since Ω is a 4-cell topologically, the solution of the Cousin II problem **[8]** gives a holomorphic function $H(z, w)$ in Ω whose zero set is exactly V. For s real, set $H_s(z, w) = H(a, w - is)$. Then H_s is holomorphic on Ω and its zero set is $V + (0, is)$.

Choose t_0 maximal such that $V + (0, it_0)$ has a non-empty intersection with $\widehat{G(f)}$. We have $t_0 \geq t - F_n(p') > 0$. Say $(q', v) \in \widehat{G(f)} \cap (V + (0, it_0))$ for $q' \in \mathbf{R}^3$, $v \in \mathbf{R}$. We have seen that $\|q'\| < 1$.

Let $0 < r < 1$. Set $S_r = \{(z, u) \in \mathbf{R}^3\colon |z|^2 + u^2 = r^2\}$ and set $X_r = \widehat{G(f)} \cap \pi^{-1}(S_r)$. As $\widehat{G(f)} \cap \pi^{-1}(S_1) = G(f)$, it follows that $X_r \to G(f)$ as $r \to 1$. Also

$$(V + (0, it_0)) \cap S_r \to G(f_n + t_0) \cap (\overline{V} + (0, it_0)) \quad \text{as } r \to 1.$$

Since $t_0 \geq t - F_n(p') > \|f - f_n\|_{S^2}$ we have $f \neq f_n + t_0$ on S^2. Thus for $r < 1$ sufficiently close to 1, X_r is disjoint from $V + (0, it_0)$. Fix such an r such that $\|q'\| < r$. Then $H_{t_0} \neq 0$ on X_r and so $|H_{t_0}| > \delta$, for some $\delta > 0$, on X_r. Then $|H_s| > \delta$ on X_r for $t_0 \leq s \leq t_1$ if $t_1 > t_0$ is sufficiently close to t_0. Consider $\{H_s\colon t_0 \leq s \leq t_1\}$. Let $V_s = \{(z, w) \in \Omega\colon H_s(z, w) = 0\} = V + (0, is)$. Then V_s is disjoint from X_r for $t_0 \leq s \leq t_1$; V_{t_0} contains (q', v); V_{t_1} is disjoint from $\widehat{G(f)}$ and therefore disjoint from $\widehat{X}_r \subseteq \widehat{G(f)}$. It follows from Oka's characterization of hulls ([**10**], [**12**]) that $(q', v) \notin \widehat{X}_r$. But, as $\|q'\| < r$, the local maximum modulus principle implies that $(q', v) \in \widehat{X}_r$. Contradiction.

6. Proof of Theorem 5. Since φ, ψ are pluriharmonic on B_2 we have holomorphic functions Φ and Ψ on B_2 such that $\varphi = \operatorname{Re}\Phi$ and $\psi = \operatorname{Im}\Psi$. Write $\Phi = \varphi + i\tilde{\varphi}$ where $\tilde{\varphi} = \operatorname{Im}\Phi$. Set $\Lambda = \Phi - \Psi$ on B_2. Fix $\lambda_0 \in B_2$, let $\Lambda(\lambda_0) = c = a + ib$ with a, b real. Define $V = \{\lambda \in B_2 \colon \Lambda(\lambda) = c\}$.

We claim that $F = \Phi - ib$ on V. In fact, on V, $\varphi = \operatorname{Re}\Psi + a$ and $\tilde{\varphi} = \psi + b$. Hence $F = \varphi + i\psi = \varphi + i\tilde{\varphi} - ib = \Phi - ib$ on V. Either V is one-dimensional subvariety of V_2 or V equals B_2 and $F \equiv \Phi - ib$. In the latter case F is holomorphic on B_2 and the theorem follows. In the former case the image of the map $V \to G(F)$ given by $\lambda \mapsto (\lambda, \Phi(\lambda) - ib)$ is the desired subvariety through $(\lambda_0, F(\lambda_0))$.

7. Proof of Theorem 6. We assume that $c = F(\lambda_0)$ is a regular value of F. Let $\Sigma = \{\lambda \in B_2 \colon F(\lambda) = c\}$. Then Σ is a smooth 3-manifold. Let T be the tangent plane to Σ at λ_0 and let π be orthogonal projection of $\mathbf{C}^2$ to T; we identify T with $\mathbf{R}^3$. In a neighborhood of λ_0 Σ is a graph over T. So for small δ, Σ is locally a graph over $E = \{\lambda \in T \colon \|\lambda - \lambda_0\| \leq \delta\}$. Set $S = \{x \in T \colon \|\lambda - \lambda_0\| - \delta\}$. Let $Q = \Sigma \cap \pi^{-1}(S)$, where π is restricted to a neighborhood of λ_0. Then Q is a graph over S of a smooth function and by the Bedford-Klingenberg theorem [**4**], $\widehat{Q}$ is a union of analytic disks. Let $\mathscr{V} = \{V\}$ be this set of analytic disks.

Fix $V \in \mathscr{V}$. Consider the map $V \to \mathbf{C}^3$, $\lambda \mapsto (\lambda, c)$. The image is a disk W such that $bW \subseteq G(F)$ since $bV \subseteq Q \subseteq \Sigma$ and so $F = c$ on bV. Since $G(F)$ is polynomially convex, $W \subseteq G(F)$ by the maximum principle; i.e., $(\lambda, c) \in W$ implies $F(\lambda) = c$. Hence $\lambda \in V$ implies $(\lambda, c) \in W$ implies $F(\lambda) = c$ implies $\lambda \in \Sigma$. Thus $V \subseteq \Sigma$ for $V \in \mathscr{V}$.

Thus $\Sigma \cap \pi^{-1}(E) \supseteq (\Sigma \cap \pi^{-1}(S))^\frown$. Since both sets are graphs over E (locally) we conclude that they are equal. Hence there exists $V_0 \in \mathscr{V}$ such that $\lambda_0 \in V_0$.

Let $\mathscr{S}$ be the set of all analytic disks which are contained in Σ. Define an equivalence relation on $\mathscr{S}$ as follows: if V, $V' \in \mathscr{S}$, say $V \sim V'$ if there exists a chain $V = V_1, V_2, \ldots, V' = V_n$ in $\mathscr{S}$ such that $V_i \cap V_{i+1} \neq \varnothing$ for $i = 1, 2, \ldots, n-1$. Let C_0 be the equivalence class of the disk V_0 containing λ_0.

Define a Riemann surface R_0 as follows. As a set

$$R_0 = \bigcup\{V : V \in C_0\} \subseteq \Sigma.$$

Topologize R_0 as follows: by the theorem of Bedford and Klingenberg **[4]** if $V_1, V_2 \in C_0$ and if $\lambda \in V_1 \cap V_2$ then there exists $V_3 \in C_0$ such that $\lambda \in V_3 \subseteq V_1 \cap V_2$. This means that C_0 forms a basis for a topology on R_0. For coordinate charts use the inverses of the imbeddings $f: U \to V$ for $V \in C_0$ where U is the unit disk. Then R_0 is a (connected) Riemann surface. Moreover the image of R of R_0 under the map $\lambda \mapsto (\lambda, c)$ contains $(\lambda_0, F(\lambda_0))$ and is contained in $G(F)$.

It remains to show that $R \subseteq (\overline{R} \cap \pi^{-1}(bB_2))\hat{}$. Since $w \equiv c$ on R, it suffices to show that $R_0 \subseteq (\overline{R}_0 \cap bB_2)\hat{}$ where we identify the Riemann surface R_0 with its image $\mathrm{id}(R_0) \subseteq B_2$ and $\overline{R}_0$ is the closure of R_0 as a subset of $\mathbf{C}^2$.

Note that each equivalence class $\mathbf{C}$ of $\mathscr{S}$ gives rise to a connected Riemann surface S and that Σ is a disjoint union of these Riemann surfaces (we identify S with $\mathrm{id}(S) \subseteq \Sigma$).

Let g be a polynomial in $\mathbf{C}^2$ and let $M = \sup\{|g(\lambda)| : \lambda \in R_0\}$. Then there exist $\{\lambda_n\} \subseteq R_0$ such that $\lambda_n \to p \in \mathbf{C}^2$ and $|g(\lambda_n)| \to M = |g(p)|$. If $\|p\| = 1$ then $p \in \overline{R}_0 \cap bB_2$ and $M = \sup\{|g(\lambda)| : \lambda \in \overline{R}_0 \cap bB_2\}$, as desired.

Suppose that $\|p\| < 1$. Then $p \in \Sigma$. Let S_0 be the Riemann surface in Σ which contains p and is associated to some equivalence class of $\mathscr{S}$. We claim that $S_0 \subseteq \overline{R}_0$. In fact $p \in S_0 \cap \overline{R}_0$ and $S_0 \cap \overline{R}_0$ is a closed non-empty subset of the connected set S_0. It is also an open subset of S_0. To see this we repeat the local construction at λ_0 above, now at the point p. As before we get a neighborhood of p in Σ of the form $\Sigma \cap \pi^{-1}(E)$ which is a union of analytic disks $\{V\} = \mathscr{V}$. By taking δ small we can assume that $\Sigma \cap \pi^{-1}(S)$ is totally real. There exist $V_n \in \mathscr{V}$ such that $\lambda_n \in V_n$ for $n = 1, 2, \ldots$ and $V \in \mathscr{V}$ with $p \in V$. Then $V_n \subseteq R_0$ for all n and $V \subseteq S_0$. Since $\lambda_n \to p$ it follows from the construction of **[4]** (cf. **[3]**) that $V_n \to V$. Hence $V \subseteq \overline{R}_0$, i.e., $S_0 \cap \overline{R}_0$ is open in S_0. As S_0 is connected, $S_0 \cap \overline{R}_0 = S_0$ and $S_0 \subseteq \overline{R}_0$.

Moreover, $|g| \le M$ on R_0 implies $|g| \le M$ on S_0. As $|g(p)| = M$ we conclude that $g \equiv g(p)$ on S_0. It follows that S_0 is an analytic component of $\{\lambda \in B_2 : g(\lambda) = g(p)\}$. Therefore $\overline{S}_0 \backslash S_0 \subseteq \overline{S}_0 \cap bB_2$ and $\overline{S}_0 \cap bB_2$ is non-empty. Hence, as $\overline{S}_0 \cap bB_2 \subseteq \overline{R}_0 \cap bB_2$, we get $\sup\{|g(\lambda)| : \lambda \in \overline{R}_0 \cap bB_2\} \ge |g(p)| = M$. This proves the theorem.

8.

8.1. *Some examples.* Let $\rho\colon bB_2 \to \mathbf{P}^1$ be the map $\rho(\lambda_1, \lambda_2) = \lambda_1/\lambda_2$ where we identify $\mathbf{P}^1$ with $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Let $h\colon \mathbf{P}^1 \to \mathbf{C}$ be continuous and set $X = h(\mathbf{P}^1)$. Define $f \in C(bB_2)$ by $f = h \circ \rho$. Recall that $\widehat{G(f)}_\lambda :\equiv \{w \in \mathbf{C}\colon (\lambda, w) \in \widehat{G(f)}\}$ for $\lambda \in \mathbf{C}^2$.

PROPOSITION. $\widehat{G(f)}_0 = \widehat{X}$.

Proof. Let $\alpha \in bB_2$. Set $l_\alpha = \{(\zeta\alpha, f(\alpha))\colon |\zeta| \le 1\}$. Then $bl_\alpha \subseteq G(f)$ since $f(\zeta\alpha) = f(\alpha)$ if $|\zeta| = 1$. By the maximum principle, $l_\alpha \subseteq \widehat{G(f)}$. Hence $f(\alpha) \in (l_\alpha)_0 \subseteq \widehat{G(f)}_0$; i.e., $X \subseteq \widehat{G(f)}_0$. Hence $\widehat{X} \subseteq \widehat{G(f)}_0$.

In the other direction, $G(f) \subseteq bB_2 \times X$ implies $\widehat{G(f)} \subseteq \widehat{bB_2} \times \widehat{X} = \overline{B}_2 \times \widehat{X}$. Hence $\widehat{G(f)}_0 \subseteq \widehat{X}$.

8.2. Consider the special case $h_0(\zeta) = |\zeta|/(1+|\zeta|^2)^{1/2}$ for $\zeta \in \overline{\mathbf{C}}$. Then $f(\lambda_1, \lambda_2) = |\lambda_1|$ for $\lambda \in bB_2$ and $\widehat{G(f)}_0 = [0, 1] \subseteq \mathbf{R}$ since $h_0(\mathbf{P}^1) = [0, 1]$. In particular $\widehat{G(f)}$ is not a graph. This shows that the condition $g \ne 0$ in Theorem 2(b) cannot be dropped.

8.3. Let h_0 be as in the last paragraph. Set $h = e^{2\pi i h_0}$ and $h_n = e^{2\pi i(1-1/n)h_0}$. Then $h(\mathbf{P}^1) = \{\zeta\colon |\zeta| = 1\} \equiv \gamma$ and $h_n(\mathbf{P}^1) = \{\zeta\colon \zeta = e^{i\theta}, 0 \le \theta \le 2\pi(1-\frac{1}{n})\} \equiv \gamma_n$. Set $f = h \circ \rho$ and $f_n = h_n \circ \rho$.

Then $h_n \to h$ uniformly on $\mathbf{P}^1$ and $f_n \to f$ uniformly on bB_2. Thus $G(f_n) \to G(f)$, but $\widehat{G(f_n)} \not\to \widehat{G(f)}$. In fact, $\widehat{G(f)}_0 = \hat{\gamma} = \{\zeta\colon |\zeta| \le 1\}$ and $\widehat{G(f_n)}_0 = \hat{\gamma}_n = \gamma_n$ and $\widehat{G(f_n)} \subseteq \overline{B}_2 \times \gamma_n$. Hence $\widehat{G(f_n)} \not\to \widehat{G(f)}$. Thus, in the corollary to Theorem 1, the assumption that $\widehat{G(f)}$ be a graph cannot be dropped.

This also shows that the fibers $\widehat{G(f)}_\lambda$ need not be convex, since $\widehat{G(f_n)}_0 = \gamma_n$. Related examples have also been found by J. Wermer and by Z. Slodkowski; see [**18**, §6].

9. Open questions. Let $f \in C(bB_2)$. Special cases would be when f is real-valued and/or smooth.

(a) What can be said about the rationally convex hull of $G(f)$? In some cases the rational hull coincides with the polynomial hull. Is this always the case? Or, can $G(f)$ be rationally convex?

(b) Are the fibers $\widehat{G(f)}$ connected? If not, can $\widehat{G(f)}$ be n-sheeted over B_2; 2-sheeted?

(c) To what extent does $\widehat{G(f)}$ contain analytic structure? What if we assume that $\widehat{G(f)}$ is a graph $G(F)$? What can be said about F? Special case: F smooth.

(d) Does there exist a real-valued function S^2 as in Theorem 4 such that the hull of its graph in $\mathbf{C}^2$ contains no analytic structure? By the work of Bedford and Klingenberg this cannot happen if the function is smooth.

REFERENCES

[1] H. Alexander, *A note on polynomial hulls*, Proc. Amer. Math. Soc., **33** (1972), 389–391.

[2] H. Alexander and J. Wermer, *Polynomial hulls with convex fibers*, Math. Ann., **271** (1985), 99–109.

[3] E. Bedford and B. Gaveau, *Envelopes of holomorphy of certain 2-spheres in* $\mathbf{C}^2$, Amer. J. Math., **105** (1983), 975–1009.

[4] E. Bedford and W. Klingenberg, *On the envelope of holomorphy of a 2-sphere in* $\mathbf{C}^2$, preprint, 1988.

[5] E. Bishop, *Holomorphic completions, analytic continuations and the interpolation of semi-norms*, Ann. of Math., **78** (1963), 468–500.

[6] A. Browder, *Cohomology of maximal ideal spaces*, Bull. Amer. Math. Soc., **67** (1961), 515–516.

[7] T. Gamelin, *Uniform Algebras*, Prentice-Hall, Englewood Cliffs, N.J., 1969.

[8] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, Van Nostrand Reinhold, N.Y., 1966.

[9] G. Lupacciolu, *A theorem on holomorphic extensions of CR-functions*, Pacific J. Math., **124** (1986), 177–191.

[10] K. Oka, *Sur les fonctions analytiques de plusieurs variables* II. *Domaines d'holomorphie*, J. Sci. Hiroshima, **7** (1937), 115–130.

[11] Z. Slodkowki, *Polynomially convex hulls with convex sections and interpolating spaces*, Proc. Amer. Math. Soc., **96** (1986), 255–260.

[12] G. Stolzenberg, *Polynomially and rationally convex sets*, Acta Math., **109** (1963), 259–289.

[13] ——, *A hull with no analytic structure*, J. Math. Mech., **12** (1963), 103–112.

[14] E. L. Stout, *Analytic continuation and boundary continuity of functions of several complex varibles*, Proc. Royal Soc. of Edinburgh, **89** (1981), 63–74.

[15] ——, *The Theory of Uniform Algebras*, Bogden and Quigley, Tarrytown-on-Hudson, N.Y., 1971.

[16] J. Wermer, *On algebras of continuous functions*, Proc. Amer. Math. Soc., **4** (1953), 866–869.

[17] ——, *Polynomially convex hulls and analyticity*, Arkiv för Mat., **20** (1982), 129–135.

[18] Z. Slodkowski, *Complex interpolation of normed and quasinormed spaces in several dimensions*, II. *Properties of harmonic interpolation*, preprint.

Received May 15, 1989.

UNIVERSITY OF ILLINOIS AT CHICAGO
CHICAGO, IL 60680